Exercise 11.1. (a) How many different MZVs of given weight \( k \) exists? 

(b) Compute the limit of \( d_k^{1/k} \) as \( k \to \infty \) for the sequence \( d_k \) constructed in Conjecture 5.

(c) Any polynomial in single zeta values,

\[
(\pi^2)^{s_0} \zeta(3)^{s_1} \zeta(5)^{s_2} \cdots \zeta(2l+1)^{s_l}, \quad s_0, s_1, s_2, \ldots, s_l \in \mathbb{Z}_{\geq 0},
\]

belongs to the linear space \( \mathcal{Z}_k \) of MZVs of weight

\[ k = 2s_0 + 3s_1 + 5s_2 + \cdots + (2l + 1)s_l. \]

Assuming Conjecture 1, all these polynomials are linearly independent over \( \mathbb{Q} \). Denote by \( c_k \) the total number of such polynomials of given weight \( k \). Compute \( c_k \) for small values of \( k \) (namely, for \( k \leq 12 \)) and show that \( c_k < d_k \) for \( k \geq 8 \). (In other words, the algebra of MZVs cannot be fully generated by single zeta values.)

(d) For the sequence \( c_k \) from part (c), find a general analytic formula and compute the limit of \( c_k^{1/k} \) as \( k \to \infty \).

Although proving Conjectures 4–6 in the form they are given is hopeless at the present time, the ‘true’ MZVs in \( \mathbb{R} \) are the images under a \( \mathbb{Q} \)-linear map of certain ‘motivic’ MZVs which are defined purely algebraically. The Terasoma and Goncharov–Deligne bound \( \dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k \), as well as Conjecture 4 about disjointness of the subspaces \( \mathcal{Z}_k \), are shown to be true for this algebraic version of MZVs. Terasoma and Goncharov established the bound by showing that all MZVs are periods of so-called mixed Tate motives that are unramified over \( \mathbb{Z} \). Another well-known conjecture in the area states the converse, that is, that all periods of mixed Tate motives over \( \mathbb{Z} \) can be expressed as linear combinations (over \( \mathbb{Q}[(2\pi i)^{\pm 1}] \)) of MZVs. Equivalently, this says that the dimension of the space of motivic MZVs of weight \( k \) is exactly \( d_k \).

The result obtained by Brown was a proof of the latter conjecture and also of the fact that the motivic MZVs from Hoffman’s conjectural basis in Conjecture 6 form a basis of the corresponding \( \mathcal{Z}_k \). In his proof Brown assumes certain quite specific properties of certain coefficients occurring in the relations over \( \mathbb{Q} \) of some special MZVs. Specifically, he shows that the special MZVs

\[
\xi(m, n) := \zeta(\{2\}^m, 3, \{2\}^n), \quad n, m \geq 0,
\]

which are part of Hoffman’s basis, are \( \mathbb{Q} \)-linear combinations of products \( \pi^{2\mu} \zeta(2\nu + 1) \) with \( \mu + \nu = m + n + 1 \). His proof, which used motivic ideas, did not yield an explicit formula for these linear combinations, but numerical evidence suggested several properties satisfied by their coefficients (and, in particular, of the coefficient of \( \zeta(2m + 2n + 3) \)) which he could show were sufficient to imply the truth of both Hoffman’s conjecture and the statement about motivic periods. The next section contains a statement and proof of an explicit formula expressing the numbers \( \xi(m, n) \) in terms of single zeta values, as well as confirmation of the numerical properties that were required for Brown’s proof.
12. Zagier’s identity for $\xi(m, n)$

Before giving the formula for the numbers $\xi(m, n)$, we first recall the much easier formula from the family (5.4),

$$\xi(n) := \zeta\left(\{2\}^n\right) = \frac{\pi^{2n}}{(2n+1)!}, \quad n \geq 0,$$

(12.1)

for the simplest of the Hoffman basis elements.

**Theorem 12.1** (Zagier). For all integers $m, n \geq 0$, we have

$$\xi(m, n) = 2 \sum_{r=1}^{m+n+1} (-1)^{r-1} \left(1 - \frac{1}{2^{2r}}\right) \left(\frac{2r}{2m+1}\right) - \left(\frac{2r}{2n+2}\right) \xi(m+n-r+1)\zeta(2r+1),$$

(12.2)

where the value of $\xi(m + n - r + 1)$ is given by (12.1). Conversely, each product $\xi(\mu)\zeta(k - 2\mu)$ of odd weight $k$ is a rational combination of numbers $\xi(m, n)$ with $m + n = (k - 3)/2$.

**Remark.** The second part of the theorem, which we only discuss as Exercise 12.6 below, gives rise to several other open questions.

The coefficients in the expressions for the products $\xi(\mu)\zeta(k - 2\mu)$ as linear combinations of the numbers $\xi(m, n)$ do not seem to be given by any simple formula. For example, the inverse of the $5 \times 5$ matrix

$$\begin{pmatrix}
3 & -\frac{15}{2} & \frac{189}{16} & -\frac{255}{16} & \frac{4603}{256} \\
0 & -\frac{15}{2} & \frac{315}{8} & -\frac{1753}{16} & \frac{9585}{64} \\
0 & 0 & \frac{157}{16} & -\frac{889}{16} & \frac{10689}{128} \\
0 & 2 & -30 & \frac{1985}{16} & -\frac{11535}{64} \\
-2 & 12 & -30 & 56 & \frac{17925}{256}
\end{pmatrix}$$

expressing the vector $\{\xi(m, n) : m + n = 4\}$ in terms of the vector $\{\zeta(2m+3)\xi(n) : m + n = 4\}$ is

$$\frac{1}{2555171} \begin{pmatrix}
11072595 & 19354609 & 23488575 & 22114173 & 15331307 \\
59984880 & 122931470 & 160083660 & 147349978 & 89977320 \\
246001728 & 508012288 & 669540272 & 613537008 & 369002592 \\
494939520 & 1022542528 & 1349936640 & 1236102000 & 742409280 \\
300405248 & 620662272 & 819546624 & 750355968 & 450607872
\end{pmatrix},$$

in which no simple pattern can be discerned and in which even the denominator (prime 2555171) cannot be recognised. This shows that the Hoffman basis, although it works over $\mathbb{Q}$, is very far from giving a basis over $\mathbb{Z}$ of $\mathbb{Z}$-linear span of MZVs, and suggests the question of finding better basis elements.

The following question is supported by numerical data for $m + n \leq 30$, but remains open.
Exercise 12.1. Denote $M_k$ the matrix from (12.2) expressing the vector $\{\xi(m, n) : m + n = k\}$ in terms of the vector $\{\zeta(2m + 3)\xi(n) : m + n = k\}$, that is,

$$M_k = \left(2(-1)^{\mu} \left(1 - \frac{1}{2^{\mu+2}}\right) \left(\frac{2\mu + 2}{2m + 1}\right) - \left(\frac{2\mu + 2}{2k - 2m + 2}\right)\right)_{0 \leq m, \mu \leq k}. \quad (12.3)$$

Show that all the entries of the inverse matrix $M_k^{-1}$ are strictly positive.

The strategy to prove Theorem 12.1 is to compare the two generating functions

$$F(x, y) = \sum_{m, n \geq 0} (-1)^{m+n+1} \xi(m, n) x^{2m+1} y^{2n+2} \quad (12.4)$$

and

$$\hat{F}(x, y) = \sum_{m, n \geq 0} (-1)^{m+n+1} \hat{\xi}(m, n) x^{2m+1} y^{2n+2}, \quad (12.5)$$

where

$$\hat{\xi}(m, n) = 2 \sum_{r=1}^{m+n+1} (-1)^{r-1} \left(1 - \frac{1}{2^{2r}}\right) \left(\frac{2r}{2m + 1}\right) - \left(\frac{2r}{2n + 2}\right) \xi(m+n-r+1) \zeta(2r+1)$$

denotes the expression occurring on the right-hand side of (12.2). Of course, if the two expressions were the same, we would be done, but in fact they are completely different, one involving a generalized hypergeometric function

$$\binom{p+1}{b_1, \ldots, b_p}(a_0, a_1, \ldots, a_p | z) = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{z^n}{n!} \quad (12.6)$$

(cf. Section 5), and the other a complicated linear combination of the digamma functions, $\psi(x) = \Gamma'(x)/\Gamma(x)$. We therefore have to proceed indirectly, showing that both $F$ and $\hat{F}$ are entire functions (of order 1) in $x$ and $y$ and that they agree whenever $x = y$ or $x$ or $y$ is an integer (the details of this comparison will be however skipped). This will imply the equality $F = \hat{F}$, and hence Theorem 12.1. There is however a belief (that is, an open problem!) that the use of known hypergeometric identities could lead to a direct proof of $F = \hat{F}$; this would considerably simplify Brown’s proofs mentioned above.

Lemma 12.1. The generating function $F(x, y)$ can be expressed as the product of a sine function and a hypergeometric function:

$$F(x, y) = \frac{\sin \pi x}{\pi} \frac{\partial}{\partial z} \binom{3}{1+x, 1-x | 1} \bigg|_{z=0}. \quad (12.7)$$
Proof. The proof is similar to that for (12.1):

\[ F(x, y) = \sum_{m,n \geq 0} (-1)^{m+n+1} \zeta(\{2\}^m,3,\{2\}^n)x^{2m+1}y^{2n+2} \]

\[ = -xy^2 \sum_{r=1}^{\infty} \prod_{k=1}^{r-1} \left( 1 - \frac{y^2}{k^2} \right) \cdot \frac{1}{r^3} \prod_{l=r+1}^{\infty} \left( 1 - \frac{x^2}{l^2} \right) \]

\[ = \frac{\sin \pi x}{\pi} \sum_{r=1}^{\infty} \frac{(-y)_r(y)_r}{(1-x)_r(1+x)_r} \frac{1}{r}, \]

and this formula is seen to be equivalent to (12.7).

\[ \Box \]

Lemma 12.2. The generating function \( \hat{F}(x,y) \) can be expressed as an integral linear combination of fourteen functions of the form

\[ \psi \left( 1 + \frac{u}{2} \right) \frac{\sin \pi v}{2\pi} \quad \text{with} \quad u \in \{ \pm x \pm y, \pm 2x \pm 2y, \pm 2x \}, \quad v \in \{ x, y \}. \]

Proof. From the definition of \( \hat{F}(x,y) \) and (12.1) we find

\[ \hat{F}(x,y) = 2 \sum_{m,n \geq 0} (-1)^{m+n}x^{2m+1}y^{2n+2} \sum_{r=1}^{m+n+1} (-1)^r \left( 1 - 2^{-2r} \right) \left( \frac{2r}{2m+1} \right) \]

\[ \left\{ \begin{array}{c}
\left( 2r \right) \\
(2n+2) \end{array} \right\} \frac{\pi^{2(m+n-r+1)} m} {2(2n+r+1) + 1} \zeta(2r + 1) \]

\[ = 2 \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} \zeta(2r+1) \left( \frac{2r}{(2n+2)} \right) x^{2(2r+s+n)} y^{2n+2} \]

\[ - (1 - 2^{-2r}) \sum_{m=0}^{r-1} \left( \frac{2r}{2m+1} \right) x^{2m+1} y^{2(2r+n)} \]

\[ = \frac{2 \sin \pi x}{\pi} \sum_{r=1}^{\infty} \zeta(2r+1) \sum_{n=0}^{r-1} \left( \frac{2r}{2n+2} \right) x^{2(r+n-1)} y^{2(n+1)} \]

\[ \quad - \frac{2 \sin \pi y}{\pi} \sum_{r=1}^{\infty} (1 - 2^{-2r}) \zeta(2r+1) \sum_{m=0}^{r-1} \left( \frac{2r}{2m+1} \right) x^{2m+1} y^{2(r+m-1)} \]

\[ = \frac{\sin \pi x}{\pi} \sum_{r=1}^{\infty} \zeta(2r+1)[(x+y)^{2r} + (x-y)^{2r} - 2x^{2r}] \]

\[ \quad - \frac{\sin \pi y}{\pi} \sum_{r=1}^{\infty} (1 - 2^{-2r}) \zeta(2r+1)[(x+y)^{2r} - (x-y)^{2r}] \]

\[ = \frac{\sin \pi x}{\pi} (A(x+y) + A(x-y) - 2A(x)) - \frac{\sin \pi y}{\pi} (B(x+y) - B(x-y)), \]
where (cf. the final part of Section 5)

\[ A(t) = \sum_{r=1}^{\infty} \zeta(2r+1)t^{2r} = \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{2r}}{n^{2r+1}} = \sum_{n=1}^{\infty} \frac{t^2}{n(n^2 - t^2)}, \]

\[ B(t) = \sum_{r=1}^{\infty} (1 - 2^{-2r})\zeta(2r+1)t^{2r} = \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2r}}{n^{2r+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^2}{n(n^2 - t^2)}. \]

Decomposing the summands into partial fractions allows us to represent the generating functions \(A\) and \(B\) in terms of the digamma function:

\[ A(t) = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n-t} + \frac{1}{n+t} - \frac{2}{n} \right) = \psi(1) - \frac{1}{2} \left( \psi(1+t) + \psi(1-t) \right), \]

\[ B(t) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n-t} + \frac{1}{n+t} - \frac{2}{n} \right) = A(t) - A\left(\frac{t}{2}\right). \]

Substituting these expressions into the previous derivation gives an expression for \(\hat{F}\) of the form stated in the lemma. \(\square\)

**Exercise 12.2.** Show the equality \(F(x, y) = \hat{F}(x, y)\) directly by using the representations in Lemmas 12.1 and 12.2.

As mentioned above, Exercise 12.2 is an open problem.

The following change of exercises sketches the remaining ingredients of the proof of Theorem 12.1.

**Exercise 12.3.** Show that both \(F(x, y)\) and \(\hat{F}(x, y)\) are entire functions on \(\mathbb{C}^2\) and are bounded by a constant multiple of \(e^{\pi X \log X}\) as \(X = \max\{|x|, |y|\} \to \infty\), and also by a multiple (depending on \(y\)) of \(e^{\pi |\text{Im } x|}\) as \(|x| \to \infty\) with \(y \in \mathbb{C}\) fixed.

**Remark.** The derivation makes use of analytic estimates of the coefficients of both \(F(x, y)\) and \(\hat{F}(x, y)\) but also of certain ‘standard’ theorems of complex analysis, like the Phragmén–Lindelöf theorem (an extension of the maximum modulus principle to functions which are analytic in sector domains and strips).

**Exercise 12.4.** Show that for \(x \in \mathbb{C}\) the following equality holds:

\[ F(x, x) = -\frac{\sin \pi x}{\pi} A(x) = \hat{F}(x, x), \]

where \(A(x)\) is the meromorphic function defined in the proof of Lemma 12.2.

**Exercise 12.5.** (a) Prove that for all \(n \in \mathbb{Z}_{>0}\) and \(x \in \mathbb{C}\),

\[ F(x, n) = \frac{\sin \pi x}{\pi} \sum_{|k| \leq n}^{\ast} \frac{\text{sgn } k}{x - k} = \hat{F}(x, n), \]

where the asterisk means that the terms \(k = \pm n\) are to be weighted with a factor 1/2.
(b) Prove that for all \( m \in \mathbb{Z}_{>0} \) and \( y \in \mathbb{C} \),
\[
F(m, y) = (-1)^m + \frac{\sin \pi y}{\pi} \sum_{|k| \leq m} (-1)^{m-k} \frac{1}{k-y} = \hat{F}(m, y),
\]
with the same convention about the asterisk.

Finally, we make use of the following result.

**Lemma 12.3.** An entire function \( f: \mathbb{C} \to \mathbb{C} \) that vanishes at all integers and satisfies \( f(z) = O(e^{\pi \Im z}) \) as \( |z| \to \infty \) is a constant multiple of \( \sin \pi z \).

**Proof.** Because \( |\Im z| \leq |z| \), the estimate implies \( f(z) = O(e^{\pi |z|}) \) as \( |z| \to \infty \); in particular, \( f(z) \) has order 1, and so does the function \( g(z) = f(z)/\sin \pi z \) (which is indeed entire as it does not have poles). The growth hypothesis on \( f \) implies that \( g \) is bounded outside a strip of finite width around the real axis, and then it follows from the Phragmén–Lindelöf theorem that it is also bounded inside this strip (since it has finite order), so that \( g \) is constant by Liouville’s theorem. \( \square \)

**Proof of Theorem 12.1.** We can now complete the proof of the main equality 12.2 as follows. We have shown that \( F(x, y) \) and \( \hat{F}(x, y) \) are entire functions of \( x \) and \( y \) satisfying certain (same) estimates, and that they agree whenever \( x = y \) or either \( x \) or \( y \) is an integer. (The latter fact follows from Exercise 12.5 and the fact that both \( F(x, y) \) and \( \hat{F}(x, y) \) are odd functions of \( x \) and even functions of \( y \) and vanish when \( y = 0 \).) It follows that, for fixed \( y \), the function \( f(x) = F(x, y) - \hat{F}(x, y) \) is an entire function which vanishes at all integers and satisfies \( f(x) = O(e^{\pi \Im x}) \) as \( |x| \to \infty \), so that by Lemma 12.3 it is a multiple of \( \sin \pi x \),
\[
F(x, y) - \hat{F}(x, y) = h(y) \sin \pi x,
\]
for a certain entire function \( h(y) \). Substituting \( y = x \) into the equality we get \( h(x) = 0 \) identically, so that indeed \( F(x, y) - \hat{F}(x, y) = 0 \) for all \( x \) and \( y \), implying \( \xi(m, n) = \hat{\xi}(m, n) \) as required. \( \square \)

**Exercise 12.6.** Prove the second statement of the theorem (that is, the invertibility of matrix \( M_k \) in (12.3)) by computing the 2-adic valuation of the entries of the matrix.

### 13. Double zeta values and products of single zeta values

In this section we fix an odd number \( k = 2l + 1 \geq 3 \) and discuss the relationship between the double zeta values \( \zeta(m, n) \), the zeta products \( \zeta(m)\zeta(n) \), and our latest heroes \( \xi(\mu, \nu) \), all of weight \( m+n = 2(\mu + \nu) + 3 = k \).

It was already found by Euler (explicitly for \( k \) up to 13) that all double zeta values of odd weight are rational linear combinations of products of single zeta values.

**Theorem 13.1.** The double zeta value \( \zeta(m, n) \) (with \( m \geq 2 \) and \( n \geq 1 \)) of weight \( m+n = k = 2l+1 \) is given in terms of the products \( \zeta(2s)\zeta(k-2s) \), \( s = 0, 1, \ldots, l-1 \),...
by

$$\zeta(m, n) = (-1)^n \sum_{s=0}^{l-1} \left( \binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{m,2s} + (-1)^n \delta_{s,0} \right) \zeta(2s) \zeta(k-2s).$$

(13.1)

**Proof.** The harmonic and shuffle products in the case of single zeta values result in

$$\zeta(r) \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(k), \quad \text{where } r + s = k, \ r, s \geq 2,$$

(13.2)

$$\zeta(r) \zeta(s) = \sum_{m=2}^{k-1} \left( \binom{m-1}{r-1} + \binom{m-1}{s-1} \right) \zeta(m, k-m), \quad \text{where } r + s = k, \ r, s \geq 2,$$

(13.3)

In both cases we can express without loss of generality that $r \leq s$, since both sides of the equations are symmetric in $r$ and $s$. This will give us only $2l-1$ equations for the $2l-1$ unknowns $\zeta(m, k-m)$, $2 \leq m \leq k-1$. However, both (13.2) and (13.3) remain true if we fix any value $T$ (that is, any regularized) for the divergent zeta value $\zeta(1)$ (here 0 or Euler’s constant $\gamma$ would be natural choices but we can also simply take $T$ to be an indeterminate) and use one of them to define the divergent double zeta value $\zeta(1, k-1)$, so that this gives $2l-1$ equations in $2l-1$ unknowns. To solve them, we introduce the generating functions

$$P(x, y) = \sum_{r,s \geq 1 \atop r+s=k} \zeta(r) \zeta(s) x^{r-1} y^{s-1} \quad \text{and} \quad Q(x, y) = \sum_{m,n \geq 1 \atop m+n=k} \zeta(m, n) x^{m-1} y^{n-1},$$

with the convention $\zeta(1) = T$ and $\zeta(1, k-1) = \zeta(k-1) T - \zeta(k) - \zeta(k-1, 1)$. Then the (double shuffle) relations (13.2) and (13.3) translate into equations

$$P(x, y) = Q(x, y) + Q(y, x) + \zeta(k) \frac{x^{k-1} - y^{k-1}}{x-y} = Q(x, x+y) + Q(y, x+y).$$

Using $Q(-x, -y) = -Q(x, y)$ (for $k$ odd), allows us to solve for $Q$:

$$Q(x, y) = R(x, y) + R(x-y, -y) + R(x-y, x),$$

where $R(x, y) = \frac{1}{2} \left( P(x, y) + P(-x, y) - \zeta(k) \frac{x^{k-1} - y^{k-1}}{x-y} \right)$.

This is equivalent (because of $\zeta(0) = -\frac{1}{2}$) to (13.1).

\[\square\]

Either of the double shuffle relations (13.2) and (13.3) permits us to express the single zeta products $\zeta(2r) \zeta(k-2r)$ in terms of all double zeta values of weight $k$, but we would like to do this using

(a) only the ‘odd-even’ values $\zeta(k-2r, 2r)$, where we also include $\zeta(k)$ to have the right number of quantities, or

(b) only the ‘even-odd’ double zeta values $\zeta(k-2r-1, 2r+1)$. 


This turns out to be possible only in case (a), as we now show.

Since in case (a) we have taken \( \zeta(k) \) as one of the basis elements, we can omit it from the basis and work modulo \( \zeta(k) \) in the right-hand side of (13.1), which simplifies to

\[
\zeta(k - 2r, 2r) \equiv \sum_{s=1}^{l-1} \left( \binom{2l - 2s}{2l - 2r} + \binom{2l - 2s}{2r - 1} \right) \zeta(2s) \zeta(k - 2s), \quad 1 \leq r \leq l - 1, \tag{13.4}
\]

where the congruence is modulo \( \mathbb{Q}\zeta(k) \).

**Theorem 13.2.** For odd \( k = 2l + 1 \geq 3 \), the products \( \zeta(2s) \zeta(k - 2s), \quad 1 \leq s \leq l - 1 \), are expressible in terms of double zeta values \( \zeta(k - 2r, 2r), \quad 1 \leq r \leq l - 1 \).

**Proof.** Let \( N_k \) be the \((l-1) \times (l-1)\) matrix whose \((r,s)\)-entry is the sum of binomials in (13.4). It is sufficient to show that the determinant of the matrix is non-zero.

Any binomial coefficient \( \binom{m}{n} \) with \( m \) even and \( n \) odd is even, because in this case \( \binom{m}{n} = m \binom{m-1}{n-1} \).

Thus, the matrix \( N_k \) is congruent modulo 2 to a unipotent triangular matrix and hence has odd determinant. \( \square \)

**Remark.** The immediate consequence of Theorems 12.1 and 13.2 is the following result. For each odd \( k = 2l + 1 \geq 3 \), the \( l \) numbers \( \zeta(k) \) and \( \zeta(k - 2r, 2r), \quad 1 \leq r \leq l - 1 \), span the same space over \( \mathbb{Q} \) as the \( l \) numbers

\[
\{ \xi(m,n) : m + n = l - 1 \} \quad \text{or} \quad \{ \pi^{2r} \zeta(k - 2r) : 0 \leq r \leq l - 1 \}.
\]

Zagier made several experimental observations about the matrix \( N_k \) which we give here as open problems.

**Exercise 13.1.** For \( k = 2l + 1 \geq 3 \) and the matrix \( N = N_k \) defined above, show the following.

(a) \( \det N = \pm (k-2)!! \), where \( (k-2)!! = 1 \cdot 3 \cdot 5 \cdots (k-2) \) is the ‘double factorial’ and the sign is \(-1\) if \( l \equiv 3 \pmod{4} \) and \( +1 \) otherwise.

(b) The entries of the inverse matrix \( N^{-1} \) are explicitly given by either of the two expressions

\[
(N^{-1})_{s,r} = \begin{cases} 
-2 & \sum_{n=0}^{k-2s} \binom{k-2r-1}{k-2s-n} \binom{n+2s-2}{n} B_n, \\
2 & \sum_{n=0}^{k-2s} \binom{2r-1}{k-2s-n} \binom{n+2s-2}{n} B_n, 
\end{cases} \quad 1 \leq s, r \leq l - 1,
\]

where \( B_n \) denotes the \( n \)th Bernoulli number (see Section 1).