16. q-Analogues of MZVs

The classical idea of introducing an additional parameter to an expression or formula we wish to deal with, is quite fruitful in many situations. This may simplify a proof of the corresponding identity or lead to a more general identity which have several other useful specializations of the introduced parameter. We have already experienced the usefulness of the method on the example of functional models of generalised polylogarithms in Section 4 and of (no name) function in Section 7. They were used for proving the shuffle and stuffle relations of MZVs, respectively. Because they (are expected to) satisfy only 'half' of relations of MZVs, we can hardly use them as a live imitation of the latter numbers.

The story of introducing the parameter q (or, the 'quantum' parameter) often has a different flavor. Note that the basic idea is simply to replace a number n (not necessarily an integer!) by the function $[n] = [n]_q := (1 - q^n)/(1 - q)$; this is, of course, nothing else but a polynomial for positive $n \in \mathbb{Z}$. The actual motivation of the replacement has strong analytic grounds:

$$\lim_{\substack{q\to 1\\ 0<q<1}} [n]_q = n,$$

so that the (sometimes formal) limit as $q \to 1$ produces back the original limits. Note however that this is only a part of the recipe, as multiplying the 'q-number' $[n]_q$ by any power of q makes exactly the same job as $q \to 1$. *Getting the right exponents of q is an art.*

The main requirement from a q-model of MZVs (or MZSVs) is a better understanding of the structure of linear and algebraic relations between the corresponding numbers. An important advantage of the q-model is that proving the absence of such relations and guessing their existence are usually a much easier task: for example, the linear independence of any version of q-MZVs (and much more) is known, while just the irrationality of odd single zeta values seems to be hard. On the other hand, showing that some relations hold is normally easier for numbers than for functions. The main problem here is finding an appropriate q-analogue which is often dictated by already existing proofs of the corresponding original identities.

An unfortunate thing about MZVs is that there is no uniform q-generalization of the multiple zeta (star) values. Having however several q-analogues in mind and a simple way to pass from one q-model to another gives one a very natural parallel between the numbers and their q-analogues.

There are very good reasons to believe that the most perfect q-extension of MZVs is given by

$$\zeta_q(s_1, s_2, \dots, s_l) = \sum_{n_1>n_2>\dots>n_l\ge 1} \frac{q^{n_1(s_1-1)+n_2(s_2-1)+\dots+n_l(s_l-1)}}{[n_1]^{s_1}[n_2]^{s_2}\cdots[n_l]^{s_l}}, \tag{16.1}$$

where conditions on the multi-index $\boldsymbol{s} = (s_1, \dots, s_l)$ are exactly the same as for the MZVs (1.5) (that is, the multi-index is admissible). The corresponding q-analogues of the values of Riemann's zeta function are in this case as follows:

$$\zeta_q(s) = \sum_{n\ge1} \frac{q^{n(s-1)}}{[n]^s}.$$

The q-model (16.1) inherits many relations available for MZVs $\zeta(\boldsymbol{s})$. There is a version of stuffle relations, which is based on the identity

$$\frac{q^{n(s-1)}}{[n]_q^s}\,\frac{q^{m(r-1)}}{[m]_q^r}\bigg|_{m=n} = (1-q)\frac{q^{n(s+r-2)}}{[n]_q^{s+r-1}} + \frac{q^{n(s+r-1)}}{[n]_q^{s+r}};$$

there is however no reasonably nice version of shuffle relations. The following result of Okuda and Takeyama, which includes numerous implications, is a convincing argument to count the q-MZVs (16.1) appropriate enough. In order to state it, we define the *height* $m = m(\boldsymbol{s})$ of a multi-index $\boldsymbol{s} = (s_1, \dots, s_l)$ to be the number of components satisfying $s_j > 1$; for an admissible $\boldsymbol{s}$ we have $s_1 > 1$, so that $m(\boldsymbol{s}) \ge 1$. Denote the set of admissible multi-indices of fixed weight $w = |\boldsymbol{s}|$, length $l = \ell(\boldsymbol{s})$ and height $m = m(\boldsymbol{s})$ by $I_0(w, l, m)$, and set

$$\Phi_q(x,y,z) := \sum_{w,l,m=0}^{\infty} x^{w-l-m} y^{l-m} z^{m-1} \sum_{\boldsymbol{s}\in I_0(w,l,m)} \zeta_q(\boldsymbol{s}).$$

Theorem 16.1. *The generating function Φ_q is given by*

$$\begin{aligned} 1 + (z-xy)\Phi_q(x,y,z) &= \prod_{n=1}^{\infty} \frac{([n]_q - \alpha q^n)([n]_q - \beta q^n)}{([n]_q - x q^n)([n]_q - y q^n)} \\ &= \exp\biggl(\sum_{k=2}^{\infty} \frac{x^k + y^k - \alpha^k - \beta^k}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta_q(j)\biggr), \qquad (16.2) \end{aligned}$$

where α and β are determined by

$$\alpha + \beta = x + y + (q-1)(z-xy), \qquad \alpha\beta = z.$$

In particular, the sum of the multiple q-zeta values of fixed weight, length and height is a polynomial in q and single q-zeta values.

The limiting case $q \to 1$ was established earlier by Ohno and Zagier.

Corollary 1. *We have the generating function identity*

$$\begin{aligned} &\sum_{s,r=0}^{\infty} x^{s+1} y^{r+1} \zeta_q(s+2, \{1\}^r) \\ &\qquad = \exp\biggl(\sum_{k=2}^{\infty} \frac{x^k + y^k - (x+y+(1-q)xy)^k}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta_q(j)\biggr). \end{aligned}$$

In particular, because of the symmetry in x and y,

$$\zeta_q(s+2, \{1\}^r) = \zeta_q(r+2, \{1\}^s).$$

Proof. The identity follows by taking $z = 0$ in (16.2). □

Corollary 2 (Sum theorem)**.** *The sum of all admissible multiple q-zeta values of fixed weight w and fixed length is equal to $\zeta_q(w)$,*

$$\sum_{\boldsymbol{s}:|\boldsymbol{s}|=w,\ \ell(\boldsymbol{s})=l} \zeta_q(\boldsymbol{s}) = \zeta_q(w).$$

Proof. This derivation is more subtle. Taking the limit as $z \to xy$ in (16.2) gives

$$\begin{aligned}
\Phi_q(x,y,xy) &= \sum_{r=1}^{\infty} \frac{q^r}{([r]_q - xq^r)([r]_q - yq^r)} \\
&= \sum_{r=1}^{\infty} \frac{q^r}{[r]_q^2}\biggl(1 - \frac{xq^r}{[r]_q}\biggr)^{-1}\biggl(1 - \frac{yq^r}{[r]_q}\biggr)^{-1} \\
&= \sum_{m,n=0}^{\infty} x^m y^n \zeta_q(m+n+2) = \sum_{w>l\ge 1} x^{w-l-1} y^{l-1} \zeta_q(w).
\end{aligned}$$

On the other hand, it follows directly from definition that

$$\Phi_q(x,y,xy) = \sum_{w,l=0}^{\infty} x^{w-l-1} y^{l-1} \sum_{\boldsymbol{s}:|\boldsymbol{s}|=w,\ \ell(\boldsymbol{s})=l} \zeta_q(\boldsymbol{s}).$$

It remains to compare the coefficients in the two representations of $\Phi_q(x,y,xy)$. □

Exercise 16.1. For an indeterminate z, show

$$\sum_{n_1>\dots>n_l\ge 1} \frac{q^{n_1}}{[n_1]_q} \prod_{j=1}^{l} \frac{1}{[n_j]_q - zq^{n_j}} = \sum_{n=1}^{\infty} \frac{q^{ln}}{[n]_q^l([n]_q - zq^n)}.$$

Hint. This is equivalent to the sum theorem in Corollary 2. □

In spite of the above 'naturalness' of the q-MZVs (16.1), there are other variations, and we indicate more in what follows. The main difficulty of all these q-models occurs when we look for a reasonable q-generalization of the shuffle product from Theorem 3.1, the product originated from the differential equations for the multiple polylogarithms (4.1). Lemma 4.1 tells us that

$$\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Li}_{s_1,s_2,\dots,s_l}(z) = \begin{cases} \dfrac{1}{z}\operatorname{Li}_{s_1-1,s_2,\dots,s_l}(z) & \text{if } s_1 \ge 2, \\ \dfrac{1}{1-z}\operatorname{Li}_{s_2,\dots,s_l}(z) & \text{if } s_1 = 1, \end{cases} \tag{16.3}$$

and this comes from the *fundamental theorem of calculus*,

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigl(f(z)g(z)\bigr) = \frac{\mathrm{d}}{\mathrm{d}z}f(z)\cdot g(z) + f(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}g(z). \tag{16.4}$$

The differential equations (16.3) give rise to an integral representation of the polylogarithms (4.1) (hence, of the multiple zeta values), where the participating differential forms $\mathrm{d}z/z$ and $\mathrm{d}z/(1-z)$ are assigned as two non-commutative letters, so that the integrals themselves are interpreted as words on these letters.

The q-analogue of (16.4) reads as

$$D_q\big(f(z)g(z)\big) = D_qf(z)\cdot g(z) + f(z)\cdot D_qg(z) - (1-q)z\cdot D_qf(z)\cdot D_qg(z), \quad (16.5)$$

where

$$D_qf(z) = \frac{f(z)-f(qz)}{(1-q)z}.$$

Defining a q-analogue of the multiple polylogarithms (4.1) as

$$\operatorname{Li}_{s_1,\dots,s_l}(z;q) = \sum_{n_1>\dots>n_l\ge 1} \frac{z^{n_1}}{[n_1]^{s_1}\cdots[n_l]^{s_l}}, \quad (16.6)$$

from (16.5) we deduce the following analogue of (16.3):

$$D_q \operatorname{Li}_{s_1,s_2,\dots,s_l}(z;q) = \begin{cases} \dfrac{1}{z}\operatorname{Li}_{s_1-1,s_2,\dots,s_l}(z;q) & \text{if } s_1\ge 2,\\ \dfrac{1}{1-z}\operatorname{Li}_{s_2,\dots,s_l}(z;q) & \text{if } s_1 = 1. \end{cases}$$

This q-model of the multiple polylogarithms, together with classical formulae in the theory of basic hypergeometric series (which we 'touch' below), were used in the derivation of Theorem 16.1 by Okuda and Takeyama. This is a reason to believe that the q-multiple polylogarithms (16.6) are 'motivated' q-analogues of (4.1), and that their values at $z=q$,

$$\begin{aligned}\mathfrak{z}_q(s_1,s_2,\dots,s_l) &= (1-q)^{-|\boldsymbol{s}|}\operatorname{Li}_{s_1,s_2,\dots,s_l}(q;q)\\ &= \sum_{n_1>n_2>\dots>n_l\ge 1}\frac{q^{n_1}}{(1-q^{n_1})^{s_1}(1-q^{n_2})^{s_2}\cdots(1-q^{n_l})^{s_l}}, \end{aligned} \quad (16.7)$$

are reasonable q-analogues of multiple zeta values. Note the normalization factor $(1-q)^{-|\boldsymbol{s}|}$ in the latter specialization; it makes many formulae for q-MZVs 'cleaner' and could be also used for the q-model (16.1).

Although the rule (16.5) might be interpreted as a shuffle product of a suitable functional q-model of the multiple polylogarithms and the corresponding q-MZVs, these models are different from and even 'incompatible' with already given models. For example, the q-analogue of the formula

$$\operatorname{Li}_1(z)^r = r!\operatorname{Li}_{\{1\}_r}(z)$$

(cf. Exercise 4.2 (a)) in terms of (16.6) involve certain undesired 'parasites': if $r=2$, from

$$D_q\big(\operatorname{Li}_1(z;q)\operatorname{Li}_1(z;q)\big) = \frac{1}{1-z}\operatorname{Li}_1(z;q) + \operatorname{Li}_1(z;q)\frac{1}{1-z} - (1-q)\frac{z}{(1-z)^2}$$

we have

$$\operatorname{Li}_1(z;q)^2 = 2\operatorname{Li}_{1,1}(z;q) - (1-q)\sum_{n=1}^{\infty}\frac{(n-1)z^n}{[n]},$$

where the latter series cannot be expressed by means of (16.6).

A related problem is a q-generalization of Euler's decomposition formula

$$\zeta(r)\zeta(s)=\sum_{i=0}^{r-1}\binom{s-1+i}{i}\zeta(s+i,r-i)+\sum_{i=0}^{s-1}\binom{r-1+i}{i}\zeta(r+i,s-i) \tag{16.8}$$

(which follows from the double shuffle relations (13.2), (13.3)), since the known proofs make use (explicitly or not) of the shuffle relations. It seems that a way to overcome this difficulty is to extend the algebra of q-MZVs *differentially*, that is, to consider a differential algebra of q-MZVs and all their δ-derivatives of arbitrary order, where $\delta=q\frac{\mathrm{d}}{\mathrm{d}q}$. Although it is hard to justify this claim, let us see how the problem may be fixed on the example of a q-analogue of (16.8) when $r=s=2$,

$$\zeta(2)^2=2\zeta(2,2)+4\zeta(3,1), \tag{16.9}$$

by means of (16.7). As Bradley shows, even this particular case involves something, which is not expressible by means of q-MZVs (16.1).

We start with the partial-fraction identity

$$\frac{1}{(1-x)(1-y)}=\frac12\bigl(f(x,y)+f(y,x)\bigr), \qquad \text{where} \quad f(x,y)=\frac{1+x}{(1-x)(1-xy)},$$

and differentiate both sides with respect to x and y,

$$\frac{\partial f(x,y)}{\partial x\,\partial y}=\frac{2}{(1-x)^2(1-xy)^2}+\frac{4}{(1-x)(1-xy)^3}-\frac{4}{(1-x)(1-xy)^2}-\frac{1+xy}{(1-xy)^3}.$$

Multiplying the result by xy, substituting $x=q^n$ and $y=q^m$, and using

$$\begin{aligned}\sum_{n,m=1}^{\infty}\frac{xy(1+xy)}{(1-xy)^3}\bigg|_{x=q^n,\,y=q^m}&=\sum_{l=1}^{\infty}(l-1)\frac{q^l(1+q^l)}{(1-q^l)^3}\\&=\delta\sum_{l=1}^{\infty}\frac{q^l}{(1-q^l)^2}-\sum_{l=1}^{\infty}\frac{q^l(1+q^l)}{(1-q^l)^3}=\delta\mathfrak{z}_q(2)-2\mathfrak{z}_q(3)+\mathfrak{z}_q(2),\end{aligned}$$

we finally arrive at

$$\mathfrak{z}_q(2)^2+\delta\mathfrak{z}_q(2)=2\mathfrak{z}_q(2,2)+4\mathfrak{z}_q(3,1)-4\mathfrak{z}_q(2,1)+2\mathfrak{z}_q(3)-\mathfrak{z}_q(2),$$

which is the desired q-analogue of (16.9).

One can also use Ramanujan's system of differential equations (16.11) to get rid of the term $\delta\mathfrak{z}_q(2)$. Namely, using

$$\delta\mathfrak{z}_q(2)=\mathfrak{z}_q(2)-5\mathfrak{z}_q(3)+5\mathfrak{z}_q(4)-2\mathfrak{z}_q(2)^2$$

we obtain

$$\mathfrak{z}_q(2)^2=-2\mathfrak{z}_q(2,2)-4\mathfrak{z}_q(3,1)+4\mathfrak{z}_q(2,1)+5\mathfrak{z}_q(4)-7\mathfrak{z}_q(3)+2\mathfrak{z}_q(2),$$

which is also a q-analogue of (16.9). But for a general q-analogue of (16.8) we do expect terms involving $\delta\mathfrak{z}_q(s)$ and $\delta\mathfrak{z}_q(t)$, hence working in the δ-*differential* algebra generated by the multiple q-zeta values (16.7). Is there a nice form of double shuffle relations in this differential algebra?

There is also an arithmetically motivated q-model, but for single (non-multiple) zeta values:

$$\tilde{\zeta}_q(s)=\sum_{n=1}^{\infty}\sigma_{s-1}(n)q^n=\sum_{n=1}^{\infty}\frac{n^{s-1}q^n}{1-q^n}\,,\qquad s=1,2,\dots, \tag{16.10}$$

where $\sigma_{s-1}(n)=\sum_{d|n}d^{s-1}$ denotes the sum of powers of the divisors. These can be readily recalculated in terms of the q-zeta values (16.1) and (16.7) with $l=1$, because

$$\tilde{\zeta}_q(1)=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\,,\quad \tilde{\zeta}_q(2)=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\,,\quad \tilde{\zeta}_q(3)=\sum_{n=1}^{\infty}\frac{q^n(1+q^n)}{(1-q^n)^3}\,,$$
$$\tilde{\zeta}_q(4)=\sum_{n=1}^{\infty}\frac{q^n(1+4q^n+q^{2n})}{(1-q^n)^4}\,,\quad \tilde{\zeta}_q(5)=\sum_{n=1}^{\infty}\frac{q^n(1+11q^n+11q^{2n}+q^{3n})}{(1-q^n)^5}$$

and, in general,

$$\tilde{\zeta}_q(k)=\sum_{n=1}^{\infty}\frac{q^n\rho_k(q^n)}{(1-q^n)^k}\,,\qquad k=1,2,3,\dots,$$

where the polynomials $\rho_k(x)\in\mathbb{Z}[x]$ are determined recursively by the formulae

$$\rho_1=1,\qquad \rho_{k+1}=(1+(k-1)x)\rho_k+x(1-x)\rho_k' \quad\text{for}\quad k=1,2,\dots.$$

The latter imply $\rho_{k+1}(1)=k!$ that results in the limiting relations

$$\lim_{\substack{q\to1\\0<q<1}}(1-q)^s\tilde{\zeta}_q(s)=(s-1)!\cdot\zeta(s),\qquad s=2,3,\dots.$$

If $s\ge2$ is even, then the series $E_s(q)=1-2s\zeta_q(s)/B_s$, where the Bernoulli numbers $B_s\in\mathbb{Q}$ are defined in (1.2), are known as the *Eisenstein series*. This circumstance allows to prove the coincidence of the rings

$$\mathbb{Q}[q,\tilde{\zeta}_q(2),\tilde{\zeta}_q(4),\tilde{\zeta}_q(6),\tilde{\zeta}_q(8),\tilde{\zeta}_q(10),\dots]\quad\text{and}\quad\mathbb{Q}[q,\tilde{\zeta}_q(2),\tilde{\zeta}_q(4),\tilde{\zeta}_q(6)];$$

the fact can be viewed as a q-analogue of the coincidence of the numerical rings

$$\mathbb{Q}[\zeta(2),\zeta(4),\zeta(6),\zeta(8),\zeta(10),\dots]\quad\text{and}\quad\mathbb{Q}[\zeta(2)]=\mathbb{Q}[\pi^2]$$

which we proved in Lemma 1.2. Even more, the ring $\mathbb{Q}[q,\tilde{\zeta}_q(2),\tilde{\zeta}_q(4),\tilde{\zeta}_q(6)]$ is *differentially stable* because of Ramanujan's system of differential equations

$$\delta E_2=\frac{1}{12}(E_2^2-E_4),\quad \delta E_4=\frac{1}{3}(E_2E_4-E_6),\quad \delta E_6=\frac{1}{2}(E_2E_6-E_4^2), \tag{16.11}$$

where, as before, $\delta=q\frac{\mathrm{d}}{\mathrm{d}q}$.

There are other examples of q-generalizations of both MZVs and generalised polylogarithms, motivated by the theory of modular forms, basic (q-) hypergeometric series and mathematical physics. They are not yet systematically investigated. A basic example here is related to the *q-exponential function*

$$e(z)=e_q(z)=\frac{1}{\prod_{m=0}^{\infty}(1-zq^m)}=\frac{1}{(z;q)_\infty}, \tag{16.12}$$

where $z \in \mathbb{C}$, $|z| < 1$. Here we use the standard q-Pochhammer notation (cf. Section 5)

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{m=0}^{n-1}(1-aq^m) & \text{if } n \ge 1, \end{cases}$$

which, of course, has perfect sense for $n = \infty$ as well, because $|q| < 1$. The similarity with the classical exponential function comes from the expansion

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n},$$

which is the special case $x = 0$, $y = z$ of the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(x;q)_n}{(q;q)_n} y^n = \frac{(xy;q)_\infty}{(y;q)_\infty}. \tag{16.13}$$

The q-polynomials

$$[n]_q! = \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q} = \frac{(q;q)_n}{(1-q)^n},$$

and so the product $(q;q)_n$, are regarded as natural q-extensions of $n!$. Moreover, the function $e(z)$ satisfies the 'standard' exponential functional identity

$$e(X+Y) = e(X)e(Y),$$

if $e(X) = e_q(X)$, $e(Y) = e_q(Y)$ and $e(X+Y) = e_q(X+Y)$ are viewed as elements in the algebra $\mathbb{C}_q[[X,Y]]$ of formal power series in two elements X, Y linked by the commutation relation $XY = qYX$. This noncommutative combinatorial interpretation was given by Schutzenberger in the 1950s.

On the other hand, from (16.12) we have the asymptotic behaviour

$$\begin{aligned} \log e(z) &= \sum_{n=0}^{\infty}\bigl(-\log(1-q^n z)\bigr) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{q^{mn} z^m}{m} = \sum_{m=1}^{\infty} \frac{z^m}{m(1-q^m)} \\ &= \frac{1}{1-q} \sum_{m=1}^{\infty} \frac{z^m}{m[m]_q} \sim \frac{-1}{\log q} \sum_{m=1}^{\infty} \frac{z^m}{m^2} \qquad \text{as} \quad q \to 1 \end{aligned} \tag{16.14}$$

(known already to Ramanujan), since $-\log q \sim 1-q$ as $q \to 1$. This allows to think of $\log e(z)$ as of a q-analogue of the dilogarithm function

$$\mathrm{Li}_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2},$$

the *quantum dilogarithm*. This analogy is much deeper than just the asymptotics above because it is not hard to check that the q-binomial theorem (16.13) is equivalent to the so-called quantum pentagonal identity

$$e(X)e(Y) = e(Y)e(-YX)e(X), \tag{16.15}$$

where as before $e(X) = e_q(X)$, $e(Y) = e_q(Y)$ and $e(-YX) = e_q(-YX)$ are elements in the algebra $\mathbb{C}_q[[X,Y]]$ of formal power series in two elements X, Y linked by the commutation relation $XY = qYX$. It seems that Richmond and Szekeres were the first to realise that the limiting case $q \to 1$ of certain q-hypergeometric identities (actually, they considered the Andrews–Gordon generalisation of the Rogers–Ramanujan identities) produces non-trivial identities for the dilogarithm values; the argument was later exploited by Loxton and rediscovered in the context of (16.13), (16.15) by Faddeev and Kashaev.

Theorem 16.2. *The limiting case $q \to 1$ of the q-binomial theorem* (16.13) *is the equality*

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(y) = \operatorname{Li}_2\Big(\frac{x}{1-y}\Big) + \operatorname{Li}_2\Big(\frac{y}{1-x}\Big) - \operatorname{Li}_2\Big(\frac{xy}{(1-x)(1-y)}\Big) - \log(1-x)\log(1-y), \qquad 0<x<1, \quad 0<y<1. \tag{16.16}$$

Remark. Although we prove relation (16.16) for x and y restricted to the interval $(0,1)$, and this positivity is always crucial in application of the allied asymptotical formulae, the identity remains valid for $x, y \in \mathbb{C} \setminus (1, +\infty)$ by analytic continuation.

Formula (16.16) is due to Abel but an equivalent formula was published by Spence nearly twenty years earlier. Another equivalent form of (16.16) (see (16.27) below) was given by Rogers.

Proof. Without loss of generality assume that q is sufficiently close to 1, namely, that

$$\max\{x, y, 1-y(1-x)\} < q < 1.$$

The easy part of the theorem is the asymptotics of the right-hand side in (16.13):

$$\log\frac{(xy;q)_\infty}{(y;q)_\infty} = \log\frac{e(y)}{e(xy)} \sim \frac{1}{\log q}\big(\operatorname{Li}_2(xy) - \operatorname{Li}_2(y)\big) \qquad \text{as} \quad q \to 1, \tag{16.17}$$

which is obtained on the basis of (16.14).

For the left-hand side of (16.13), write

$$\sum_{n=0}^{\infty}\frac{(x;q)_n}{(q;q)_n}y^n = \sum_{n=0}^{\infty} c_n, \qquad \text{where} \quad c_n = \frac{(x;q)_n}{(q;q)_n}y^n > 0. \tag{16.18}$$

Then the sequence

$$d_n = \frac{c_{n+1}}{c_n} = \frac{1-xq^n}{1-q^{n+1}}y > 0, \qquad n = 0, 1, 2, \dots, \tag{16.19}$$

satisfies

$$\frac{d_{n+1}}{d_n} = \frac{(1-xq^{n+1})(1-q^{n+1})}{(1-xq^n)(1-q^{n+2})} = 1 - \frac{q^n(1-q)(q-x)}{(1-xq^n)(1-q^{n+2})} < 1 - q^n(1-q)(q-x), \qquad n = 0, 1, 2, \dots \tag{16.20}$$

(we use $0 < x < q < 1$), hence it is strictly decreasing. On the other hand, $1 - y(1-x) < q$ implies

$$d_0 = \frac{c_1}{c_0} = \frac{1-x}{1-q}y > 1,$$

while

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{1-xq^n}{1-q^{n+1}} y = y < 1;$$

thus, there exists the unique index $N \ge 1$ such that

$$d_{N-1} = \frac{c_N}{c_{N-1}} \ge 1 \qquad \text{and} \qquad d_N = \frac{c_{N+1}}{c_N} < 1. \tag{16.21}$$

Solving the inequality $c_{n+1}/c_n < 1$ or, equivalently, $(1-xq^n)y < 1-q^{n+1}$ we obtain $n > T$, where

$$T = \frac{1}{\log q} \cdot \log \frac{1-y}{q-xy}, \tag{16.22}$$

hence $N = \lfloor T \rfloor$, the integral part of T. From (16.19)–(16.21) we conclude that c_N is the main term contributing the sum in (16.18), namely,

$$1 < \frac{\sum_{n=0}^{\infty} c_n}{c_N} < \text{const}\,.$$

This implies

$$\begin{aligned} \log \sum_{n=0}^{\infty} c_n \sim \log c_N &= \log\biggl(\frac{e(q)e(xq^N)}{e(x)e(q^{N+1})} y^N \biggr) \\ &\sim \log\biggl(\frac{e(q)e(xq^T)}{e(x)e(q^{T+1})} y^T \biggr) \qquad \text{as} \quad q \to 1. \end{aligned} \tag{16.23}$$

Note now that from (16.22)

$$q^T = \frac{1-y}{q-xy},$$

whence the asymptotics in (16.23) may be continued as follows:

$$\begin{aligned} \log \sum_{n=0}^{\infty} c_n &\sim \log e(q) + \log e\biggl(x\frac{1-y}{q-xy} \biggr) - \log e(x) - \log e\biggl(q\frac{1-y}{q-xy} \biggr) \\ &\qquad + \frac{\log y}{\log q} \cdot \log \frac{1-y}{q-xy} \\ &\sim \frac{1}{\log q} \biggl(\mathrm{Li}_2(x) + \mathrm{Li}_2\biggl(\frac{1-y}{1-xy} \biggr) - \mathrm{Li}_2(1) - \mathrm{Li}_2\biggl(x\frac{1-y}{1-xy} \biggr) \\ &\qquad + \log y \cdot \log \frac{1-y}{1-xy} \biggr) \qquad \text{as} \quad q \to 1, \end{aligned} \tag{16.24}$$

where (16.14) is used.

Comparing the asymptotics (16.17) and (16.24) of the both sides of (16.13) we arrive at the identity

$$\begin{aligned} \mathrm{Li}_2(x) + \mathrm{Li}_2\biggl(\frac{1-y}{1-xy} \biggr) - \mathrm{Li}_2(1) - \mathrm{Li}_2\biggl(x\frac{1-y}{1-xy} \biggr) + \log y \cdot \log \frac{1-y}{1-xy} \\ = \mathrm{Li}_2(xy) - \mathrm{Li}_2(y). \end{aligned} \tag{16.25}$$

Take $x = 0$ in (16.25) to get

$$\mathrm{Li}_2(y) + \mathrm{Li}_2(1-y) - \mathrm{Li}_2(1) + \log y \cdot \log(1-y) = 0.$$

This identity, in particular, implies

$$\mathrm{Li}_2\biggl(\frac{1-y}{1-xy}\biggr)-\mathrm{Li}_2(1)=-\mathrm{Li}_2\biggl(1-\frac{1-y}{1-xy}\biggr)-\log\frac{1-y}{1-xy}\cdot\log\biggl(1-\frac{1-y}{1-xy}\biggr). \quad (16.26)$$

Substituting (16.26) into (16.25) results in

$$\mathrm{Li}_2(xy)+\mathrm{Li}_2\biggl(\frac{x(1-y)}{1-xy}\biggr)+\mathrm{Li}_2\biggl(\frac{y(1-x)}{1-xy}\biggr)+\log\frac{1-y}{1-xy}\cdot\log\frac{1-x}{1-xy}$$
$$=\mathrm{Li}_2(x)+\mathrm{Li}_2(y). \quad (16.27)$$

Finally, changing variable $\tilde{x}=x(1-y)/(1-xy)$, $\tilde{y}=y(1-x)/(1-xy)$, hence

$$1-\tilde{x}=\frac{1-x}{1-xy},\quad 1-\tilde{y}=\frac{1-y}{1-xy},\qquad x=\frac{\tilde{x}}{1-\tilde{y}},\quad y=\frac{\tilde{y}}{1-\tilde{x}},$$

reduces identity (16.27) to the required form (16.16). □

A similar 'mixed' q-extension of the multiple zeta values might be possible. The following example is due to Zagier.

Theorem 16.3. *The following identity is valid:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn(m+n)}\,\frac{q^{m+n}}{(1-q^m)(1-q^n)(1-q^{m+n})}=\sum_{m=1}^{\infty}\frac{1}{6m^3}\,\frac{q^{2m}(3-q^m)}{(1-q^m)^3}. \quad (16.28)$$

Remark. The limiting case as $q\to 1$ of the identity reads

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n^2(m+n)^2}=\frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^6}=\frac{\pi^6}{2835}\,.$$

Although the left-hand side here is not a standard MZV, the identity reduces it to a single zeta value.

By comparing the coefficients of q^N, we see that (16.28) is equivalent to the number-theoretic identity

$$\sum_{\substack{m,n,r,s>0\\ mr+ns=N}}\frac{\min(r,s)}{mn(m+n)}=\frac{\sigma_5(N)-\sigma_3(N)}{6N^3},\qquad N\in\mathbb{N}.$$

Lemma 16.1. *Let $\{\alpha(m,n)\}_{m,n\in\mathbb{N}}$ be a collection of complex numbers which can be written in the form*

$$\alpha(m,n)=\beta(m,n)-\beta(m+n,n)-\beta(m,m+n),\qquad m,n\in\mathbb{N}, \quad (16.29)$$

where $\sum_{m,n>0}\beta(m,n)$ is absolutely convergent. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha(m,n)=\sum_{m=1}^{\infty}\beta(m,m).$$

Proof. It follows from (16.29) that

$$\sum_{m,n>0}\alpha(m,n)=\biggl(\sum_{m,n>0}-\sum_{m>n>0}-\sum_{n>m>0}\biggr)\beta(m,n)=\sum_{m=n>0}\beta(m,n),$$

which is the wanted identity. □

Proof of Theorem 16.3. It is interesting that the summand on the left-hand side of (16.28) cannot be given in the form (16.29). However, there is an identity of this type at the level of *derivatives*, namely

$$q\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{1}{mn(m+n)}\,\frac{q^{m+n}}{(1-q^m)(1-q^n)(1-q^{m+n})}\right)=\beta(m,n)-\beta(m+n,n)-\beta(m,m+n)$$

with

$$\beta(m,n)=\frac{1}{m}\,\frac{q^m}{(1-q^m)^2}\cdot\frac{1}{n}\,\frac{q^n}{(1-q^n)^2},$$

and now the required identity follows from Lemma 16.1 and

$$\beta(m,m)=q\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{q^{2m}(3-q^m)}{6m^3(1-q^m)^3}\right)$$

after integration. ∎