The Douglas Rachford Reflection Method and Generalizations

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Feasibility Problem

Given closed sets $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ the feasibility problem asks

$$\text{find } x \in \bigcap_{j=1}^{N} C_j.$$ 

Many problems can be cast in this form. Three examples:

1. **Linear systems** “$Ax = b$”: $C_j = \{x : \langle a_j, x \rangle = b_j\}$.
2. **Phase retrieval**: $C_1 = \{f : |\hat{f}| = m \text{ a.e.}\}$ and $C_2 = \{f : f = 0 \text{ on } D\}$.
3. **Matrix completion problems**: more on this later!

Projection algorithms are a popular approach to solving feasibility problems. They work on the following principle:

1. While the intersection might be difficult to deal with directly, the individual constraint sets are sufficiently “simple”.
2. “Simple” means we can efficiently compute nearest points.
3. Use an iterative scheme which employs nearest points to individual constraint sets at each stage, and obtain a solution in the limit.
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2. “Simple” means we can efficiently compute nearest points.
3. Use an iterative scheme which employs nearest points to individual constraint sets at each stage, and obtain a solution in the limit.
Let $S \subseteq \mathcal{H}$ be non-empty. The (nearest point) projection onto $S$ is the (set-valued) mapping,

$$P_S x := \left\{ s \in S : \|x - s\| \leq \inf_{s \in S} \|x - s\| \right\}.$$

If $S$ is closed and convex then projections exists uniquely with

$$P_S(x) = p \iff \langle x - p, s - p \rangle \leq 0 \text{ for all } s \in S.$$

The reflection w.r.t. $S$ is the (set-valued) mapping,

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Algorithmic Building Blocks

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The Douglas–Rachford Algorithm

Given an initial point \( x_0 \in \mathcal{H} \), the Douglas–Rachford method is the fixed-point iteration given by

\[
x_{n+1} \in T_{C_1, C_2} x_n \quad \text{where} \quad T_{C_1, C_2} := \frac{\text{Id} + R_{C_2} R_{C_1}}{2}.
\]

We hope that \( (x_n) \) converges to a fixed point of \( T_{C_1, C_2} \).

\[C_1 = \{ x \in \mathcal{H} : \|x\| \leq 1 \}, \quad C_2 = \{ x \in \mathcal{H} : \langle a, x \rangle = b \}.\]
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Why Fix $T_{C_1, C_2}$? Assuming single-valueness of $R_{C_1}$ and $R_{C_2}$ we have:

$$x \in \text{Fix } T_{C_1, C_2} \iff x = \frac{x + R_{C_2}R_{C_1}x}{2}$$

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$$\implies P_{C_1}x \in C_1 \cap C_2.$$  

The same argument for the set-valued case yields:

- If $x \in T_{C_1, C_2}x$ then there is an element of $P_{C_1}x$ contained in $C_1 \cap C_2$. 

Jonathan Borwein (CARMA, University of Newcastle)
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Let $T : \mathcal{H} \to \mathcal{H}$. Then $T$ is:

- **nonexpansive** if
  $$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$  

- **firmly nonexpansive** if
  $$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$
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**Proposition (Nonexpansive properties)**

The following are equivalent.

- $T$ is firmly nonexpansive.
- $I - T$ is firmly nonexpansive.
- $2T - I$ is nonexpansive.
- $T = \alpha I + (1 - \alpha)R$, for $\alpha \in (0, 1/2]$ and some nonexpansive $R$.
- Many other characterisations.
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**Nonexpansive properties of projections**

Let $C_1, C_2 \subseteq \mathcal{H}$ be closed and convex. Then

- $P_{C_1} := \arg\min_{c \in C_1} \|c - c\|$ is firmly nonexpansive.
- $R_{C_1} := 2P_{C_1} - I$ is nonexpansive.
- $T_{C_1, C_2} := \frac{1}{2}(I + R_{C_2}R_{C_1})$ is firmly nonexpansive.

Nonexpansive maps are closed under composition, convex combinations, etc. **Firmly nonexpansive maps need not be.** E.g., Composition of two projections onto subspace in $\mathbb{R}^2$ (Bauschke–Borwein–Lewis, 1997).
asymptotically regular if, for all \( x \in \mathcal{H} \),

\[
\| T^{n+1}x - T^n x \| \to 0.
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Any firmly nonexpansive mapping with at least one fixed point is asymptotically regular.

A useful Theorem for building iterative schemes:

**Theorem (Opial, 1967)**

Let \( T : \mathcal{H} \to \mathcal{H} \) be nonexpansive and asymptotically regular. Set \( x_{n+1} = Tx_n \). Then \( x_n \overset{w}{\rightharpoonup} x \) such that \( x \in \text{Fix } T \).

→ Design a non-expansive operator with a useful fixed point set.

Before proving this theorem, we require the following lemma.
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Proof of Opial’s Theorem

Lemma (Demiclosedness)

Let \( T : \mathcal{H} \to \mathcal{H} \) be nonexpansive and denote \( x_n := T^n x_0 \) for some initial point \( x_0 \in \mathcal{H} \). Suppose \( x_n \rightharpoonup x \) and \( x_n - Tx_n \to 0 \). Then \( x \in \text{Fix } T \).

Proof. Since \( T \) is nonexpansive,

\[
\|x - Tx\|^2 = \|x_n - Tx\|^2 - \|x_n - x\| - 2\langle x_n - x, x - Tx \rangle \\
= \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, Tx_n - Tx \rangle + \|Tx_n - Tx\|^2 \\
- \|x_n - x\|^2 - 2\langle x_n - x, x - Tx \rangle \\
\leq \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, Tx_n - Tx \rangle - 2\langle x_n - x, x - Tx \rangle.
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Since \( x_n \rightharpoonup x \) and \( x_n - Tx_n \to 0 \), it follows that each term tends to 0. \( \blacksquare \)
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Proof of Opial’s Theorem

Proof (Opial’s Theorem). Since $T$ is non-expansive, for any $y \in \text{Fix } T$, we have

$$\|T^{n+1} x - y\| \leq \|T^nx - y\| \leq \cdots \leq \|x - y\|.$$ 

Whence the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Fejér monotone w.r.t the closed convex set $\text{Fix } T$. By Th. 4.5.10(iii) of Lect. I (Properties of Fejér monotone sequences) the sequence $\{x_n\}_{n \in \mathbb{N}}$ has at most one weak cluster point in $\text{Fix } T$. To complete the proof it suffices to show: (i) $\{x_n\}_{n \in \mathbb{N}}$ has at least one cluster point; and (ii) that every cluster point of $\{x_n\}_{n \in \mathbb{N}}$ is contained in $\text{Fix } T$.

Indeed, as $\{x_n\}$ is bounded, it contains at least one weak cluster point. Let $z$ be any weak cluster point and denote by $\{x_{n_k}\}_{k \in \mathbb{N}}$ a subsequence weakly convergent to $z$. Since $T$ is asymptotically regular,

$$\|x_{n_k} - Tx_{n_k}\| \to 0.$$ 

By the Demiclosedness Lemma, $z \in \text{Fix } T$. This completes the proof. •
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The basic result which we have prove is the following.

**Theorem (Douglas–Rachford ‘56, Lions–Mercier ‘79, Eckstein–Bertsekas ‘92, . . .)**

Suppose $C_1, C_2 \subseteq \mathcal{H}$ are closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := T_{C_2, C_1} x_n$$

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Then $(x_n)$ converges weakly to some $x \in \text{Fix } T_{C_1, C_2}$ with $P_{C_1} x \in C_1 \cap C_2$.

- If the intersection is empty the iterates diverge: $\|x_n\| \to \infty$.
- Hesse *et al.* & Bauschke *et al.* (2014): Convergence is strong for subspaces, and the rate is linear whenever their sum is closed.
- Phan (arXiv:1401.6509v3): If $\dim \mathcal{H} < \infty$ and $\text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset$ then convergence in linear.
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- Phan (arXiv:1401.6509v3): If $\dim \mathcal{H} < \infty$ and $\text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset$ then convergence in linear.
The Douglas–Rachford Algorithm

The following generalization include potentially empty intersections. Let

\[ V := \overline{C_1 - C_2}, \quad v := P_V(0), \quad F := C_1 \cap (C_2 + v). \]

**Theorem (Bauschke–Combettes–Luke 2004)**

Suppose \( C_1, C_2 \subseteq H \) are closed and convex. Given \( x_0 \in H \) define 
\[ x_{n+1} := T_{C_2, C_1} x_n. \]
Then the following hold.

(a) \( x_n - x_{n+1} = P_{C_1} x_n - P_{C_2} R_{C_1} \to v \) and \( P_{C_1} x_n - P_{C_2} P_{C_1} \to v \).

(b) If \( C_1 \cap C_2 \neq \emptyset \) then \((x_n)\) converges weakly to a point in

\[ \text{Fix } T_{C_1, C_2} = C_1 \cap C_2 + N_V(0); \]

otherwise, \( \|x_n\| \to +\infty \).

(c) Exactly one of the following alternatives holds:

(i) \( F = \emptyset, \|P_{C_1} x_n\| \to +\infty \) and \( \|P_{C_2} P_{C_1} x_n\| \to +\infty \).

(ii) \( F \neq \emptyset \), the sequence \((P_{C_1} x_n)\) and \((P_{C_2} P_{C_1} x_n)\) are bounded and their weak cluster points are best approximation pairs relative to \((C_1, C_2)\).
Recall the moment problem from Lecture I for linear map \( A : X \to \mathbb{R}^M \) and a point \( y \in \mathbb{R}^M \) has constraints:

\[
C_1 := \mathcal{H}^+, \quad C_2 := \{ x \in \mathcal{H} : A(x) = y \}.
\]

The following theorem gives conditions for norm convergence.

**Theorem (Borwein–Sims–Tam 2015)**

Let \( \mathcal{H} \) be a Hilbert lattice, \( C_1 := \mathcal{H}^+ \), \( C_2 \) be a closed affine subspace with finite codimensions, and \( C_1 \cap C_2 \neq \emptyset \). For \( x_0 \in \mathcal{H} \) define \( x_{n+1} = T_{C_1,C_2}x_n \). Let \( Q \) denote the projection onto the subspace parallel to \( C_2 \). Then \( (x_n) \) converges in norm whenever:

(a) \( C_1 \cap \text{range}(Q) = \{0\} \),

(b) \( Q(C_2 - C_1) \subseteq C_1 \cup (-C_1) \) and \( Q(C_1) \subseteq C_1 \).

(c) \( C_2 \) has codimension 1.

For codimension greater than 1?
Recall the moment problem from Lecture I for linear map $A : X \to \mathbb{R}^M$ and a point $y \in \mathbb{R}^M$ has constraints:

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(a) $C_1 \cap \text{range}(Q) = \{0\}$,
(b) $Q(C_2 - C_1) \subseteq C_1 \cup (-C_1)$ and $Q(C_1) \subseteq C_1$.
(c) $C_2$ has codimension 1.

For codimension greater than 1?
Pierra’s Product Space Reformulation

For our constraint sets $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ we define

$$D := \{(x, x, \ldots, x) \in \mathcal{H}^N : x \in \mathcal{H}\}, \quad C := \prod_{j=1}^{N} C_j.$$ 

We now have an equivalent two set feasibility problem since

$$x \in \bigcap_{j=1}^{N} C_j \subseteq \mathcal{H} \iff (x, x, \ldots, x) \in D \cap C \subseteq \mathcal{H}^N.$$ 

Moreover the projections onto the new sets can be computed whenever $P_{C_1}, P_{C_2}, \ldots, P_{C_N}$. Denote $x = (x_1, x_2, \ldots, x_N)$ they are given by

$$P_D x = \left( \frac{1}{N} \sum_{j=1}^{N} x_j \right)^N \quad \text{and} \quad P_C x = \prod_{j=1}^{N} P_{C_j} x_j.$$
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Moreover the projections onto the new sets can be computed whenever $P_{C_1}, P_{C_2}, \ldots, P_{C_N}$. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ they are given by

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$$
A Many-set Douglas–Rachford Scheme?

Is there a Douglas–Rachford variant which can be used to solve the problem in the original space? *i.e.*, Without recourse to a product space formulation?

An obvious candidate is the following: Given \( x_0 \in \mathcal{H} \) define

\[
x_{n+1} = T_{A,B,C}x_n \quad \text{where} \quad T_{A,B,C} = \frac{I + R_C R_B R_A}{2}.
\]

A similar argument shows:

- \((x_n)\) converges weakly to a point \( x \in \text{Fix} \, T_{A,B,C} \).
- Unfortunately, it is possible that \( P_A x, P_B x, P_C x \not\in A \cap B \cap C \).
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Jonathan Borwein (CARMA, University of Newcastle)
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The Douglas Rachford Reflection Method and Generalizations
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A similar argument shows:

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A Many-set Douglas–Rachford Scheme?

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Let \( x_0 = (-\sqrt{3}, -1) \) & \( 2 \leq \alpha \leq \infty \).
Define constraints:

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\begin{align*}
A &= \{ \lambda(0,1) : |\lambda| \leq \alpha \}, \\
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Theorem (Borwein–T 2013)

Let $C_1, \ldots, C_N \subseteq \mathcal{H}$ be closed convex sets with nonempty intersection, let $T_j : \mathcal{H} \to \mathcal{H}$ and denote $T := T_M \ldots T_2 T_1$. Suppose the following three properties hold.

(i) $T$ is nonexpansive and asymptotically regular,

(ii) $\text{Fix } T = \cap_{j=1}^M \text{Fix } T_j \neq \emptyset$,

(iii) $P_{C_j} \text{Fix } T_j \subseteq C_{j+1}$ for each $j = 1, \ldots, N$.

Then, for any $x_0 \in \mathcal{H}$, the sequence $x_n := T^n x_0$ converges weakly to some $x$ such that $P_{C_1} x = P_{C_2} x = \cdots = P_{C_N} x$. In particular, $P_{C_1} x \in \bigcap_{i=1}^N C_i$.

Proof sketch. Denote $C_{N+1} := C_1$.

1. (i) + (ii) \implies (x_n) converges weakly to some $x \in \cap \text{Fix } T$.

2. (iii) + convex projection inequality yields

\[ \langle x - P_{C_{j+1}} x, P_{C_j} x - P_{C_{j+1}} x \rangle \leq 0 \text{ for all } j \]
A Common Framework

**Theorem (Borwein–T 2013)**

Let \( C_1, \ldots, C_N \subseteq \mathcal{H} \) be closed convex sets with nonempty intersection, let \( T_j : \mathcal{H} \rightarrow \mathcal{H} \) and denote \( T := T_M \cdots T_2 T_1 \). Suppose the following three properties hold.

1. \( T \) is nonexpansive and asymptotically regular,
2. Fix \( T = \cap_{j=1}^M \text{Fix } T_j \neq \emptyset \),
3. \( P_{C_j} \text{Fix } T_j \subseteq C_{j+1} \) for each \( j = 1, \ldots, N \).

Then, for any \( x_0 \in \mathcal{H} \), the sequence \( x_n := T^n x_0 \) converges weakly to some \( x \) such that \( P_{C_1} x = P_{C_2} x = \cdots = P_{C_N} x \). In particular, \( P_{C_1} x \in \bigcap_{i=1}^N C_i \).

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\[
\langle x - P_{C_{j+1}} x, P_{C_j} x - P_{C_{j+1}} x \rangle \leq 0 \text{ for all } j
\]
To complete the proof observe

\[
\frac{1}{2} \sum_{j=1}^{N} \| P_{C_{j+1}}x - P_{C_j}x \|^2
\]

\[
= \langle x, 0 \rangle + \frac{1}{2} \sum_{j=1}^{N} \left( \| P_{C_{j+1}}x \|^2 - 2 \langle P_{C_{j+1}}x, P_{C_j}x \rangle + \| P_{C_j}x \|^2 \right)
\]

\[
= \left\langle x, \sum_{j=1}^{N} (P_{C_j}x - P_{C_{j+1}}x) \right\rangle - \sum_{j=1}^{N} \langle P_{C_{j+1}}x, P_{C_j}x \rangle + \sum_{j=1}^{N} \| P_{C_{j+1}}x \|^2
\]

\[
= \sum_{i=1}^{N} \left\langle x, (P_{C_j}x - P_{C_{j+1}}x) \right\rangle - \sum_{j=1}^{N} \langle P_{C_{j+1}}x, P_{C_j}x - P_{C_{j+1}}x \rangle
\]

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= \sum_{j=1}^{N} \langle x - P_{C_{j+1}}x, P_{C_j}x - P_{C_{j+1}}x \rangle \leq 0.
\]
Corollary (Borwein–T 2013)

Let \( C_1, C_2, \ldots, C_N \subseteq \mathcal{H} \) be closed and convex with non-empty intersection. Given \( x_0 \in \mathcal{H} \) define

\[
x_{n+1} := (T_{C_N, C_1} T_{C_{N-1}, C_N} \cdots T_{C_2, C_3} T_{C_1, C_2}) x_n \text{ where } T_{C_j, C_{j+1}} = \frac{I + R_{C_{j+1}} R_{C_j}}{2}.
\]

Then \( (x_n) \) converges weakly to a point \( x \) such that \( P_{C_1} x = \cdots = P_{C_N} x \).

- **Using Hundal (2004):** There exists a hyperplane and convex cone with nonempty intersection such that convergence is not strong.
- **Bauschke–Noll–Phan (2014):** If \( \dim \mathcal{H} < \infty \) and \( \bigcap_{j=1}^N \text{ri } C_j \neq \emptyset \) then convergence is linear.
- **Bauschke–Noll–Phan (2014):** If \( \text{Fix } T_{[1 \ 2 \ \ldots \ N]} \) is bounded linearly regular and \( C_j + C_{j+1} \) is closed, for each \( j \), then convergence is linear.
Consider the following examples with $C_2 := 0 \times \mathbb{R}$, and

$$C_1 := \text{epi}(\exp(\cdot) + 1) \text{ or } \text{epi}(\cdot^2 + 1).$$
The following variant lends itself to parallel implementation.

**Corollary (Borwein-Tam 2013)**

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := \frac{1}{N} \left( \sum_{j=1}^{N} T_{C_j, C_{j+1}} \right) x_n \quad \text{where} \quad T_{C_j, C_{j+1}} = \frac{I + R_{C_{j+1}} R_{C_j}}{2}.$$  

Then $(x_n)$ converges weakly to a point $x$ such that $x = \cdots = P_{C_N} x$.

**Proof sketch.** For $x_0 \in \mathcal{H}$, set $x_0 = (x_0, \ldots, x_0) \in \mathcal{H}^N$. Apply the theorem to the product-space iteration

$$x_{n+1} = P_D \left( \prod_{i=1}^{N} T_{C_i, C_{i+1}} \right) x_n.$$
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x_{n+1} = P_D \left( \prod_{i=1}^{N} T_{C_i, C_{i+1}} \right) x_n.
\]
Cyclically Anchored Douglas–Rachford Method

Choose the first set $C_1$ to be the anchor set, and think of

$$\bigcap_{j=1}^{N} C_j = C_1 \cap \left( \bigcap_{j=2}^{N} C_j \right).$$

**Theorem (Bauschke–Noll–Phan 2014)**

Let $C_1, C_2, \ldots, C_N \subseteq H$ be closed and convex with non-empty intersection. Given $x_0 \in H$ define

$$x_{n+1} := \prod_{j=2}^{N} T_{C_1, C_j} x_n \quad \text{where} \quad T_{C_1, C_j} = \frac{I + R_{C_j} R_{C_1}}{2}.$$

Then $(x_n)$ converges weakly to a point $x$ such that $P_{C_1} x \in \bigcap_{j=1}^{N} C_j$.

- **Bauschke–Noll–Phan (2014):** If $\dim H < \infty$ and $\bigcap_{j=1}^{N} \text{ri } C_j \neq \emptyset$ then convergence is linear.
- **Bauschke–Noll–Phan (2014):** For subspaces, if $\text{Fix } T_{C_1, C_j}$ is bounded linearly regular and $C_1 + C_j$ is closed then convergence is linear.
Averaged Anchored Douglas–Rachford Method

The scheme also has a parallel counterpart:

**Theorem**

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := \frac{1}{N} \left( \sum_{j=1}^{N} T_{C_1, C_j} \right) x_n$$

where

$$T_{C_1, C_j} = \frac{I + R_{C_j} R_{C_i}}{2}.$$

Then $(x_n)$ converges weakly to a point $x$ such that $P_{C_1} x \in \bigcap_{j=1}^{N} C_j$.

**Proof sketch.** Use the product space (as we did for the averaged DR iteration) up the iteration:

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\]
The (classical) Douglas–Rachford method better than theory suggests performance on non-convex problems. Consequently many variants and extensions have recently been proposed. Even in the convex setting there are many subtleties and open questions.

- Norm convergence for realistic moment problems with codimension greater than 1?

- Experimental comparison of the variants needed.
1. Let $T_j : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive, for $j = 1, \ldots, r$, and define $T := T_r \ldots T_2 T_1$. If $\text{Fix } T \neq \emptyset$ show that $T$ is asymptotically regular.

2. Show that the cyclic DR method becomes MAP in certain cases. Hence find an example where convergence in cyclic DR is only weak.

3. (Hard) Prove or disprove: The Douglas–Rachford algorithm converges in norm for the moment problem when the affine set has codimension 2.


Many resources available at: