OPTIMIZATION WITH RESPECT TO PARTIAL ORDERINGS

by

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ABSTRACT

The thesis is primarily concerned with optimization problems which have objective functions which do not take values on the real line.

In chapter one convexity properties are investigated for functions in partially ordered vector spaces.

In chapter two the concept of a tangent cone is introduced and the previously little used notion of a weak tangent cone is defined. Properties of these cones are investigated and various differential relationships are proved.

Chapter three establishes several transposition theorems and Farkas lemmas both for linear and non-linear systems. These results have some applications in later chapters.

In chapter four the concept of a subgradient is extended and related to the tangent cone results of chapter two.

The fifth chapter establishes Kuhn-Tucker and Fritz John type necessary and sufficient conditions for general non-linear programming problems to have solutions with respect to various notions of minimization. These conditions are given in tangent cone terms. They include one sided derivative and subgradient results.

Chapter six includes a variety of multiplier theorems for convex and quasiconvex programmes in partially
ordered spaces. A minimax theorem is included.

Chapter seven uses tangent cones to generalize known second order conditions to more general problems and spaces.

Finally, in chapter eight, the results of chapter five are specialized to Hilbert space using pseudoinverses and projections on convex sets. The chapter also contains a section on variational inequalities which centres around the non-linear complementarity problem.
INTRODUCTION

In the thesis I have attempted to extend many of the standard results of nonlinear programming theory in the following directions:

(1) The objective functions are not generally supposed real valued,

(2) The various convexity assumptions usually associated with sufficiency conditions have been generalized and weakened,

(3) The notion of a tangent cone, and of constraint qualifications given in tangent cone terms, has been extended.

Using these three extensions I have attempted to both unify and enlarge what theory does exist for optimization with respect to partial orderings. I have found it possible to state and prove a number of fairly general theorems.

The two notions of minimization that are used have both been studied before, but not within the mainstream of abstract optimization theory as represented by the tangent cone investigations of Varaiya, Guignard and others. I hope that I have partially rectified this situation.

Some remarks on format seem in order. The chapters are arranged so that almost all the preliminary results are proved in chapters one through four and are applied and investigated further in chapters five through eight. The chapters are divided internally into numbered
paragraphs, which are also used for purposes of cross reference. I have tried to keep the notation as uniform as possible. For example bilinear forms are used only in chapter eight where it seemed unavoidable. Otherwise the value of a linear functional $x^+$ at a point $x$ has been denoted $x^+(x)$ not $(x^+,x)$.

Finally, the bibliography is arranged chronologically within each author's listing and all references are referred to simply by name and date.
Chapter One

CONVEX TYPE FUNCTIONS
This chapter is devoted to a survey and extension of results on convex type functions. In the literature these functions are, except in the case of constraint functions, usually taken to be real valued. As will be seen this restriction is often unnecessary.

Certain preliminary definitions are necessary. For the sake of convenience all spaces are assumed to be real vector spaces. All general topological and functional analytic definitions are used as in Robertson and Robertson (1964) if they are not defined explicitly.

[1] A set $C$ in $X$ is convex if $\lambda x_1 + (1 - \lambda) x_2 \in C$ whenever $x_1, x_2 \in C$ and $0 \leq \lambda \leq 1$.

[2] A topological vector space is a vector space with an associated topology in which the vector operations are continuous. Such a topology is said to be compatible.

[3] A (locally) convex (topological vector) space is a topological vector space in which there is a base of neighbourhoods of the origin consistent of convex sets.
[4] A cone \( S \) in \( X \) is a set with the property that \( \lambda x \in S \) whenever \( x \in S \) and \( \lambda \geq 0 \). \( S \) is a convex cone if it is convex in addition.

[5] **Partial orderings**

Each convex cone \( S \) in \( X \) determines a relation \( \preceq \)

which is transitive and reflexive and which is given by

\[ x \preceq_y \iff x - y \in S. \]

When there is no ambiguity \( \preceq \) will be denoted simply by \( \geq \).

This relationship is compatible with the vector structure.

That is

1. \( x \geq 0 \) and \( y \geq 0 \) implies \( x + y \geq 0 \).
2. \( x \geq 0 \) and \( \lambda \geq 0 \) implies \( \lambda x \geq 0 \).

The relation determined by the cone \( S \) is called the vector (partial) ordering of \( X \) and the said \( (X, S) \) or \( (X, \geq) \) is a partially ordered vector space.

[6] Conversely if \( \geq \) is a symmetric and transitive relation satisfying (1) and (2) then \( S = \{ x \mid x \in X \text{ and } x \geq 0 \} \) is a convex cone in \( S \) and \( \geq \) is exactly the ordering on \( X \) induced by \( S \).

[7] In some cases only (2) is required of the cone which need not then be convex. If \( S_0 = \{ 0 \} \) then \( S \) is said to be pointed or proper. It is clear that \( S \) is pointed if and only if the induced ordering is anti-symmetric, that is if \( x \geq 0 \) and \( x \leq 0 \) \( (-x \geq 0) \) then \( x = 0 \).
With these definitions it is possible to turn to an investigation of the elementary properties of convex type functions. It will be useful to list the eight function types of primary interest.

[8] Let \( f : X \to Y \) be convex with respect to \( S \) a convex cone in \( Y \) if
\[
f(\lambda x + (1-\lambda)y) \leq_s \lambda f(x) + (1-\lambda)f(y)
\]
for \( x, y \in X; 0 \leq \lambda \leq 1 \).

If \(-f\) is convex \( f \) is said to be concave. Similar remarks apply to the following definitions.

[9] Let \( f: X \to Y \) be quasi-convex with respect to \( S \), if for \( 0 \leq \lambda \leq 1 \).
\[
f(\lambda x + (1-\lambda)y) \leq f(y) \text{ whenever } f(x) \leq f(y) \quad x, y \in X.
\]

[10] Let \( f: X \to Y \) be strongly quasi-convex with respect to \( S \) if
\( S(z) = \{ x \mid f(x) \leq_s z \} \) is a convex set for each \( z \) in \( Y \).

\( S(z) \) is called a level set of \( f \) with respect to \( S \).

[11] Let \( f: X \to Y \) be absolutely quasi-convex with respect to \( S \) if
\[
f(\lambda x + (1-\lambda)y) \geq f(y)
\]
for some \( 0 \leq \lambda \leq 1 \), one has
\[
f(x) \geq f(y).
\]

[12] If \( S \) is a cone in a topological vector space and \( S^0 \neq \emptyset \), it is possible to define a strict inequality (denoted by \( >_S \)) by
\[
x >_S y \quad \text{if and only if} \quad x - y \in S^0.
\]
Clearly \( x > y \) implies \( x \geq y \). Equally clearly if \( S = \mathbb{R}^+ \), the nonnegative real axis, then \( x >_S y \) has the usual meaning.
with this extra definition to sharpen the previous ones:

\[ f : X \rightarrow Y \text{ is strictly convex with respect to } S \text{ if } \]
\[ f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y) \quad x, y \in X; \quad 0 \leq \lambda \leq 1. \]

\[ f : X \rightarrow Y \text{ is strictly quasiconvex with respect to } S \text{ if } \]
\[ f(\lambda x + (1 - \lambda) y) < f(y) \text{ whenever } f(x) \leq f(y) \quad x \neq y \text{ and } 0 < \lambda < 1. \]

\[ f : X \rightarrow Y \text{ is strongly strictly quasiconvex with respect to } S \]
\[ \text{if whenever } 0 < \lambda < 1, \ x \neq y, \text{ and } f(x) \leq z, f(y) \leq z \text{ then } \]
\[ f(\lambda x + (1 - \lambda) y) < z. \]

\[ \text{Ponstein (1967) has introduced, in the real case, a property} \]
\[ \text{which will be called } (P) \text{ strict quasiconvexity and which is} \]
\[ \text{weaker than } [14] \text{ or } [15] \text{ but suffices for some basic propositions.} \]
\[ f : X \rightarrow Y \text{ is } (P) \text{ strictly quasiconvex with respect to } S \text{ if } \]
\[ f(\lambda x + (1 - \lambda) y) < f(y) \text{ whenever } f(x) < f(y) \text{ and } 0 < \lambda < 1. \]

\[ \text{Relationships between the above definitions} \]
\[ \text{The properties will be referred to by their respective numbers} \]
\[ \text{since these are directly above. All the relationships follow} \]
\[ \text{directly from the definitions.} \]

\(1\) [13] → [8] → [10] → [9]
\(2\) [13] → [15] → [14] → [16]
\(3\) [13] → [15] → [10] → [9]
\(4\) [8] → [11], [8] → [16]
The next proposition will be useful in the sequel.

[18] **Proposition:** If $S$ is closed and $S^0 \neq \emptyset$, strong quasiconvexity is equivalent to $T(z) = \{ x \mid f(x) < z \}$ being convex $\forall z \in Y$.

**Proof:** \( \Rightarrow \) Let $N$ be a convex neighbourhood in $S$. Suppose that $x, y \in T(z)$. Then $f(x) - z \in -S^0$; $f(y) - z \in -S^0$ or equivalently, for any $a \in N$ one has
\[
f(x) \leq z - a, \quad f(y) \leq z - a.
\]
Since $f$ is strongly quasiconvex $f(\lambda x + (1 - \lambda) y) \leq z - a$ for all $a \in N$ and this is the same as
\[
f(\lambda x + (1 - \lambda) y) < z.
\]
\( \Leftarrow \) If $T(z)$ is convex and $f(x) \leq z$, $f(y) \leq z$ then when $a \in S^0$, $a/ \in S^0$ for any $n \in N$ and
\[
f(x) < z + a/n, \quad f(y) < z + a/n.
\]
By the convexity of $T(z) \forall z$,
\[
f(\lambda x + (1 - \lambda) y) < z + a/n
\]
and taking limits
\[
f(\lambda x + (1 - \lambda) y) - z \in -S^0 = S.
\]
When $(Y, S) = (\mathbb{R}, \mathbb{R}^+)$ or more generally any linearly ordered space more can be said about the definitions.

[19] **Proposition:** Strong quasiconvexity and quasiconvexity coincide.


**Proof:** $[10] \Rightarrow [9]$ is general. Conversely if $f(x) \leq z$ and $f(y) \leq z$ then $z \geq z_1 = \max (f(x), f(y))$. By $[9]$ one has
\[
f(\lambda x + (1 - \lambda) y) \leq z_1 \leq z
\]
In a linearly ordered space the cone generally has no interior. One can, however, define \( x \lessdot y \) by
\[
x \lessdot y \text{ if } x \leq y \text{ and } x \neq y.
\]
This definition is good for any cone and in the case of a linear ordering it can be used to give meaning to strict inequality. Replacing \( < \) by \( \lessdot \) one has:

**Proposition:** If \((X, S)\) is linearly ordered

1. Strict quasiconvexity \([14]\) and strong strict quasiconvexity \([15]\) coincide.
2. Absolute quasiconvexity \([11]\) and \((P)\) strict quasiconvexity \([16]\) agree.

**Proof:** \([15] \Rightarrow [14]\) and \([16] \Rightarrow [11]\) follow directly from the relevant definitions and \([20] \Rightarrow [14]\) \(\Rightarrow [15]\) follows as does \([9] \Rightarrow [10]\) in \([19]\).

In general, the strong quasiconvex types are actually stronger than the corresponding quasiconvex types. As an example one has the following.

**Example:** Let \( f: \mathbb{R} \to \mathbb{R}^2 \) with the ordering in \( \mathbb{R}^2 \) the orthant ordering \((x_1, x_2) \geq (y_1, y_2) \) if \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \). Let
\[
f(x) = \begin{cases} 
(1,0) & x \geq 1 \\
(2,2) & 1-x \leq 1 \\
(0,1) & x \leq -1
\end{cases}
\]
Suppose \( f(x) \leq f(y) \).

1. \( y \geq 1 \), then \( f(x) \leq (1,0) \) and \( x \in \{ r \mid 1 \leq r < \infty \} \)
2. \( |y| < 1 \), then \( f(x) \leq (2,2) \) and \( x \in \mathbb{R} \)
3. \( y \leq -1 \), then \( f(x) \leq (0,1) \) and \( x \in \{ r \mid -\infty < r < -1 \} \).
In each case \( \{ x \mid f(x) \leq f(y) \} \) is convex and thus \( f \) is quasiconvex with respect to the orthant ordering. However, if \( z = (1,1) \), \( S(z) = \{ x \mid f(x) \leq (1,1) \} \) is not convex and \( f \) is not strongly quasiconvex.

More sophisticated examples could be given but the above serves to indicate the reason for the divergence of definitions in general.

Although (P) strict quasiconvexity does not in general imply quasiconvexity, as is shown by \( f : \mathbb{R} \to \mathbb{R} \) with \( f(0) = 1 \) and \( f(x) = 0, x \neq 0 \), only the mildest of continuity conditions is necessary for the implication to hold when \( (Y,S) = (\mathbb{R},\mathbb{R}^+) \).

**Definition:** \( f : X \to Y \) has the one point exclusion property with respect to \( S \) if \( f(x_0) \leq z \) whenever \( f(x) \leq z \) for all \( x \) in a line segment \([x_1, x_2]\) containing \( x_0 \) and not equal to \( x_0 \).

A continuity condition which is clearly stronger is:

**Definition:** \( f : X \to Y \) is lower semicontinuous with respect to \( S \) if \( S(z) = \{ x \mid f(x) \leq z \} \) is closed for all \( z \in Y \).

Upper semicontinuity is defined similarly. On the line this reduces to the usual definition. In general, however, one also has:

**Definition:** \( f : X \to Y \) is fully lower semicontinuous with respect to \( S \), a cone with interior, if for all \( z \) in \( Y \)

\[
F(z) = \{ x \mid f(x) > z \}
\]

is an open set.

These notions are related by the following proposition.
[26] **Proposition:** Suppose $S$ is a closed convex cone with interior in a convex space $Y$. Then $f : X \to Y$ is lower semicontinuous whenever $f$ is fully lower semicontinuous.

**Proof:** Suppose $x_0 \in S(z)$. Let $\{x_t \mid t \in T\}$ be a net in $S(z)$ with limit $x_0$. Let $s \in S^0$ and let $m \in N$. Since $f(x_0) > f(x_0) - s/m$ and [25] holds there is some $t_0 \in t$ such that $f(x_{t_0}) > f(x_0) - s/m$ when $t \geq t_0$. Since $Y$ is a convex space $s/m$ tends to zero in $Y$ as $m$ tends to infinity and one has

$$z \geq f(x_{t_0}) > f(x_0) - s/m$$

and $f(x_0) \leq z$. Thus $S(z)$ is closed. is needed

It becomes apparent that some mechanism for relating the order-bounded sets, those $\{x \mid y \leq x \leq z\}$, and the original topology. The following which is taken from Kelley and Namioka (1963) is sufficient for present purposes.

[27] **Definition:** A convex cone $S$ in a pseudo normed space with pseudonorm $p$, is said to be **normal** if whenever $x$ and $y$ belong to $S$ and have pseudo norms greater than or equal to 1 one has $p(x + y) \geq e$ where $e$ is some fixed positive number.

Equivalently one has the requirement that the set

$$(B + S) \cap (B - S)$$

is bounded where $B = \{x \mid p(x) \leq 1\}$. The following theorem holds in a pseudo normed space.

[28] **Theorem:** (Kelley and Namioka 23.7). If $S$ is normal each order-bounded set is bounded. If $x_0 \in S^0$ then $S$ is normal if and only if $\{y \mid -x_0 \leq y \leq x_0\}$ is bounded.

One has for closed cones and finite dimensional spaces the following:
Proposition: If $X$ is a finite dimensional pseudonormed Hausdorff space (and hence normable) $S$ is pointed i.f.f. $S$ is normal.

**Proof**: $\Leftarrow$ If $S$ is not pointed there is some $x$ with $x \in S \setminus -S$ and $\|x\| = 1$ and, hence, $\|x + (-x)\| = 0$ and $S$ is not normal. $\Rightarrow$ Let $\{x_n\}, \{y_n\}$ be sequences in $S$ with $\|x_n\| \geq 1$ $\|y_n\| \geq 1$ and, in contradiction of normality, with $\|x_n + y_n\| = 1/n$. Let $k_n = x_n/\|x_n\|$ then since $X$ is finite dimensional one can suppose $k_n$ is convergent to $k_0$ which will be non zero. Now, since $\|x_n\| \geq 1$,

$$\|k_n + y_n\| = 1/n \|x_n\| \leq 1/n \|x_n\| \leq 1/n,$$

and this means that $k_n + \|x_n\|^{-1} y_n \rightarrow 0$. Since $k_n \rightarrow k_0 \neq 0$ $\|x_n\|^{-1} y_n \rightarrow -k_0 \neq 0$. Since the cone is closed and $x_n, y_n \in S$ both $-k_0 \in S$ and $k_0 \in S$ contradicting pointedness.

This result does not hold true in general Banach spaces.

Example: Define the cone $S$ in $\ell_1$ by

$$S = \left\{ \left\{ x_k \right\} \mid x_1 \geq 0, x_k + x_{k+1} \geq 0, \text{ for } k \geq 0 \right\}$$

$S$ is pointed since $x \in S \setminus -S$ implies that $x_1 = 0$ and $x_k + x_{k+1} = 0$. That is $\left\{ x_k \right\} = 0$.

$S$ is not normal. Suppose $e$ is the constant of definition $[\ell_1]$. Let $\{x_k\}, \{y_k\}$ have

$$x_k = \begin{cases} \frac{(-1)^{k+1}}{n} & k = 3, \ldots, 2n+3 \\ 0 & \text{otherwise} \end{cases}$$

and
\[ y_k = \begin{cases} 
\left(\frac{-1}{n}\right)^k & k = 2, \ldots, 2n+2 \\
0 & \text{otherwise}. 
\end{cases} \]

Then \( \|\{x_k^2\}\| = \sum_{k=2}^{2n+2} \left| \frac{-1}{n} \right|^k \geq 1 \) and \( \{x_k^2\} \in S \)

\[ \|\{y_k^2\}\| = \sum_{k=2}^{2n+2} \left| \frac{-1}{n} \right|^k \geq 1 \] and \( \{y_k^2\} \in S \)

\[ \|\{x_k^2 + y_k^2\}\| = \frac{1}{n} \]

Taking \( n \) sufficiently large, \( \frac{1}{n} < e \).

**Proposition:** If both \( f \) and \(-f\) are fully lower semicontinuous with respect to a normal cone \( S \) with \( S \) closed and with interior then \( f \) is continuous.

**Proof:** Let \( z_1, z_2 \in Y \). By hypothesis the set

\[ \{x \mid z_1 < f(x) < z_2\} \]

is open. Since \( S \) is normal and has interior the norm bounded sets in \( Y \) generate the norm topology. Thus the inverse images of open sets are open and \( f \) is continuous. Since any finite dimensional pointed cone is normal this result is a true extension of the standard relationship.

Returning now to the discussion of quasiconvexity in \([22]\):

**Proposition:** If \( f : X \to \mathbb{R} \) is (P) strict quasiconvex \([16]\) and satisfies \([23]\) then \( f \) is quasiconvex \([9]\).

**Proof:** Suppose \( f(x) \leq f(y) \). If \( f(x) < f(y) \) then

\( f(\lambda x + (1-\lambda)y) \leq f(y) \) by \([16]\).
If \( f(x) = f(y) \) suppose that \( x_0, x_1 \in [x, y] \) with
\[ f(x_0) > f(x) \text{ and } f(x_1) > f(x) \].
Then either \( x_1 \in [x_0, y] \) or \( x_1 \in [x, x_0] \). In either case by \([16]\) one has
\[ f(x_1) < f(x_0) \].

Symmetrically \( f(x_0) < f(x_1) \) since \( x_0 \in [x_1, y] \) or \([x, x_1] \)
which gives \( f(x_0) < f(x_0) \). Thus there can only be one
exceptional point on the line segment \([x, y] \). This is
excluded by hypothesis. Thus \( f(\lambda x + (1 - \lambda)y) \leq f(x) \)
if \( 0 \leq \lambda \leq 1 \).

In similar vein to Proposition [32] is:

**Proposition:** If \( f : X \to \mathbb{R} \) is \( (P) \) strictly quasiconvex and
satisfies the one point exclusion property then if \( f(x) \leq c \) and \( f(y) < c \)
and \( f(\lambda x + (1 - \lambda)y) < c \); \( 0 \leq \lambda \leq 1 \).

**Proof:** Suppose \( f(y) < c \) and \( f(x) \leq c \). If \( f(y) < f(x) \)
then \( f(\lambda x + (1 - \lambda)y) < f(x) \leq c \) by \([16]\), while if
\( f(x) \leq f(y) \) one has \( f(\lambda x + (1 - \lambda)y) \leq f(y) < c \) if
\( 0 \leq \lambda < 1 \) since, by \([32]\), \( f \) is quasiconvex.

The next few results are concerned with properties of convex
functions which for the most part do not generalise to quasiconvex
ones.

Since the sums of quasiconvex functions are generally not
quasiconvex (even if one is convex) it seems worth noting some
of the quasiconvexity maintaining operations before turning
to convex functions.
Proposition: If $f : X \to \mathbb{R}$ is quasiconvex and $g : X \to \mathbb{R}$ is the indicator function of a convex set $C$ (that is $g(x) = 0$ if $x \in C$ and $g(x) = \infty$ if $x \notin C$) then $f + g$ is quasiconvex.

Proof: $S(r) = \{ x \mid f(x) + g(x) \leq r \} = \{ x \mid f(x) \leq r \} \cap C$.
Since both $C$ and $\{ x \mid f(x) \leq r \}$ are convex $f + g$ is quasiconvex.

The pointwise supremum of a family of real valued quasiconvex functions is quasiconvex.

A function $M$ mapping $(Y, S)$ into $(Z, T)$ such that $x - y \in S$ if and only if $M(x) - M(y) \in T$ will be called cone monotone.

Proposition: (1) If $f : X \to Y$ is quasiconvex with respect to $S \subset Y$ and $M : Y \to Z$ is cone monotone then $g = Mf$ is quasiconvex with respect to $T \circ Z$.

(2) If $f$ is strongly quasiconvex with respect to $S$ and $M$ is cone monotone and surjective then $Mf$ is strongly quasiconvex with respect to $T$.

Proof: Let $g = Mf$.

(1) $\{ x \mid g(x) \leq_T g(y) \} = \{ x \mid M(f(x)) \leq_T M(f(y)) \} = \{ x \mid f(x) \leq_S f(y) \}$ since $M$ is cone monotone. Since $f$ is quasiconvex with respect to $S$ this last set is convex.

(2) $\{ x \mid g(x) \leq_T z \} = \{ x \mid M(f(x)) \leq_T M(y) \}$ since $M$ is assumed surjective. Then as in (1) $\{ x \mid M(f(x)) \leq_T M(y) \}$ is equal to $\{ x \mid f(x) \leq y \}$ which is convex $\forall y \in Y$ since $f$ is strongly quasiconvex.
If $A : X \to Y$ is linear and $f : Y \to Z$ is quasiconvex with respect to $S$ then $(fA) (x) = f(Ax)$ defines a function $fA : X \to Z$ which is quasiconvex with respect to $S$. If in addition $A$ is surjective and $f$ is strongly quasiconvex, so is $fA$.

If $A : X \to X$ is linear and $f : X \to R$ is quasiconvex then $Af : X \to R$ defined by $(Af)(x) = \inf \{ f(x) \mid Ax = y \}$ is quasiconvex.

Rockafellar (1970a) gives an exhaustive list of convexity preserving operations for real valued functions. Note that if $f$ is taken to be convex in [37] or [38], the composite mapping is also convex.

The following characterization of convex type functions is very useful.

Proposition: $f : X \to Y$ is strongly quasiconvex with respect to a closed cone $S$ if $f$ is lower semicontinuous and

$$f \left( \frac{1}{2} x + \frac{1}{2} y \right) \leq z \text{ whenever } f(x) \leq z \text{ and } f(y) \leq z.$$  

Proof: Assume inductively that for $0 \leq m \leq 2^{n-1}$ and $n \leq n_0 - 1$ one has

\begin{equation}
(1) f \left( \frac{m}{2^n} + \left( 1 - \frac{m}{2^n} \right) y \right) \leq z \text{ when } f(x) \leq z; f(y) \leq z.
\end{equation}

For $n_0 = 2$ this is true by hypothesis. Now

\begin{equation}
(2) f \left( \frac{m + 2^{n_0} - 1}{2^{n_0}} x + \left[ 1 - \frac{m + 2^{n_0} - 1}{2^{n_0}} \right] y \right)
\end{equation}

\begin{equation}
= f\left( \frac{1}{2} x + \frac{1}{2} \left( \frac{1 - m}{2^{n_0} - 1} y + \frac{m}{2^{n_0} - 1} x \right) \right)
\end{equation}
By the hypothesis of the theorem this last term is \( \leq z \) if (1) holds. Exchanging \( x \) and \( y \) in (2) one sees that for \( 0 \leq k \leq 2^n \)

\[
(3) \quad f \left( \frac{k}{2^n} x + \frac{1-k}{2^n} y \right) \leq z \text{ if } f(x) \leq z, f(y) \leq z.
\]

Using the semicontinuity of \( f \) with respect to \( S \) and the fact that the dyadic rationals are dense in \([0,1]\) one obtains the desired result. \( \square \) Similarly:

**Proposition:** \( f : X \to Y \) is convex with respect to \( S \) if \( f \) is lower semicontinuous with respect to \( S \) and satisfies

\[
f \left( \frac{1}{2} x + \frac{1}{2} y \right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y).
\]

The analogous results hold for strict convex type functions.

If \( X' \) denotes the topological dual of \( X \) then the dual cone \( S^+ \) is defined by

\[
S^+ = \left\{ x^+ \in X' \mid x^+(x) \geq 0 \quad \forall x \in S \right\}.
\]

\( S^+ \) is a closed convex cone even if \( S \) is an arbitrary set.

The second dual \((S^+)^+\) is defined by

\[
(S^+)^+ = \left\{ x \in X \mid x^+(x) \geq 0 \quad \forall x^+ \in S^+ \right\}.
\]

If \( S \) is a closed convex cone a standard separation argument (for cones) shows that \((S^+)^+ = S\). The dual cone can be used to give the following important characterization of convexity.

**Proposition:** \( f : X \to Y \) is convex with respect to a closed convex cone \( S \) if and only if \( u^+ f : X \to \mathbb{R} \) is convex for every \( u^+ \in S^+ \).
Proof: \( \Rightarrow \) If \( f \) is convex with respect to \( S \), then for \( 0 \leq \lambda \leq 1 \), and for \( u^+ \in S^+ \)
\[
  u^+ \left[ \lambda f(x) + (1 - \lambda) f(y) \right] \geq u^+ \left[ f(\lambda x + (1 - \lambda) y) \right]
\]
which asserts the convexity of \( u^+ f \).
\( \Leftarrow \) If \( u^+ f \) is convex \( \forall u^+ \in S^+ \) then
\[
  u^+ \left[ \lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y) \right] \geq 0
\]
and
\[
  f(\lambda x + (1 - \lambda) y) - \left[ \lambda f(x) + (1 - \lambda) f(y) \right] \in (-S^+)^+
\]
which since \( S \) is a closed convex cone means that \( f \) is convex with respect to \( S = (S^+)^+ \). Unfortunately the same characterization of strongly quasiconvex functions breaks down.

\[43\] Example: Let \( f : \mathbb{R} \rightarrow \mathbb{R}^2 \) be defined by \( f(x) = (x, -x^3) \) with the orthant ordering \( P \). Then \( f \) is strongly quasiconvex but if \( u^+ = (1, 1) \in P^+ = P \)
\[
  u^+ f = x - x^3
\]
which is clearly not quasiconvex.

\[44\] Another property of convex functions (or sets) is that of local convexity. A set \( A \subset X \) is said to be locally convex if for each \( x \in A \) there is a neighbourhood \( N \) of \( x \) with \( A \cap N \) convex. A result of Kelley and Namioka (1965) says that in a Hausdorff topological vector space a closed connected locally convex set is convex.

\[45\] Defining the epigraph of \( f : X \rightarrow Y \) with respect to \( B \) by
\[
  \text{Epi}_B f = \{ (x, y) \mid y - f(x) \in B \}
\]
one has:
Proposition: \( f : X \to Y \) is convex with respect to \( S \) if and only if \( \text{Epi}_S f \) is a convex set.

Proof: \( \Rightarrow \) Suppose \((x_1, y_1)\) and \((x_2, y_2) \in \text{Epi}_S f\). Then \( f(x_1) \leq y_1 \), \( f(x_2) \leq y_2 \) and since \( f \) is convex

\[
 f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(y_1) \leq \lambda y_1 + (1 - \lambda) y_2 \]

and \((\lambda x_1 + (1 - \lambda) x_2, \lambda y_1 + (1 - \lambda) y_2) \in \text{Epi}_S f\) which must be convex.

\( \Leftarrow \) \((x_1, f(x_1))\) and \((x_2, f(x_2))\) belong to \( \text{Epi}_S f \) which being convex means that

\[
(\lambda x_1 + (1 - \lambda) x_2, \lambda f(x_1) + (1 - \lambda) f(x_2)) \in \text{Epi}_S f.
\]

By the definition of \( \text{Epi}_S f \)

\[
f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).
\]

Using the above proposition and the result of Kelley and Namioka quoted in [46] one derives:

Proposition: \( f : X \to Y \) is convex with respect to \( S \) if and only if \( f \) satisfies

\[
f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)
\]

whenever \( x_1 \) and \( x_2 \) belong to some neighbourhood \( N \).

Proof: The condition is equivalent to the local convexity of \( \text{Epi}_S f \).

This result is clearly not true for quasiconvex functions as is seen by \( f(x) = \begin{cases} 1 - x^2 & |x| > 1 \\ 0 & |x| \leq 1 \end{cases} \)

In fact with \( N = \{x \mid |x| \leq 1\} \) \( f \) is locally quasiconvex but it is not quasiconvex.
The definition in 45 could be rephrased to include a restricted domain C. Since this can equally well be done by redefining f, it seems simpler to leave it as it is.

**Proposition:** f : X → Y is strongly quasiconvex if and only if Epi f has the following property:

If (y, x_1) and (y, x_2) ∈ Epi f then (y, λ x_1 + (1 - λ) x_2) does for 0 ≤ λ ≤ 1.

**Proof:** This is immediate from the definitions of the epigraph and quasiconvexity.

A set B ⊂ Y is said to be minimizable with respect to a cone S if there is some z ∈ Y with b - z ∈ S ∀ b ∈ B.

Maximizability is defined dually.

**Proposition:** If S is a cone with non empty interior in a convex space then all bounded sets are minimizable and maximizable with respect to S. The maximizing and minimizing points can be taken in S.

**Proof:** Since B is bounded it is absorbed by all neighbourhoods. In particular if s ∈ S^0, there is an open set N with λ_0 > 0 and

s + λ_0B ⊂ s + N ⊂ S

where the first containment follows from the boundedness of B and the second from s ∈ S^0. Since S is a cone

λ_0^{-1} s + B ⊂ S

which rewritten says b ≥ - λ_0^{-1} s whenever b ∈ B.

Applying the same argument to -B one sees that B is maximizable.
The results of [27] and [52] can be used to generalise the standard result (Iuenberger (1969)) that a convex function on $R$ is continuous throughout the relative interior of its domain if it is continuous anywhere. For simplicity $X$ and $Y$ are assumed normed. By the relative interior of a set $C$ (denoted $\text{ri}C$) one means the interior with respect to the smallest closed affine subspace containing $C$. It is a standard result that for a finite dimensional convex set $C, \text{ri}C \neq \emptyset$.

**Proposition:** Let $f : X \to Y$ be convex with respect to $S$ which is normal with interior. Then if $C$ is a convex set in $\text{dom} f$ and $f$ is continuous at $x_0 \in \text{ri}C$ $f$ is continuous throughout $\text{ri}C$.

**Proof:** Suppose without loss that $f(0) = 0, x_0 = 0$ and $\text{ri}C = C^0$. Let $\varepsilon > 0$ be given. Then $\exists \delta_0$ such that $\| x \| < \delta_0$ implies that $\| f(x) \| < \varepsilon_0$ since $f$ is continuous at $0$.

Let $y_0 \in C^0$. $\exists \beta > 1$ with $\beta y_0 \in C^0$.

If $\| z - y_0 \| < (1 - \beta^{-1})\delta_0$ then $z = y_0 + (1 - \beta^{-1})x$ with $\| x \| < \delta_0, x \in C$ and $f(z) \leq \beta^{-1} f(\beta y_0) + (1 - \beta^{-1}) f(x)$.

Since, $\| f(x) \| \leq \varepsilon_0$ when $\| x \| < \delta_0$ there is, using [52], some $d$ such that $f(x) \leq d \forall x$ with $\| x \| < \delta_0$ and $f(z) \leq \beta^{-1} f(\beta y_0) + (1 - \beta^{-1}) d = d(y_0)$ if $\| y_0 - z \| < (1 - \beta^{-1})\delta_0$. For $0 < \varepsilon < 1$ one has $(z + y_0) = \varepsilon (y_0 + \varepsilon^{-1} z) + (1 - \varepsilon) y_0$.
and by convexity
\[ f(z + y_o) - f(y_o) \leq \varepsilon \left[ f(\varepsilon^{-1} z + y_o) - f(y_o) \right] . \]

So if \( ||z|| \leq \varepsilon (1 - \beta^{-1}) \delta_o \)

(1) \( f(z + y_o) - f(y_o) \leq \varepsilon [d(y_o) - f(y_o)] = \varepsilon \alpha \).

Moreover, since \( y_o = \frac{1}{1+\varepsilon} (z + y_o) + (1 - \frac{1}{1+\varepsilon}) (-\varepsilon^{-1} z + y_o) \)

\[ f(y_o + z) - f(y_o) \geq - \varepsilon [f(y_o - \varepsilon^{-1} z) - f(y_o)] \]

which as before gives

(2) \( f(y_o + z) - f(y_o) \geq - \varepsilon \alpha \)

Combining (1) and (2) gives \( a \varepsilon \leq f(y_o + z) - f(y_o) \leq a \varepsilon \)

whenever \( ||z|| \leq \varepsilon (1 - \beta^{-1}) \delta_o \).

By the result of [28]

the set \( A = \{ x \in \mathbb{R}^n \mid a \leq x \leq a \} \) is bounded because \( S \) is normal and one has

\[ f(y_o + z) - f(y_o) \in \varepsilon A \text{ when } (||z|| < \varepsilon (1 - \beta^{-1}) \delta_o) \]

and \( f \) is continuous throughout \( \text{ri} C \).

[54] By the result in [29] any pointed cone in \( \mathbb{R}^n \) is normal and in particular [53] generalises Iuenberger's proposition. Also the condition that \( S^0 \neq \emptyset \) is needed for [52] and it plus the boundedness of \( A \) are, by [28] equivalent to normality of \( S \).

[55] The next proposition relates continuity off and interior points of \( \text{Epi}_S f \) when \( X \) and \( Y \) are normed spaces.

**Proposition:** If \( S \) is a normal cone with interior and \( \text{ri}(\text{dom } f) \neq \emptyset \),

then \( f \) is continuous at \( x_o \) if and only if \( (x_o, y_o) \in \text{ri} \text{Epi}_S f \) for some \( y_o \).
Proof: Without loss \( x_0 = o = f(o) \) and replacing \( X \) by \( V(o) \) the variety (affine subspace) spanning dom\( f \) one can assume that \( (\text{dom} f)^0 \neq \emptyset \). Suppose \( (0, y_o) \in \text{ri Epi} f^* \), then since

\[
V(\text{Epi} f^*) = V(o) \times Y
\]

\((0, y_o)\) may be supposed interior to \( \text{Epi} f^* \).

Thus, \( \delta_1 \) and \( \delta_2 > 0 \) exist with \( (x, y) \in \text{Epi} f^* \) when

\[
\|x - o\| < \delta_1 \quad \text{and} \quad \|y_o - y\| < \delta_2.
\]

As in \([53]\)

\[
-\lambda f(-\lambda^{-1} x) \leq f(x) \leq \lambda f(\lambda^{-1} x).
\]

Also

\[
f(x) \leq y \quad \text{when} \quad \|y - y_o\| < \delta_2 \quad \text{and} \quad \|x\| < \delta_1.
\]

Since neighbourhoods in normed spaces are bounded and \( S^0 \neq \emptyset \), \( N = \{y \mid \|y - y_o\| \leq \delta_2\} \) is maximizable and there is some \( a_o \) with \( f(x) \leq a_o \) if \( \|x\| < \delta_1 \).

Thus if \( \|x\| < \lambda \delta_1 \),

\[
-\lambda a_o \leq f(x) \leq \lambda a_o.
\]

Since \( S \) is normal the same argument as in \([53]\) shows that \( f \) is continuous at \( o \). \( \Rightarrow \) Let \( \varepsilon > 0 \) be given and suppose \( f \) is continuous at \( o \in (\text{dom} f)^0 \). Then \( \|x\| < \delta \) implies that

\[
\|f(x)\| < \varepsilon/2 \quad \text{and by} \quad [52] \quad \text{there is some} \quad s_2 \in S \quad \text{with}
\]

\[
-s_2 \leq f(x) \leq s_2.
\]

Let \( s_1 \in S^0 \) with \( N \in S \) such that \( s_1 + N \in S \). Then \( f(x) < s_1 + s_2 = s_o \in S \) when \( \|x\| < \delta \).

If \( \|y - s_o\| \leq \varepsilon/2 \) then

\[
f(x) - y = (f(x) - s_o) + (s_o - y)
\]

\[
\subset N \varepsilon/2 - s_o + N \varepsilon/2 \subset N \varepsilon - s_o \subset -S
\]
Thus if \( \| x - x_0 \| \leq \delta \) and \( \| y - s_0 \| \leq \varepsilon / \alpha \) \( (x, y) \in Epi_{1} f \) and \( (x_0, s_0) \in \text{ri } Epi_{1} f \) as desired.

The continuity results above can be used to generalise a theorem of Rockafellar’s (1970).

Theorem: Let \( f : X \rightarrow Y \) be convex with respect to a convex cone \( S \) with \( S^0 \neq \emptyset \). Let \( X, Y \) be normed spaces with \( X \) reflexive and let \( B \) be a weakly compact subset of the relative interior of \( \text{dom} f (W) \) in the weak topology. If \( f \) is weakly continuous on \( W \) then \( f \) is Lipschitzian relative to \( B \). In particular if \( X \) is finite dimensional and \( f \) is continuous at some point of \( \text{ri}(\text{dom} f) \) the result holds.

Proof: By restricting attention to the variety spanned by \( \text{dom} f \) one can assume that \( W \) is the weak interior of \( \text{dom} f \).

Let \( U \subset X \) be the unit ball; then \( U \) is weakly compact since \( X \) is reflexive and hence so is

\[
\frac{1}{n} U + B \quad \forall n \in \mathbb{N}.
\]

Moreover

\[
\bigcap_{n \in \mathbb{N}} \left( \frac{1}{n} U + B \right) \cap \overline{W} = \emptyset.
\]

This is the intersection of weakly compact sets so there is some \( n \) with \( B + \frac{1}{n} U \subset W \).

Since \( f \) is assumed continuous on \( \text{dom} f \) and \( B + \frac{1}{n} U \) is weakly compact \( f(B + \frac{1}{n} U) \) is weakly bounded and thus both minimizable and maximizable, since \( S^0 \neq \emptyset \).

Let \( b_1 \leq f(B + \frac{1}{n} U) \leq b_2 \) and let \( x, y \in B \) \( x \neq y \). Then

\[
z = y + \frac{1}{n} \| y - x \|^{-1} (y - x) \in B + \frac{1}{n} U \quad \text{and} \quad y = \lambda z + (1 - \lambda) x
\]

with \( \lambda = \| y - x \| / \frac{1}{n} + \| y - x \| \). Since \( f \) is convex with respect to \( S \).
\[ f(y) \leq (1 - \lambda) f(x) + \lambda f(z) \]

and

\[ f(y) - f(x) \leq \lambda (b_2 - b_1) . \]

Since \( x, y \) are interchangable

\[ f(y) - f(x) \in \{ w \mid -\lambda (b_2 - b_1) \leq w \leq \lambda (b_2 - b_1) \} \]

This last set is contained in \( D \) where \( D \) is

\[ D = \{ w \mid -||x - y|| k \leq w \leq (x - y) k \} \]

and \( k = n(b_2 - b_1) \).

By a result of Kelley and Namioka (1963) there is \( L > 0 \) (since \( S \) is normal) with

\[ \alpha (||w|| \leq k ||x - y||) \quad \forall w \in \mathbb{W} . \]

In particular for \( x, y \in E \)

\[ ||f(x) - f(y)|| \leq L^{-1} k ||x - y|| . \]

The second conclusion follows from [53] and [55].

Rockafellar's initial result was the case \( X = \mathbb{R}^m \)
\( (Y, S) = (\mathbb{R}, R^+) \). If \( X = \mathbb{R}^m \) and \( Y = \mathbb{R}^n \) with \( S \) any pointed cone with interior then it is clear that \( f \) is Lipschitz on any closed bounded set in \( \text{ri}(\text{dom} f) \).

[57] **Examples of discontinuous behaviour**

1. Let \( D = \{ x \in C [0, 1] \mid d x/dt \in C [0, 1] \} \) and let \( A : D \rightarrow C [0, 1] \) be defined by \( Ax = dx/dt \). \( A \) is discontinuous but is convex with respect to any cone \( S \).

2. \( f : \mathbb{R} \rightarrow \{ \infty \} \) (with the orthant ordering) given by \( f(x) = x^{2n} \). \( f \) is continuous with \( \text{dom} f = \{ x \mid x \leq 1, \} \). In this case \( f \) is continuous if \( |x| < 1 \) but is discontinuous at 1 since

\[ ||f (1 - \frac{1}{n}) - f(1)|| = \sup_k ||(1 - \frac{1}{n})^{2k} - 1|| \] which is 1.
Although convex functions share the property of [53] with linear functions a convex function can be continuous at a point and not weakly continuous.

Let \( f : \ell_2 \to \ell_2 \) (with the orthonormal orderings) be given by \( f(\{x_n\}) = \{\overline{x_n^{2n}}\} \). \( f \) is clearly convex with \( \text{dom} f = \ell_2 \).

It is reasonably simple to verify continuity especially at the origin. \( f \) is not weakly continuous since \( \{\{x_{nk}\}\} = \{\{2\delta_{nk}\}\} \) is weakly convergent but \( \{\{x_{nk}\}\} = \{\{2\delta_{nk}^{2n}\}\} \) is not since \( u^+ = \{\frac{1}{n}\} \in \ell_2^+ = \ell_2 \) and

\[
 u^+ - f(\{x_{nk}\}) = \sum_n \frac{1}{n} (2\delta_{nk})^{2n} = 2^{-k/2} \to \infty.
\]

Note that \( S \) is normal but has no interior.

The following partial analogues of the linear situation do hold:

**Proposition:** If \( f : X \to Y \) is strongly quasiconvex with respect to \( S \) and lower semicontinuous over \( C \), a closed convex set then \( f \) is weakly lower semicontinuous over \( C \).

**Proof:** \( \{ x \, | \, f(x) \leq z \} \cap C \) is a closed convex set and thus weakly closed.

More interestingly one has

**Proposition:** If \( f : X \to Y \) is strongly quasiconvex with respect to \( S \) and lower semicontinuous sequentially on any convex set \( C \) then \( f \) is weakly lower semicontinuous on \( C \).

**Proof:** Let \( \{ x_n \} \subset C \) be a sequence with limit \( x_0 \in C \). Since \( x_0 \) belongs to the weak closure of the convex hull of \( \{ x_{nk}^{\infty} \} \) for any \( n \), \( x_0 \) actually belongs to the closure of the convex hull. Thus there is for each \( n \) a point \( z_n \) and scalars \( \lambda_{nk} \geq 0 \) with

\[
 \sum_{k=1}^{\infty} \lambda_{nk} z_n = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_{nk} z_n = 1.
\]
\[ z_n = \sum_{k=n}^{m} x_k \rightarrow x_0. \]

Since \( f \) is strongly quasiconvex over \( C \)
\[ f(z_n) \leq \max_{n \leq k \leq m} f(x_k). \]

Hence if \( f(x_k) \leq z_0 \) and \( x_k \rightarrow x_0 \) one has \( f(z_n) \leq z_0 \) when \( n > n_0 \).

Because \( z_n \rightarrow x_0 \) and \( f \) is sequentially semicontinuous one has \( f(x_0) \leq z \) and \( f \) is weakly sequentially lower semicontinuous on \( C \).

This last result generalizes Daniel (1971) and provides at least a partial analogue to the equivalence of weak and strong continuity of linear maps.

[61] Since in the result of [53] it is only to prove that bounded sets are maximizable that \( S^0 \neq \emptyset \) is used, one could have required only the former condition. This condition would not be much weaker because the following partial converse to [52] holds.

**Proposition:** If \( S \subset Y \) is a generating cone, that is \( S - S = Y \), and \( Y \) is normed then \( S^0 \neq \emptyset \).

**Proof:** Since \( Y \) is normed the unit ball \( U \) is bounded. Let \( y_0 \in Y \) be a maximizer for \( U \). Then
\[ y_0 \succ u \quad \forall u \in U. \]

Since \( S - S = Y \) \( y_0 = s_1 - s_2 \), \( s_1, s_2 \in S \) and
\[ s_1 + U \subset s_2 + S \subset S \]
so that \( s_1 \in S^0 \).

[62] In further reference to [53], [55] it is apparent that if \( S \) is not pointed it is unreasonable to expect continuity since in the nonpointed case there is at least one direction in which the behaviour of \( f \) is not restricted at all. It is also apparent that
no sort of simple continuity result should be expected for quasiconvex mappings even on the line. Certainly dense countable sets of discontinuities can exist because any monotone real valued mapping is quasiconvex. On the line this is the worst that can happen because:

**Proposition:** (Stoer and Witzgall (1970)) A function $f: C \rightarrow R$, where $C$ is convex, is (quasi)convex if and only if its restriction to any line segment is (quasi)convex. Moreover, $f: R \rightarrow R$ is quasiconvex if and only if there is a partition of $R$ into two (possibly disjoint) intervals $(I_1, I_2)$ with $f$ non-increasing on $I_1$ and non-decreasing on $I_2$.

In particular this means that if $f: R \rightarrow R$ is quasiconvex it is of bounded variation and has at worst countable discontinuities. This also implies that the set of discontinuities of any real valued quasiconvex map has no interior.

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**63 Boundedness of level sets**

It is a well known result in $R^n$ that any closed unbounded convex $C$ contains all half lines of the form $x + th$, $t \geq 0$ where $h$ is some fixed non zero point and $x$ is any member of $C$. $h$ is called a direction of recession. An unbounded convex set in general need not have any such directions.

**Example:** Let $X$ be $L^\infty$ and let $C$ be the convex set defined by

$$C = \left\{ a_k \mid a_k \leq \frac{k}{k}, \ k = 1, 2, \ldots \right\}$$

which is clearly unbounded and closed. Suppose that $a + tb$, $t \geq 0$ was a line segment in $C$, $a \in C$, $b \neq 0$. Then for some $b_k$, $b_k$ is non zero and for $t > t_0$ $|a_k + tb_k| > k$ which means $a + tb$ is not in $C$.

---

**64** The next result, which generalizes a proof of Stoer and
Witzgall (1970) is phrased in \( \mathbb{R}^n \) because it relies on the existence of half lines in unbounded sets.

**Proposition:** Let \( f: \mathbb{R}^n \rightarrow Y \) be lower semicontinuous and convex with respect to a generating closed convex cone \( S \). Either all nonvoid level sets are bounded or they are all unbounded.

**Proof:** Suppose the level sets \( S(z) = \{ x \mid f(x) \leq z \} \) are such that \( S(z_1) \) is bounded and \( S(z_2) \) is unbounded. Since \( f \) is supposed convex and lower semicontinuous all the level sets are closed and convex in \( \mathbb{R}^n \). Hence there is \( h_2 \not\in 0 \) with \( x + th_2 \in S(z_2) \) \( \forall x \in S(z_2) \) and \( t > 0 \).

If \( z_3 \) is chosen such that \( z_3 \geq z_2, z_3 \geq z_1 \), which can be done since \( S \) is generating, \( S(z_3) \) is unbounded because it contains \( S(z_2) \).

Let \( x \in S(z_1) \). Then \( x \in S(z_3) \) and since \( h_2 \) is a direction of recession for \( S(z_2) \) it is also one for \( S(z_3) \) and \( x + th_2 \in S(z_3) \) \( \forall t > 0 \). Thus

\[
f(x + th_2) \leq \lambda f(x) + (1 - \lambda) f(x + \frac{t}{1 - \lambda} h_2) 
\leq \lambda z_1 + (1 - \lambda) z_3 \text{ if } 0 < \lambda < 1.
\]

Letting \( \lambda \rightarrow 1 \)

\[
f(x + th_2) \leq z_1 \forall t > 0, \text{ since } S \text{ is closed.}
\]

This shows that \( S(z_1) \) is unbounded and a contradiction has been established.

---

For quasiconvex functions the result in [64] is not true. The next proposition which generalizes Lemma 4.9.7. of Stoer and Witzgall (1970) is a partial clarification.

**Definition:** A chain (linear ordering) is order complete if any subset \( B \) of \( A \) which can be maximized has a supremum in \( B \). That is, there exists \( z_1 \) such that if \( z \succ b \forall b \in B \) then \( z \succ z_1 \) and
Proposition: Let \( f: \mathbb{R}^n \to Y \) be lower semicontinuous and strongly quasiconvex with respect to \( S \). Let \( A \) be an order complete chain in a sequential topological space \( Y \). Suppose that any descending convergent sequence in \( A \) has limit in \( A \). If the level sets \( \{ S(a) \mid a \in A \} \) contain both bounded and unbounded sets there is an \( a \in A \) with \( S(a) \) unbounded exactly when \( a \succ a \).

Proof: Suppose \( S(a_2) \) is unbounded and \( S(a_1) \) is bounded. Let \( \bar{a} = \inf \{ a \mid S(a) \text{ is unbounded} \} \). Then \( a_1 \leq \bar{a} \leq a_2 \). It remains to show \( S(a) \) is unbounded. Let \( T(a) \) denote the directions of recession of \( S(a) \) with norm one. These sets are compact and, since \( T(a) \subset T(b) \) if \( a \leq b \), have the finite intersection property for \( a \succ \bar{a} \). Hence, \( \bigcap_{a \succ \bar{a}} S(a) \neq \emptyset \). Let \( h \) be in this intersection.

For each \( a \succ \bar{a} \) and for \( x \in S(\bar{a}) \) one has (1) \( f(x + th) \leq a \) \( \forall t \geq 0 \).

The assumptions on \( A \) imply that there is a sequence \( \{ a_n \} \), \( a_n \to \bar{a} \), with \( a_n \to \bar{a} \). This with (1) shows \( S(\bar{a}) \) is unbounded.

Taking \( f: \mathbb{R} \to \mathbb{R} \), \( f(x) = \begin{cases} 1 & x \geq 1 \\ 0 & x < 1 \end{cases} \) one sees that \( S(r) \) is unbounded if and only if \( r \geq 1 \). Setting \( A = \{ 0 \} \cup \{ x \mid x > 2 \} \), which is an order complete chain which does not contain its limit points one has an example in which the proposition does not hold.

Differential characterizations of convex type functions

For the most part differential conditions will be introduced as they prove necessary. The following few propositions are given for the sake of completeness.

Definition: If \( f: X \to Y \) is a mapping between two convex spaces
then \( f \) is said to be \( \mathcal{B} \)-differentiable at \( x \) with respect to a family \( \mathcal{B} \) of sets in \( X \) if there is a continuous linear transformation \( f'(x_0): X \to Y \) with
\[
\lim_{t \to 0} \left[ f(x + th) - f(x) \right] - f'(x_0)(h) = 0
\]
uniformly in the topology defined by \( \mathcal{B} \).

\( f \) is said to be (1) compactly differentiable if \( \mathcal{B} \) is all sequentially compact sets and (2) boundedly differentiable if \( \mathcal{B} \) is all bounded sets. In particular these notions agree in sequential Montel spaces. In normed spaces (2) is just Fréchet differentiation.

**Proposition:** If \( f: X \to \mathbb{R} \) is boundedly or compactly differentiable then \( f \) is quasiconvex if and only if
\[
f'(x_0) (x - x_0) \preceq 0 \text{ when } f(x) \preceq f(x_0).
\]

**Proof:** This is proved in Ponstein (1967) for \( X = \mathbb{R}^n \) and Fréchet differentiation. The differences are entirely technical since any sensible \( \mathcal{B} \)-derivative will suffice in his proof. From now on when any reasonable derivative will do it will just be called differentiable.

**Proposition:** \( f: X \to Y \) is convex with respect to a closed cone \( S \) if and only if
\[
f'(x_0) (x - x_0) \preceq f(x) - f(x_0) \ \forall x, x_0 \in X.
\]

**Proof:** The result when \( Y = \mathbb{R} \) is standard. In the general case by [42] \( u^+ f \) is convex (and differentiable) \( \forall u^+ \in S^+ \).

By the linear result this is equivalent to
\[
u^+ (f(x) - f(x_0)) \succeq u^+ f'(x_0) (x - x_0) \ \forall u^+ \in S^+.
\]
Since \( S \) is closed this last inequality gives as equivalent
\[
f(x) - f(x_0) \preceq \mathcal{B} f'(x_0) (x - x_0).
\]
It does not appear that the condition 
\[ f'(x_o) (x - x_o) \leq_\circ f(x_o) \] whenever \( f(x) \leq_\circ f(x_o) \) is equivalent to (strong) quasiconvexity although by a direct derivative argument it is implied by quasiconvexity.

A condition which will be of primary importance in optimization is 
\[ f'(x_o) (x - x_o) \in S \text{ implies } f(x) - f(x_o) \in S \quad \forall x \in A \]
which is called pseudo-convexity over \( A \) at \( x_o \). It is possessed by convex functions ([69]). The function \( f: \mathbb{R} \rightarrow \mathbb{R} \) given by 
\[ f(x) = \begin{cases} -x^2 & x \in [0,1] \\ \infty & x \notin [0,1] \end{cases} \]
is quasiconvex but not pseudo-convex when the derivatives are taken to be one sided at \( 0 \) and \( 1 \).

A useful relationship which simplifies proofs given in Guignard (1969) and Cottle and Ferland (1970) is:

**Proposition:** Let \( f: X \rightarrow \mathbb{R} \) be quasiconvex on a convex set \( C \) and differentiable at \( x_o \). Suppose that for some \( y \in C \) 
\[ f'(x_o)(y-x_o) \geq 0 \]
then \( f \) is pseudo-convex at \( x_o \) on \( C \).

**Proof:** Suppose \( f'(x_o)(x-x_o) \geq 0 \quad \forall x \in C \).
Then \( (1-\lambda) f'(x_o)(x-x_o) + \lambda f'(x_o)(y-x_o) \geq 0 \quad \forall x \in C \).
Since \( x, y \in C \) 
\[ x_\lambda = \lambda y + (1-\lambda) x_o \in C \text{ and} \]
\[ f'(x_o)(x_\lambda - x_o) \geq 0. \]
Since \( f \) is supposed quasiconvex on \( C \)
\[ f(x_\lambda) \geq f(x_o). \]
The continuity of \( f \) implies that \( f(x) \geq f(x_o) \).
Guignard's case was \( C = X \) and \( f'(x_o) \lessdot 0 \) while Cottle and Ferland had \( X = \mathbb{R}^n, C = \mathbb{R}^n_+ \), \( f'(x_o) \lessdot 0 \). It is easy to show that these are both subsumed.
The natural condition
\[ f'(x_0) (x - x_0) \preceq f(x) - f(x_0) \text{ if } f(x) \preceq f(x_0) \text{ is strong enough to imply convexity in most cases. Precisely one has:} \]

**Proposition:** Let \( f: X \to \mathbb{R} \) be twice Fréchet differentiable with \( f''(x) \) continuous in \( x \) then the above condition implies that \( f \) is convex.

**Proof:** Consider \( X = \mathbb{R} \). By Taylor's theorem
\[ f(\lambda y + (1 - \lambda)x) - f(x) = \lambda f'(x)(y - x) + \frac{1}{2} \lambda^2 f''(x)(y - x)^2 \]
with \( x, y \to x \) as \( \lambda \to 0 \).

The condition of the hypothesis implies quasiconvexity ([68]) and thus for \( 0 < \lambda < 1 \) and \( f(y) \preceq f(x) \) one has
\[ f(\lambda y + (1 - \lambda)x) \preceq f(x) \text{ and } f(\lambda y + (1 - \lambda)x) - f(x) \gtrsim f'(x) \lambda(y - x). \]
Thus
\[ \frac{1}{2} \lambda^2 f''(x)(y - x)^2 \gtrsim 0. \]
On dividing by \( \lambda^2 \) and letting \( \lambda \to 0 \) one gets \( f''(x)(y - x)^2 \gtrsim 0 \).

Now, if \( f(x) \nless \inf f(y) \) there is some \( y \nless x \) with \( f(y) \preceq f(x) \).
In this case \( (y - x)^2 \gtrsim 0 \) and \( f''(x) \gtrsim 0 \).

Otherwise, let \( \{x_n\} \) be a sequence of points with \( x_n \nless x \)
\( x_n \to x \). By the definition of \( x_n \), \( f(x) \preceq f(x_n) \) and thus
\[ f''(x_n)(x_n - x)^2 \gtrsim 0. \]
As before \( f''(x_n) \gtrsim 0 \). Letting \( x_n \to x \)
\( f''(x) \gtrsim 0 \) because \( f'' \) is continuous.

Consider now \( g(\lambda) = f(x + \lambda y) \) for fixed \( x, y \in X \). It is immediately verifiable that \( g \) satisfies the conditions and hence \( g(\lambda) = f(x + \lambda y) \) is convex for any \( x, y \). This, using the first part of the proposition in [62], implies that \( f \) is convex.

**Concets of minimization with respect to cones**

There is a profusion of possible extensions to the notion of the minimum of a real valued function over a set \( A \). The two most useful and possibly most natural are defined below. \( S \) is always
assumed to be a closed convex cone.

**Definition:** $f: X \to Y$ is said to have a **strong minimum** (with respect to $S$) over $A$ at $x_0$ if $f(x) - f(x_0) \in S \quad \forall x \in A$.

**Definition:** $f: X \to Y$ is said to have a **weak minimum** (with respect to $S$) over $A$ at $x_0$ if $f(x) - f(x_0) \notin -S^o$ when $x \in A$.

If $S^o = \emptyset$ any point is a weak minimum so from now on the interior of $S$ will be assumed nonvoid when weak minima are being discussed.

**Proposition** (1) Any strong minimum is a weak minimum.

(2) If $S$ is pointed any two strong minima agree in value.

(3) If $C$ is convex and $x_0$ is a strong minimum for $f$ over $C$ and if $f$ is quasiconvex with respect to $S$, a pointed cone, then $M = \{ x \mid f(x) = f(x_0) = \text{strong min} \{ f(x) \mid x \in C \} \}$ is convex.

**Proof:** Only (3) is not immediate. If $x_1, x_2 \in C$ and $0 \leq \lambda \leq 1$ then $\lambda x_1 + (1 - \lambda) x_2 \in C$ and $f(\lambda x_1 + (1 - \lambda) x_2) \leq f(x_1)$ by quasiconvexity. By (2) and the definitions

$$f(\lambda x_1 + (1 - \lambda) x_2) = f(x_0)$$

and $\lambda x_1 + (1 - \lambda) x_2 \in M$.

**Proposition:** Let $f: X \to Y$ be fully lower semicontinuous with respect to $S$ and let $A$ be closed then the set of weak minima for $f$ over $A$ with respect to $S$ is closed.

**Proof:** Suppose $\{ x_n \mid n \in N \}$ is a net of weak minima with
Since \( x_n \rightarrow x_0 \). Since \( A \) is closed \( x_n \in A \) implies \( x_0 \in A \). Suppose for some \( x \in A \) \( f(x) < f(x_0) \). By full lower semicontinuity \( f(x) < f(x_n) \) if \( n > n_0 \) since \( S^0 \) is an open set. This contradicts the minimality of \( x_n \).

Similarly one has:

**Proposition.** If \( f: X \rightarrow Y \) is lower semicontinuous with respect to \( S \) and \( A \) is closed the set of strong minima for \( f \) over \( A \) is closed.

If one wishes to guarantee the convexity of the set of weak minima over a convex set quasiconvexity is too weak. The condition stated below seems artificial but it is equivalent to quasiconvexity when \((Y, S) = (\mathbb{R}, \mathbb{R}^+)\).

**Proposition:** The set of weak minima over \( C \) is convex if \( f: X \rightarrow Y \) satisfies: Whenever \( x, y, x_0 \in C \) and for some \( 0 < \lambda < 1 \)

\[ f(\lambda x + (1 - \lambda)y) < f(x_0) \text{ then } f(x) < f(x_0) \text{ or } f(y) < f(x_0). \]

If \( X \) is a topological space then \( x_0 \) is called a **local minimum** for \( f \) over \( A \) with respect to \( S \) if \( x_0 \) is a minimum over \( A \cap N \) for some neighbourhood \( N \). If \( A = X = N \) the minimum is called **global**.

Fonstein's (1967) result that every local minimum of a real valued \((P)\) strictly quasiconvex mapping is global has the following extensions.

**Theorem:** If \( f: X \rightarrow Y \) is \((P)\) strictly quasiconvex with respect to \( S \) then every local weak minimum with respect to \( S \) is global.

**Proof:** Suppose \( x_1 \) is a local non global minimum. Then there is some \( x_2 \in X \) with \( f(x_2) - f(x_1) \in -S^0 \). Let \( x_\lambda = \lambda x_2 + (1 - \lambda)x_1 \).
For $\lambda$ sufficiently small $x_2$ will belong to the neighbourhood over which $x_1$ is a weak minimum.

Thus $f(x_2) - f(x_1) \not\in S^0$ for $0 < \lambda < \lambda_0$.

But by (P) strict quasiconvexity ([16])

$$f(x_2) - f(x_1) \in S^0$$

since

$$f(x_2) - f(x_1) \in S^0$$

and a contradiction has been derived.  

For strong local minima one has (dually)

[81] **Theorem:** If $f$ is absolutely quasiconvex ([11]) with respect to $S$ then every strong local minimum is global.

**Proof:** Let $x_1$ be a local minimum and let $x_0 \in X$. For $0 < \lambda < \lambda_0$

$$f(\lambda x_1 + (1 - \lambda) x_0) \geq f(x_1).$$

By [11] $f(x_0) \geq f(x_1)$ and $x_1$ is a global minimum.

If $(Y,S) = (R,R^*)$ then by [21] both [80] and [81] agree with Fonstein's result. When this is the case one can in fact show that if every local minimum of a quasiconvex function is global the $f$ is (P) strictly quasiconvex.

More generally for any convex space $X$ one has

[82] **Proposition:** If $f: X \to Y$ is quasiconvex with respect to a pointed cone $S$, and $x_2$ is any nonglobal but local strong minimum then

$f(x)$ is constant on $I(\varepsilon) = \{ x | x = \lambda x_1 + (1 - \lambda) x_2, 0 \leq \lambda \leq \varepsilon \}$

where $x_1$ is any global minimum and $\varepsilon$ is some positive number.

**Proof:** Since $x_2$ is a local minimum there is $\varepsilon > 0$ with

$$f(\lambda x_1 + (1 - \lambda) x_2) \geq f(x_2) \text{ if } 0 \leq \lambda \leq \varepsilon.$$  Since

$$f(x_1) \leq f(x_2)$$

and $f$ is quasiconvex

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq f(x_2) \text{ if } 0 \leq \lambda \leq 1.$$
Thus for $x \in L(\varepsilon)$ $f(x) = f(x_2)$ since $S$ is pointed.

Weak and strong maximization are defined dually to minimization. Any convex minimization result yields a dual maximization result for concave type functions. Thus it is usually unnecessary to consider both maximization and minimization problems. The next results, however, give information about maxima of quasiconvex functions.

Theorem: If $f: X \to Y$ satisfies

1. If $f(x) \not\leq f(z)$ and $f(y) \not\leq f(z)$ then
   
   $f(\lambda x + (1 - \lambda) y) \not\leq f(z)$ for $0 < \lambda \leq 1$,

then if $f$ achieves its strong maximum over $C$, a convex set contained in $\text{dom} f$, at $x_0 \in \text{rel} C$ then $f$ is constant on $C$.

Proof: Let $z \in \text{rel} C$ with $f(z) \not> f(y) \forall y \in C$ and let $x \in C$. Then there is some $y \in C$ and $0 < \lambda_0 < 1$ with $z = \lambda_0 x + (1 - \lambda_0) y$.

(Otherwise $z$ would be a boundary point). Now, if $f(y) \leq f(z)$ and $f(x) \not= f(z)$ one has, since $f(x) \not= f(z)$, that

$f(\lambda x + (1 - \lambda) y) \not\leq f(z)$. This is impossible for $\lambda = \lambda_0$ and $f(x)$ must, therefore be equal to $f(z)$. Thus $f$ is constant on $C$.

Corollary: If $f: X \to Y$ is such that either

1. $f$ is convex

or

2. $(Y, S) = (R, R^+)$ and $f$ is (P) strictly quasiconvex and satisfies the one point exclusion property then the result holds.

Proof: For (1) it is easily verified that any convex $f$ satisfies (1) of [33], while for (2) the proposition of [33] proves that the property is satisfied.
The proof method of [83] is derived from a result of Rockafellar (1970a) which is in fact the corollary of [84] in case (1) with \((Y,S) = (R,R^+))

For any quasiconvex function it is simple to show the following result.

**Proposition:** If \(f: X \rightarrow Y\) is quasiconvex with respect to \(S\) and if \(A\) is any set over which \(f\) has a strong maximum at \(x_0\) then \(x_0\) is a strong maximum for \(f\) over the convex hull, \(C\), of \(A\).

**Proof:** \(A \subset \{x \mid f(x) \leq f(x_0)\} = S(f(x_0))\).

Since \(f\) is quasiconvex \(S(f(x_0))\) is convex and thus

\[C \subset S(f(x_0))\]

Equivalently \(f(x) \leq f(x_0)\) \(\forall x \in C\). 

Definitions have been made of convex like conditions which do not require that one be in a vector space. In particular, a function \(f: X \rightarrow R\) is called **pathwise connected** if whenever \(x, y \in X\) there is an arc \(p(t)\) with \(p(0) = x\) and \(p(1) = y\) and with \(f(p(t)) \leq \max(f(x), f(y))\). **Strict pathwise connectedness** is defined similarly. Many of the previous results hold for strict pathwise connectedness. For example local minima are still global.

The properties defined in [8] through [16] could all be extended analogously but the difficulty in verifying the connectedness of a function and its relative inutility because of this suggest that the effort is not worthwhile.

If \(f\) is both quasiconvex and quasiconcave with respect to \(S\) \(f\) is called **quasiaffine** while if it is both (P) strictly quasiconcave and quasiconvex as well it is called **strictly quasiaffine**.
If \((Y,S) = (R,R^+)\) then Stoer and Witzgall (1970) have shown that the set of maxima for \(f\) over a convex set \(C\) consists exclusively of extreme points. The result can be seen to hold for any strictly quasiaffine function with respect to a pointed cone. Note that on the line if \(f\) is \((P)\)-strictly quasiconvex and \(P\) strictly quasiconcave it is automatically strictly quasiaffine.

**Multivalued convex and quasiconvex functions**

In another direction to the notions previously discussed lies the idea of multivalued quasiconvexity and convexity. As will be seen later many standard multiplier theorems can be painlessly extended to cover multivalued functions.

The sequel gives the definitions which will be used and some propositions.

**Definition:** \(F: X \rightarrow Y\) is said to be a **multivalued convex function** with respect to \(S\) if \(F(x)\) is a subset of \(Y\) for each \(x \in X(P: X \rightarrow 2^Y)\) and if whenever \(y_1 \in F(x_1), y_2 \in F(x_2)\) and \(0 < \lambda < 1\) there is some \(y_\lambda \in F(\lambda x_1 + (1 - \lambda)x_2)\) with \(y_\lambda - [\lambda y_1 + (1 - \lambda)y_2] \in - S\).

**Definition:** \(F: X \rightarrow Y\) is **multivalued \((P)\) strictly quasiconvex** with respect to \(S\) if whenever \(0 < \lambda < 1\), \(y_1 \in F(x_1), y_2 \in F(x_2)\) with \(y_1 < y_2\) there is some \(y_\lambda \in F(\lambda x_1 + (1 - \lambda)x_2)\) and with \(y_\lambda < y_2\).

**Definition:** \(F: X \rightarrow Y\) is **multivalued quasiconvex** with respect to \(S\) if whenever \(y_1 \in F(x_1), y_2 \in F(x_2)\) with \(y_1 \leq y_2\) there is a \(y_\lambda \in F(\lambda x_1 + (1 - \lambda)x_2)\) with \(y_\lambda \leq y_2\).

The following list of facts, collected as a theorem, follow
from the definitions or from the same type of arguments as in
the single valued cases.

Theorem: (1) If \( f : X \rightarrow Y \) is quasiconvex, \((P)\) strictly quasiconvex,
or convex with respect to \( S \) it is multivalued of the same type.

(2) \( F \) is multivalued convex with respect to \( S \) if and only
if \( \text{Epl}_S F = \left\{ (x, z) \mid \exists y \in F(x) : z \succ y \right\} \) is convex.

(3) Convexity in the multivalued sense implies both
multivalued quasiconvexity and \((P)\) strict quasiconvexity.

Definition: \( x_0 \) is a weak (local) minimum for \( F \) over \( A \) if there
is some \( y_0 \in F(x_0) \) such that whenever \( y \in F(x) \) and \( y - y_0 \in -S^o \)
x \( \notin A \) (\( x \notin N \cap A \)).

Proposition: If \( F \) is multivalued \((P)\) strict quasiconvex with
respect to \( S \) every local minimum is global.

Proof: Suppose \( x_0 \) is a local minimum, then there is a
neighbourhood \( N \) and \( y_0 \in F(x_0) \) such that when \( x \in N(x_0) \) and
\( y \in F(x) \) \( y - y_0 \notin -S^o \).

Suppose that \( y_1 - y_0 \in -S^o \) and \( y_1 \in F(x_1) \). Then for
\( 0 < \lambda < \lambda_0 \) \( \lambda x_1 + (1 - \lambda)x_0 \in N(x_0) \). Since \( F \) is multivalued
\((P)\) strictly quasiconvex there is some \( y_\lambda \in F(\lambda x_1 + (1 - \lambda)x_0) \)
with \( y_\lambda < y_0 \). This contradicts the
local minimality of \( y_0 \) which asserts that no such \( y_\lambda \) can exist
for \( \lambda < \lambda_0 \).

If \( F : X \rightarrow R \) is multivalued \((quasi)\)convex and for each \( x \in F(x) \) is
a nonempty compact set then
\[
f(x) = \min \{ r | r \in F(x) \}
\]
is a (quasi)convex retraction \( f \subset F \). However, even on the line examples exist of multivalued convex and quasiconvex functions with no single valued retractions.

Examples: (1) \( F : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
F(x) = \left\{ \left[ \frac{x}{2} \right] + n \mid n \in \mathbb{N} \right\},
\]

where \( \left\lfloor x \right\rfloor \) is the greatest integer less than \( x \), is a multivalued convex function which has no everywhere defined single valued convex restriction because the graph of \( F \) is not connected.

(2) Any multivalued maximal monotone mapping (see the final chapter) \( f \) mapping \( \mathbb{R} \) into \( \mathbb{R} \) is multivalued quasiconvex but won't necessarily contain a maximal monotone single valued restriction.
Chapter 2

TANGENT AND PSEUDOTANGENT CONES
Tangent cones and pseudotangent cones

The elementary observation in calculus that a function has derivative 0 at an extreme value is in many ways the keystone of all optimization results. The following standard proposition from Luenberger (1969) is the motivation for the developments in this section.

[1] Proposition: Let $f$ be the real valued functional defined on a vector space $X$. Suppose that $\mathbf{x}_0$ minimizes $f$ on a convex set $C \subseteq X$ and that $f$ is Gateaux differentiable at $\mathbf{x}_0$. Then

$$f'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \geq 0 \quad \forall \mathbf{x} \in C.$$

Essentially it was in order to generalize this result to nonconvex sets, with the corresponding implications for more general minimization problems, that the notions of tangent cones were introduced by Varaiya (1967), Guignard (1969) and others.

[2] The set $T(E,x)$ consisting of all limits of the form $h = \lim_n \lambda_n (\mathbf{x}_n - x)$ with $\mathbf{x}_n \in E \subseteq X$, $\lambda_n \downarrow 0$ and $\mathbf{x}_n \to x$ in the topology on $X$ is called the tangent cone to $E$ at $x$. It is largely irrelevant whether the convergence is defined in terms of nets or sequences. For the purposes of simplicity all spaces will be assumed to be Hausdorff and locally convex from now on. Convergence in the given topology will be denoted by $\to$ while convergence in the weak topology, denoted $\sigma(X,X')$, will be denoted by $\rightharpoonup$ or by "wlim".

[3] The set $wT(E,x)$ consisting of all $h$ which are limits in the weak
topology of nets of the form $\lambda_n(x_n - x)$ with $\lambda_n > 0, x_n \in E$
and $x_n \rightharpoonup x$ will be called the weak tangent cone to $E$ at $x$.
This definition, which was announced by Nashed (1971), but which
does not seem to have appeared in published articles, is extremely
useful in optimization.

[4] $T(E,x)$ and $wT(E,x)$ are, respectively, closed and weakly closed
cones but need not be convex. This motivates the next definitions.

The closures of the convex hulls of $T(E,x)$ and $wT(E,x)$
are called the pseudotangent and weak pseudotangent cones,
respectively, and are denoted by $P(E,x)$ and $wP(E,x)$.

[5] A set $E \subset X$ will be said to pseudoconvex at $x_0$ with respect to
a set $F \subset X$ if $E - x_0 \subset P(F,x_0)$. Weak pseudoconvexity is defined
analogously. Guignard (1969) defined a set to be pseudoconvex
at $x_0$ if $E - x_0 \subset P(E,x_0)$ which coincides with this definition
in the case $E = F$. When this is so $E$ will merely be called
pseudoconvex at $x_0$.

This chapter is devoted to a development of the properties
of tangent cones which are useful for framing optimization
conditions but includes some results whose interest is intrinsic.

The next theorem lists some properties of tangent cones given

[6] Theorem: If $I$ is any index set then

$$ (1) \ T(\bigcap_{i \in I} A_i, x) \subset \bigcap_{i \in I} T(A_i, x) $$
(2) \( \bigcap_{i \in I} A_i, x \) \( \subset \) \( \bigcap_{i \in I} P(A_i, x) \)

(3) \( \bigcup_{i \in I} T(A_i, x) \) \( \subset \) \( T( \bigcup_{i \in I} A_i, x) \)

(4) \( \bigcup_{i \in I} P(A_i, x) \) \( \subset \) \( P( \bigcup_{i \in I} A_i, x) \)

Proof: These all follow from the definitions and the properties of closed convex cones.

The result holds also for weak cones. For the rest of the chapter results which hold by the same argument for both tangent cones and weak tangent cones will be marked simply by ' (w) also at the conclusion.

[7] A set \( A \) is said to be starshaped at \( x \in A \) if whenever \( y \in A \) and \( 0 \leq \lambda \leq 1 \), \( \lambda x + (1 - \lambda)y \in A \). Clearly any convex set is starshaped at all its members.

[8] Proposition: (1) \( A \subset B \Rightarrow T(A, x_o) \subset T(B, x_o) \); if \( x_o \in A \), \( o \in T(A, x_o) \).

(2) \( T(A, x_o) \subset wT(A, x_o) \); \( P(A, x_o) \subset wP(A, x_o) \).

(3) The union of sets each pseudoconvex at \( x_o \) is pseudoconvex at \( x_o \). (Guignard (1969))

(4) If \( x_o \in A^0 \), \( T(A, x_o) = X \).

(5) If \( A \) is starshaped at \( x_o \), \( A \) is pseudoconvex at \( x_o \).

Proof: (1), (2), (3) follow directly from the definitions.

(4) Let \( x \in X \). Since \( x_o \in A^0 \) there is an \( n_o \in N \) such that for

\( n > n_o \), \( x_n = x/n + x_o \in A \). Setting \( \lambda_n = n \), \( \lambda_n(x_n - x_o) = x \)

and since \( x \to x \), \( x \in T(A, x_o) \).

(5) Let \( x \in A \) then \( x_n = \frac{1}{n}x + (1 - \frac{1}{n})x_o \in A \) since \( A \) is starshaped at \( x_o \). Setting \( \lambda_n = n \) again

\( \lim n(x_n - x_o) = x - x_o \in T(A, x_o) \) and \( A - x_o \subset T(A, x_o) \).

Thus \( A \) is pseudoconvex at \( x_o \).
Proposition: If \( X \) is a metrizable convex space \( T(A, x_o) = T(\bar{A}, x_o) \).

Proof: If \( h \equiv 0 \) then \( h \in T(A, x_o) \).
If \( h \neq 0 \) then \( \{\lambda_n\} \) is unbounded and without loss of generality can be taken to be positive with limit \( \infty \). Since \( x_n \in \bar{A} \) there is a point \( y_n \in A \) with \( y_n - x_n \in \lambda_n^{-1} N_n \) (where \( N_n \) is a nested countable base for the topology). Then

\[
\lambda_n(y_n - x_o) = \lambda_n(x_n - x_o) + \lambda_n(y_n - x_n)
\]

and \( \lambda_n(y_n - x_o) - \lambda_n(x_n - x_o) \subset N_n \). Thus

\[
\lim \lambda_n(y_n - x_o) = h.
\]

(\( w \)) also.

This proposition allows one for the most part to examine only closed sets.

Proposition: If there is some set \( E \) with \( A \subset E \subset B \) and \( E \) pseudoconvex at \( x_o \) then \( A \) is pseudoconvex at \( x_o \) with respect to \( B \).

Proof: \( A - x_o \subset E - x_o \subset F(E, x_o) \subset F(B, x_o) \).

(\( w \)) also.

In particular this holds if there is some convex set \( C \) with \( A \subset C \subset B \).

An example of a set with a trivial tangent cone and a nontrivial weak tangent cone is given below.

Example: Let \( X = \ell_2 \) and let \( A \subset X \) be the set comprising of the element in \( \ell_2 \) and \( \{\{x_n\} \mid x_{nk} = \frac{1}{n} \text{ if } k = 1 \text{ or } n \text{ and } x_{nk} = 0 \text{ otherwise}\} \).

(1) \( T(A, o) = o \). Clearly \( x_n \rightarrow o \). Suppose \( \lambda_n x_n \rightarrow a \neq o \).

Then \( ||\lambda_n x_n|| \rightarrow o \) if \( n \rightarrow n_o \).
\[ \lambda_{n} \overline{x}_{n} \rightarrow 1/n \sqrt{2} \lambda \text{ so } \frac{\lambda_{n}}{n} \text{ does not tend to } 0. \]

Now
\[ (1, n, n, ...) \in w T(A, c) \text{ since if } \lambda_{n} = n, \]
\[ \lambda_{n} \overline{x}_{n} = (1, n, n, ..., 0, 1, n, ...) \rightarrow (1, 0, 0, ...) \]

[12] When S is a convex set alternative characterizations of T(S, x_o) are possible.

Proposition: (Yasaka (1967)) When S is convex
\[ \bigcup_{\lambda \rightarrow 0} T(S, x_o) = T(S, x_o) = \text{P}(S, x_o) \quad x_o \in S. \]

Proof: If \( h \in \bigcup_{\lambda \rightarrow 0} T(S, x_o) \), \( h = \lim_{\lambda \rightarrow 0} \lambda_n (x_n - x_o) \), \( \lambda_n \rightarrow 0 \), \( x_n \in S \).

The proof of [8] (5) shows that \( S - x_o \subseteq T(S, x_o) \) so that
\[ h_n = \lambda_n (x_n - x_o) \in T(S, x_o) \]. Since \( T(S, x_o) \) is a closed cone
\( h \in T(S, x_o) \). The converse containment is immediate. It is also clear that when S is convex \( T(S, x_o) \) is convex and thus equals
\[ \text{P}(S, x_o) \] (W) also.

[13] Proposition: If S is convex \( T(S, x_o) = w T(S, x_o) \).

Proof: By [12] \( T(S, x_o) = \bigcup_{\lambda \rightarrow 0} T(S, x_o) \) and
\[ w T(S, x_o) = \bigcup_{\lambda \rightarrow 0} (S - x_o) \text{ (the closure in the weak topology)}. \]

However, \( \bigcup_{\lambda} (S - x_o) \) is a convex set and has the same weak and initial closures (Taylor (1958)).

[14] The above proposition allows another characterization of pseudoconvexity. \[ A \] will be used to denote the convex hull of \( A \).
**Proposition:** A is pseudoconvex at \( x_0 \) if and only if

\[
T([A], x_0) = P(A, x_0).
\]

**Proof:** \( \Rightarrow \) If A is pseudoconvex at \( x_0 \) one has

\[
A - x_0 \subseteq P(A, x_0)
\]

and since \( P(A, x_0) \) is a closed convex cone

\[
T([A], x_0) = \bigcup_{\lambda > 0} (\lambda [A] - x_0) \subseteq P(A, x_0).
\]

Since \( A \subseteq [A] \) the other containment follows from

\[ P(A, x_0) \subseteq P([A], x_0) = T([A], x_0) \text{ by [12].} \]

\( \subseteq \) If \( T([A], x_0) = P(A, x_0) \)

\[ A - x_0 \subseteq [A], \quad -x_0 \subseteq T([A], x_0) \subseteq P(A, x_0) \text{ using [8] (5) and [12].} \]

[15] The previous results also give the bound

\[
P(A, x_0) \subseteq w P(A, x_0) \subseteq P([A], x_0)
\]

and

**Proposition:** If A is pseudoconvex at \( x_0 \) then \( w P(A, x_0) = P(A, x_0). \)

[16] The next theorem lists various translation properties of cones.

**Theorem:** (1) \( T(A, x_0) = T(A - x_0, 0). \)

Let \( g(x) = f(x + x_0) - f(x) \) then

(2) \( T(f(A), f(x_0)) = T(g(A - x_0), g(0)) \)

(3) \( T(g^{-1}(B), 0) = T(f^{-1}(B + f(x_0)), x_0). \)

**Proof:** These all follow by calculation from the definitions.

(W) also.

[17] **Proposition:** \( P(A, x_0) + P(B, y_0) \subseteq P(A + B, x_0 + y_0) \) with equality if A and B are pseudoconvex at \( x_0 \in A \) and \( y_0 \in B \) respectively.

**Proof:** \( P(A, x_0) = P(A + y_0, x_0 + y_0) \subseteq P(A + B, x_0 + y_0) \) if \( y_0 \in B. \)

\( P(B, y_0) = P(B + x_0, x_0 + y_0) \subseteq P(A + B, x_0 + y_0) \) if \( x_0 \in A. \)

Since \( P(A + B, x_0 + y_0) \) is a closed convex cone
\[ P(A, x_o) + P(B, y_o) \subset P(A + B, x_o + y_o). \]

If \( A \) is pseudoconvex at \( x_o \) and \( B \) is at \( y_o \) then
\[ [A] - x_o \subset P(A, x_o) \quad \text{and} \quad [B] - y_o \subset P(B, y_o). \]
Thus \( [A] + [B] - (x_o + y_o) \subset P(A, x_o) + P(B, y_o). \)

Since \( [A] + [B] \) is a convex set one derives
\[ P(A + B, x_o + y_o) \subset T([A] + [B], x_o + y_o) \subset \overline{P(A, x_o) + P(B, y_o)}. \]

\( (\dagger) \) also.

A more interesting and much more useful result is proved next.

It gives conditions under which \([6] (1)\) can be replaced by equality.

\[ [18] \quad \textbf{Theorem:} \quad \text{Suppose} \ A \text{ and} \ B \text{ satisfy the following conditions.} \]

1. \([A] \cap [B] = [A \cap B]. \]
2. \([A]^o \cap [B]^o \neq \phi. \quad \text{(or in fact} \quad \text{ri} (A) \cap \text{ri} (B) \neq \phi). \]
3. \( A \cap B \) is pseudoconvex at \( x_o. \)

Then \( P(A, x_o) \cap P(B, x_o) = P(A \cap B, x_o). \)

\textbf{Proof:} By \([6] (1)\) it suffices to show \( P(A \cap B, x_o) \supset P(A, x_o) \cap P(B, x_o). \)

Since \( A \subset [A], \ B \subset [B] \)
\[ P(A, x_o) \cap P(B, x_o) \subset P([A], x_o) \cap P([B], x_o). \]

By a theorem of Rockafellar's (1970a) which is proved in \( R^2 \)
but holds in any convex space
\[ P([A], x_o) \cap P([B], x_o) = P([A] \cap [B], x_o) \] when \((2)\) holds.

By \((1), \) therefore,
\[ P(A, x_o) \cap P(B, x_o) \subset P([A] \cap [B], x_o) = P(A \cap B, x_o) \]

and since \((3)\) holds \([14] \) shows that
\[ P(A, x_o) \cap P(B, x_o) \subset P([A \cap B], x_o) = P(A \cap B, x_o). \]

\( (\dagger) \) also.

\[ [19] \quad \text{A set} \ K \text{ is said to be polyhedrally convex if} \]
\[ K = \{ x \in X \mid x_1^+(x) \geq a_1 \quad i = 1, \ldots, n \quad x_1^+ \in X^+ \}. \]

The next result lists various properties of polyhedral sets in \( \mathbb{R}^n \). Many of these are valid more generally.

**Theorem:** (1) If \( K \) is polyhedrally convex and \( M \) is a subspace of \( \mathbb{R}^n \) then

\[ P(K, x_o) \cap M = P(K \cap M, x_o). \]

(2) If \( K \) is polyhedrally convex \( P(K, x_o) \) is polyhedral.

(3) The sums and duals of polyhedral sets are polyhedral.

(4) If \( B \) is polyhedrally convex and \( A \) is convex and \( \text{ri}(A) \cap B \neq 0 \) then \( P(A, x_o) \cap P(B, x_o) = P(A \cap B, x_o) \).

(5) If \( B \) and \( A \) are polyhedrally convex then

\[ P(A, x_o) \cap P(B, x_o) = P(A \cap B, x_o). \]

**Proof:** These results, in \( \mathbb{R}^n \), all follow from the definitions with the exception of (4) which is proved in Rockafellar (1970a).

---

[20] This paragraph gives various examples of tangent cones which in particular show reasons why the conditions in [17] and [18] are imposed.

(1) \( X = \mathbb{R}; \ A = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \cup \{ 0 \}, \ x_o = \frac{1}{2} \)

\[ B = (-\infty, \frac{1}{2}] \quad y_o = -\frac{1}{2}. \]

\( P(A, x_o) = 0 \) since \( \frac{1}{2} \) is an isolated point.

\( P(B, x_o) = (-\infty, 0] = P(A, x_o) + P(B, x_o) \) while

\[ P(A + B, x_o + y_o) \supset P(A + o, x_o + y_o) = P(A, o) = \mathbb{R}. \]

\( A \) is an example of a closed set which is pseudoconvex at 0 but is not starshaped there.

(2) \( X = \mathbb{R}; \ A \) is the rationals in \([0, 1]\); \( B \) is the irrationals in \([0, 1]\) plus \( \{ 0, 1 \} \), \( x_o = 0 \); then \( A, B \) fail only to satisfy [17] (3) and \( P(A, o) = P(B, o) = [0, \infty) \); \( P(A \cap B, o) = 0 \).
(3) $X = \mathbb{R}^2; A = \{ (x, y) \mid x^2 \leq y \}$, $B = \{ (x, y) \mid x > y, (x, y) = (0, 0) \}.
\begin{equation}
P(A, (x_0, y_0)) = \{(x, y) \mid y > 0 \} \cup \{(x, y) \mid x = 0 \text{ or } y = 0 \}
\end{equation}
\begin{equation}
P(B, (x_0, y_0)) = \{(x, y) \mid x > y \} \cup \{(x, y) \mid x = y \text{ or } x = -y \}
\end{equation}

\text{but } P(A \cap B, (x_0, y_0)) = (0, 0).$

In this case only [18](2) is violated.

(4) $X = \mathbb{R}^2; A = \{ (x, y) \mid x \geq 0, y \geq 0 \} \cup \{ (x, y) \mid x = 0 \text{ or } y = 0 \}
\begin{equation}
B = \{ (x, y) \mid x \geq 0, y \geq 0 \} \cup \{ (x, y) \mid x = y \text{ or } x = -y \}
\end{equation}

and $(x_0, y_0) = (0, 0)$.

$P(A, (x_0, y_0)) = P(B, (x_0, y_0)) = \mathbb{R}^2$.

while $P(A \cap B, (x_0, y_0)) = \{(x, y) \mid x \geq 0, y \geq 0 \}$.

In this case [18](1) is violated.

(5) $X = \mathbb{R}; A = C$, the Cantor set in $[0, 1]$, $x_0 \in C$.

Then $P(C, x_0)$ is either $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$ dependent on whether the ternary expression of $x_0$ contains finitely many twos or finitely many zeros or infinitely many of both.

\[21\] As a partial converse to [18] one has:

\textbf{Proposition:} If $P(A \cap B, x_0) = P(A, x_0) \cap P(B, x_0)$ and both $A$ and $B$ are pseudoconvex at $x_0$ so is $A \cap B$.

\textbf{Proof:} $(B \cap A) - x_0 = (B - x_0) \cap (A - x_0)$
\begin{equation}
\subseteq P(B, x_0) \cap P(A, x_0) \text{ (by pseudoconvexity)}
\end{equation}
\begin{equation}
\subseteq P(A \cap B, x_0) \text{ (by hypothesis)}
\end{equation}
and $B \cap A$ is pseudoconvex at $x_0$.

\[22\] An apparently open question is whether the condition that a closed set be pseudoconvex at all its points is equivalent to convexity. A partial answer to this is provided by the next propositions.

\textbf{Proposition:} Suppose $f: X \rightarrow \mathbb{R}$ is Fréchet differentiable then $E(p)$ is pseudoconvex at all its points exactly when $f$ is convex.
\textbf{Proof:} \Rightarrow The proof relies on the geometrically evident assertion that \( P(\text{Epif}, (x_o, f(x_o))) = \xi(x, y) \mid y \succ f'(x_o)(x) \) \( \xi \) which is straightforward but slightly tedious to prove and is thus omitted. Suppose that Epif is pseudoconvex at \((x_o, f(x_o))\) for all \( x_o \).

Then

\[
\text{Epif} \subset (x_o, f(x_o) + P(\text{Epif}, (x_o, f(x_o))))
\]
or

\[
(x - x_o, f(x) - f(x_o)) \in P(\text{Epif}, (x_o, f(x_o))) \quad \forall x \in X
\]

which, using the assertion above, shows that

\[
f(x) - f(x_o) \succ f'(x_o)(x - x_o) \quad \forall x, x_o \in X.
\]

This is equivalent, using [1.69], to convexity of \( f \).

\( \Leftarrow \) This follows immediately from [8] and [1.46].

[23] The next result links tangent cones directly with the results of chapter 1 for the first time.

\textbf{Theorem:} (1) If \( f \) is \((F)\) strictly quasiconvex and fully upper semicontinuous with respect to \( S \) then any weak minimum \( x_o \) for \( f \) over \( A \) is a weak minimum over \( x_o + T(A, x_o) \).

(2) If \( f \) is absolutely quasiconvex and upper semicontinuous with respect to \( S \) then any strong minimum \( x_o \) for \( f \) over \( A \) is a strong minimum over \( x_o + T(A, x_o) \).

\textbf{Proof:} Let \( y_o \in x_o + T(A, x_o) \). Then \( y_o = \lim y_n + x_o \).

\[
y_n = \lambda_n(x_n - x_o), \quad \lambda_n \geq 0 \quad x_n \in A \text{ and } x_n \to x_o.
\]

(1) Suppose \( f(y_o) - f(x_o) \in -S^o \).

By [1.25] \( f(x_o + y_n) \prec f(x_o) \) for \( n \) sufficiently large.

Also \( x_n = (1 - \lambda_n^{-1}) x_o + \lambda_n^{-1}(x_o + \lambda_n(x_n - x_o)) \) so that

\[
f(x_n) = f((1 - \lambda_n^{-1}) x_o + \lambda_n^{-1}(x_o + y_n)) < f(x_o)
\]
since \( f \) is assumed \((F)\) strictly quasiconvex and since

\[
f(x_o + y_n) < f(x_o).
\]

(Note that one may assume \( 0 < \lambda_n^{-1} < 1 \) for \( n \) sufficiently large.
since $x_0 \neq y_0$, $f(x_n) < f(x_0)$ contradicts the assumed minimality of $x_0$ over $A$ so that no such $y_0$ can exist and the result is true.

(2) The second case follows similarly but under the weaker continuity assumption of [1.24].

**Corollary:** If in (2) the hypothesis that $f$ is quasiconcave with respect to $S$ is added, $x_0 + T(A, x_0)$ can be used to replace $x_0 + T(A, x_0)$.

**Proof:** The quasiconcave version of [1.85] allows $A$ to be extended to [A].

These results extend, from continuous real valued convex mappings, a theorem of Norris (1971).

**Derivatives and tangent cones**

The notions previously discussed are now related to differential properties since it is in this form that they are of most use in optimization.

[24] **Theorem:** Let $f: X \rightarrow Y$ be compactly differentiable and suppose $x_0$ is a strong minimum for $f$ over $A$ with respect to a closed convex cone $S$ then

$$f'(x_0)(h) \in S \quad \forall h \in F(A, x_0).$$

**Proof:** Zlobec (1973) proves the result when $(Y, S) = (R, R^+)$. There is no difficulty in using the properties of cones and the definition of a strong minimum ([1.73]) to extend it.

[25] For a weak minimum the following theorem holds.

**Theorem:** Let $f: X \rightarrow Y$ be compactly differentiable at $x_0$ and
suppose $x_0$ is a weak minimum for $f$ over $A$ with respect to a closed convex cone with interior, then there is some $u^+ \in S^+ \setminus \{0\}$ with $u^+(f'(x_0)) \in P^+(A,x_0)$. 

**Proof:** Let $E = \{y \mid \exists h \in T(A,x_0) \text{ such that } (f'(x_0))(h) \leq y\}$ 

Suppose $h \in T(A,x_0)$ with $(f'(x_0))(h) < 0$. Now 

$$h = \lim_{n} \lambda_n (x_n - x_0), \quad x_n \in A, x_n \to x_0, \quad \lambda_n > 0.$$ 

Let $h_n = \lambda_n (x_n - x_0)$ 

$$f'(x_0)(h) = \lim_{n} \lambda_n \left[f(x_0 + \lambda_n^{-1} h_n) - f(x_0)\right]$$ 

since $f$ is compactly differentiable. Hence for $n > n_0$ 

$$\lambda_n \left[f(x_0 + \lambda_n^{-1} h_n) - f(x_0)\right] < 0$$ 

and since $-S^0$ is a cone 

$$f'(x_0) = f(x_0 + \lambda_n^{-1} h_n) < f(x_0).$$ 

This contradicts the minimality of $x_0$ as $x_n \in A$. Thus $E \cap -S^0 = \emptyset$ 

$E$ is clearly a convex set and the Hahn-Banach theorem implies the existence of a non-zero linear functional with 

$$u^+(y) > 0 \quad \forall y \in E; \quad u^+(s) < 0 \quad \forall s \in -S.$$ 

So 

$$0 \neq u^+ \in S^+ \quad \text{and} \quad u^+(f'(x_0))(h) > 0 \quad \forall h \in T(A,x_0).$$ 

By continuity and linearity of $f'(x_0)$ 

$$u^+(f'(x_0)) \in P^+(A,x_0).$$ 

While, as Zlocic, remarks the compact derivative is the natural derivative to use when dealing with tangent cones it is the bounded derivative which plays the comparable role for weak tangent cones.

26] **Theorem:** If $f$ is taken to be boundedly differentiable the results of [24] and [25] remain true with $WP(A,x_0)$ replacing $P(A,x_0)$. In [25] it is necessary for $f'(x_0)$ to be completely continuous. (See the remarks on complete continuity at the end of the chapter). $X$ is assumed to be sequential.
Proof: To mirror the previous proofs it is only necessary to observe that when $h_n = \lambda_n (x_n - x_0), x_n \to x_0, \lambda_n \to \lambda, x_n \in A$ and $h_n \to h_0$ one can assume that $\{h_n\}$ is weakly bounded since $X$ is sequential. Since the same sets are bounded in any topology of the dual pair (Robertson and Robertson (1964)) it is bounded. The definition of the bounded derivative enables one to assert that

$$\lambda_n [f(x_0 + \lambda_n^{-1} h_n) - f(x_0)] \to f'(x_0)(h)$$

and the proofs proceed as before.

To see that [26] is a genuine sharpening of [24] it is only necessary to consider the example in [11]. With $A$ and $x_0$ as in [11] and $f: \ell_2 \to R$, Zlobec's theorem says that $f'(x_0) \in P(A,x_0) = \ell_2$ while [26] requires $f'(x_0)$ to have first coordinate non negative. Note that since $\ell_2$ is a Banach space the bounded derivative is in fact Fréchet.

[27] Guignard's (1969) sufficiency condition also has extensions to weak cones and $(Y,S)$.

Theorem: Let $f: X \to Y$ be $\beta$ - differentiable at $x_0$. Suppose that

1. $f'(x_0)(h) \in S \quad \forall h \in P(A,x_0)$, 2. $A$ is pseudoconvex at $x_0$, 3. $f$ is pseudoconvex at $x_0$ with respect to $S$; then $x_0$ is a strong minimum for $f$ over $A$.

Proof: Since $A$ is pseudoconvex at $x_0$, $x - x_0 \in P(A,x_0) \quad \forall x \in A$.

Thus $f'(x_0)(x - x_0) \in S \quad \forall x \in A$, and since $f$ is supposed pseudoconvex at $x_0$

$$f(x) \succeq f(x_0) \quad \forall x \in A.$$

[28] Theorem: Suppose in [27] that (1) becomes (1)' $f'(x_0)(h) \in S$ \quad $\forall h \in wP(A,x_0)$ and (2) becomes (2)' $A$ is weakly pseudoconvex at
\( x_0 \) then the result still holds.

**Proof:** The proof is independent of the nature of the cones.

[29] A sufficiency condition of a sort can be proven for weak minima. The result guarantees a real valued equivalent problem.

**Theorem:** Suppose that for some \( u^+ \in S^+ / \{ o \} \)

\[
u^+(f^\prime(x_0)(h)) \geq o \quad \forall h \in P(A,x_0)
\]

and that \( u^+ f \) is pseudoconvex at \( x_0 \) and \( A \) is pseudoconvex at \( x_0 \) then

\[
\forall x \in A.
\]

**Proof:** This is contained in [27].

The analogous result for weak cones is contained in [28].

The pseudoconvexity of \( u^+ f \) is not implied by the pseudoconvexity of \( f \) with respect to \( S \) it is however implied by the convexity of \( f \) with respect to \( S \) ([1.42], [1.43]).

[30] That [29] actually adds to ones knowledge can be seen from the following example of a function which is quasiconvex with respect to the orthant in \( \mathbb{R}^2 \) but has no equivalent real valued map.

**Example:** Let \( f: \mathbb{R} \rightarrow \mathbb{R}^2 \), with \( S \) the coordinate cone, be given by

\[
f(x) = (x^3 + 1, -x).
\]

\( f \) is quasiconvex and every \( x \) is a weak minimum for \( f \) over \( S \) since

\[
f(x) \leq f(y) \implies x = y.
\]

Let \( u^+ = (r_1, r_2) \in S \), that is \( r_1 \geq 0, r_2 \geq 0, (r_1, r_2) \neq 0 \).

Then

\[
u^+ f(x) = r_2x^3 - r_2x + r_1
\]

which is unbounded on \( \mathbb{R} \). For any \( x_0 \), therefore, there is some \( x \) with \( u^+ f(x) < u^+ f(x_0) \) and \( f \) has no real valued equivalent map.
In keeping with the definition of Zlobec (1973), the set
(i) \( \mathcal{E} = \left\{ E \mid E \in B(X, Y) \ : \ E(F(g^{-1}(B), x_o)) \subset F(B, g(x_o)) \right\} \)

is called the **local cone** (of derivatives) where \( g : X \to Y \).

(ii) \( w\mathcal{E} = \left\{ E \mid E \in B(X, Y) \ : \ E(wF(g^{-1}(B), x_o)) \subset wF(B, g(x_o)) \right\} \)

will correspondingly be called the **weak local cone**.

Zlobec introduced the terminology of [31](i) to allow the formulation of optimality conditions when \( g \) is not differentiable.

Massam (1973) has shown that \( \mathcal{E} \) is a closed convex cone in the topology of convergence on bounded sets. It is clear that this holds for \( w\mathcal{E} \). More information on cone containments can be found in Ritter (1969a, b).

**Proposition: [Zlobec (1973)]** If \( g \) is compactly differentiable at \( x_o \) then \( g'(x_o) \in \mathcal{E} \).

**Proof:** The proof is similar to that of the next proposition.

**Proposition:** Let \( g : X \to Y \) be boundedly differentiable at \( x_o \) and let \( X \) be sequential; then \( g'(x_o) \in w\mathcal{E} \).

**Proof:** Set \( \varepsilon > 0 \). Let \( h \in wT(g^{-1}(B), x_o) \). Then

\[
\lambda_n(x_n - x_o), g(x_n) \in B, x_n \to x_o, \lambda_n \to 0 \text{ and } h_n \to h_o.
\]

Let \( u_1^+, \ldots, u_k^+ \in X^- \) and let \( N \) be any neighbourhood in \( X \) with \( u_i^+(N) \subset \varepsilon \) for \( i = 1, \ldots, k \). Because \( h_n \to h_o \) in a sequential space one can assume that \( \{h_n\} \) is in fact a sequence (relabelling if necessary). Then \( \{h_n\} \) is a bounded set because it weakly bounded. By the definition of the bounded derivative

\[
\lambda_n[g(x_o + \lambda_n h_n) - g(x_o)] - g'(x_o)(h_n) \to 0 \text{ as in [26].}
\]

Note that \( \lambda_n^{-1} \) can be assumed convergent to zero by choosing a subsequence if necessary since otherwise \( h = 0 \). From (1) one has for \( n \geq n_1 \).
(2) \( \frac{g(x_n) - g(x_0)}{h_n} - g'(x_0)(h_n) \in N \).

By the definition of differentiation \( g' \) is a continuous map and is, therefore, \( \sigma(X, Y') \) - \( \sigma(Y, Y') \) continuous (Robertson and Robertson (1964)). There is then some \( n_2 \) with

(3) \( \left| u_i^{+}(g'(x_0))(h_n) - u_i^{+}(g'(x_0))(h) \right| < \varepsilon \quad 1 = 1, 2, \ldots, k. \)

for \( n > n_2 \). From (2), (3) and the choice of \( N \)

(4) \( \left| u_i^{+} \left[ \frac{g(x_n) - g(x_0)}{h_n} - (g'(x_0))(h) \right] \right| < 2\varepsilon \)

when \( n \geq \max(n_1, n_2, n_3) \) and (4) implies the weak convergence of

\( \frac{g(x_n) - g(x_0)}{h_n} \) to \( (g'(x_0))(h) \).

Since \( g \) is continuous and \( x_n \rightarrow x_0 \), \( g(x_n) \rightarrow g(x_0) \) and \( g'(x_0)(h) \in wT(E, g(x_0)) \).

The linearity and continuity of \( g'(x_0) \) suffice to derive

\( g'(x_0)(wT(g^{-1}(B), x_0)) \subseteq wT(E, g(x_0)). \)

If the continuity assumptions on \( g \) are strengthened the conclusion can be improved.

[34] **Definition:** \( f : X \rightarrow Y \) is said to be **completely continuous** if it
is continuous from the weak topology on \( X \) to the given topology on \( Y \).

[35] **Proposition.** If \( g'(x_0) \) is completely continuous then the
conclusion of [33] can be strengthened to

\( (g'(x_0))(wT(g^{-1}(B), x_0)) \subseteq P(B, g(x_0)). \)

**Proof:** Examining the proof of [33] one sees that (3) can be
replaced by (3) \( (g'(x_0))(h_n) - (g'(x_0))(h) \in N \) for \( n > n_4 \)
where \( N \) is now any neighbourhood. One then combines (3)' and
(2) and proceeds as in the proposition.

Since \( g'(x_0) \) is automatically continuous it suffices in
addition to require that \( g'(x_0) \) map bounded sets into compact
sets. In normed spaces this last condition coincides with compactness of $g'(x_0)$. More generally compactness is stronger. Nashed (1971) has that a sufficient condition in a normed space for $g'(x_0)$ to be completely continuous is the complete continuity of $g$ itself.

[36] For a mapping $E \in \mathcal{E}$, Zlobec (1973) has defined the set

$$K(E) = \{ h | E(h) \in P(B,g(x_0)) \}$$

For a mapping $E \in w\mathcal{E}$, $wK(E)$ is defined analogously by

$$wK(E) = \{ h | E(h) \in wP(B,g(x_0)) \}$$

The following theorems give some conditions for $P(g^{-1}(E),x_0)$ to equal $K = K(g'(x_0))$. Clearly, by [32] and [33], this reduces to showing that

$$g'(x_0)^{-1} \left[ P(B,g(x_0)) \right] \subseteq P(g^{-1}(E),x_0).$$

The results are all framed in normed spaces since they all rely at some level on the implicit function theorem (Luenberger (1969)) or on similar results.

[37] Theorem: (Halkin (1972b)) Let $X$ and $Y$ be Banach spaces and $T:X$ a closed subspace. Suppose

1. $f$ is continuously differentiable in a neighbourhood $U$ of $x_0$.
2. $f'(x_0)$ is a bijection of $T$ onto $X$.

Then there is some neighbourhood $N$ of $x_0$ with a continuously differentiable mapping $j$ of $N$ into $X$ with

1. $f(x + j(x)) = f(x_0) + f'(x_0)(x-x_0)$ $\forall x \in N$
2. $\lim_{p \to 0} \sup_{x : \|x-x_0\| < p} \|j(x)\| = 0$. 

[38] Theorem: (Halkin (1972a)) Let $X$ be normed and $Y$ finite dimensional.
Suppose

(1) $f$ is differentiable in a neighbourhood $U$ of $x_0$.

(2) $f'(x_0)$ is surjective.

Then the conclusions of [37] hold except that $j$ need not be continuously differentiable.

These two results can be used to give sufficient conditions for $K = P(g^{-1}(B), x_0)$ when $B$ is a closed subspace.

[39] Proposition: Suppose $g: X \rightarrow Y$ satisfies the hypotheses of [37] or [38] and that $B \subset Y$ is a closed subspace. Then

$$K = P(g^{-1}(B), x_0) = T(g^{-1}(B), x_0).$$

Proof: Suppose $g'(x_0)(h) \in P(B, g(x_0)) = B$. Set $x_0 = o$.

For $n > n_0$ \( \frac{h}{n} \in N \) and thus by (i)

$$g(x_0 + \frac{h}{n} + j(\frac{h}{n})) = \frac{1}{n}g'(x_0)(h) + g(x_0) \in B.$$

Thus

$$x_0 + \frac{h}{n} + j(\frac{h}{n}) \in g^{-1}(B) \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{h}{n} - j(\frac{h}{n}) \in T(g^{-1}(B), x_0).$$

Using (ii) this limit is just $h$. The theorem now follows from the fact that $K$ always contains $T(g^{-1}(B), x_0)$.

Flett (1966) has proved [39] by a rather complicated implicit function theorem only using $g'(x_0)$ is onto $Y$. He does not need $T$ to exist in [37].

[40] Definition: When $K = P(g^{-1}(B), x_0)$ $g$ will be said to be regular at $x_0$. If $\text{wP}(g^{-1}(B), x_0) = wK$ $g$ will be called weakly regular at $x_0$.

These definitions generalize the standard notion of regularity.
The subject will be discussed again later.

Remarks on complete continuity of derivatives ([26], [34])

The complete continuity of \( f'(x_o) \) is necessary in any argument such as a separation argument in which one wishes to deduce from \( h \in \text{wP}(A,x_o) \) and

\[
h_n = \lambda_n (x_n - x_o) \to h_o
\]

that \( (f'(x_o))(h_o) \in \mathcal{S}^o \) implies \( f(x_n) - f(x_o) \in \mathcal{S}^o \), \( n > n_o \). This relies on the fact that

\[
\lambda_n \left[ f(x_n) - f(x_o) \right] - (f'(x_o))(h_n) \to 0
\]

which in conjunction with \( h_n \to h_o \) and \( f'(x_o) \) completely continuous gives \( \lambda_n \left[ f(x_n) - f(x_o) \right] \to (f'(x_o))(h_o) \in \mathcal{S}^o \). Then \( n > n_o \) implies, since \( \mathcal{S}^o \) is open in the topology in which one has convergence, that the implication (1) holds.

Alternatively one can define minimization in the weak topology by requiring that \( S \) has an interior \( S^{wo} \) in the weak topology, and by defining \( f \) to have a weak minimum in the weak topology at \( x_o \) over \( A \) if

\[
f(x) - f(x_o) \notin -S^{wo} \text{ when } x \in A.
\]

With this notion complete continuity is unnecessary since now \( \lambda_n \left[ f(x_n) - f(x_o) \right] \to f'(x_o)(h_o) \in -S^{wo} \) implies that \( f(x_n) - f(x_o) \in -S^{wo} \) for \( n > n_o \) as the cone now has interior in the topology in which \( \lambda_n \left[ f(x_n) - f(x_o) \right] \) converges.

These considerations lead to the following general proposition.

**Proposition:** If \( S \) is supposed to have weak interior then any separation argument which holds for (strong) tangent cones and
compact derivatives remains valid for weak tangent cones and bounded derivatives without complete continuity: provided that $x_0$ is required to be a minimum in the weak topology.

It must be emphasised that most applications of weak tangent cones will rely on [26] and [33] which require $X$ to be sequential. For this reason any bounded derivative will from now on be assumed to have a sequential domain space and the hypothesis will not be listed separately.
Chapter Three

FARKAS LEMMAS AND TRANSPOSITION THEOREMS
Farkas Lemmas and Transposition Theorems.

Two of the most useful optimization methods (other than direct arguments which usually involve the Hahn-Banach theorem) are the application of Farkas Lemmas and Transposition or Alternative Theorems. This chapter proves various results of this nature some of which will be applied subsequently. The following definitions are needed. They are taken from Berge (1959) which is a good general reference text for multivalued maps. Recall that \( F : X \to Y \) is multivalued if \( F \) maps points in \( X \) onto subsets of \( Y \).

\[1\] Definitions: (1) \( F : X \to Y \) is said to be upper semicontinuous as a multivalued mapping between topological spaces if for any neighbourhood \( V \subseteq Y \) with \( F(x_0) \cap V \) there is a neighbourhood \( U \subseteq X \) of \( x_0 \) with \( F(x) \cap V \) when \( x \in U \). (2) \( F \) is called lower semicontinuous as a multivalued mapping if for any neighbourhood \( V \subseteq Y \) with \( F(x_0) \cap V \neq \emptyset \) there is a neighbourhood \( U \subseteq X \) of \( x_0 \) with \( F(x) \cap V \neq \emptyset \) when \( x \in U \).

It is apparent that these concepts coincide when \( f \) is single valued.

\[2\] Theorem: Let \( f : X \to Y \) \( g : X \to Z \) be arbitrary (single valued) mappings. Let \( U \subseteq Y, V \subseteq Z \) be any sets such that \( U \subseteq R(f) \).

Then the following are equivalent:

1. \( f(x) \subseteq U \Rightarrow g(x) \subseteq V \)
2. \( g(x) \subseteq h \circ \; f \; (x) \quad \{h(U)\} \subseteq V \) where \( h \) is a multivalued mapping of \( Y \) into \( Z \).
Proof: \( \Rightarrow \) Let \( h(u) = \{ g(x) \mid f(x) = u \} \).

In multivalued terms \( h = g f^{-1} \). Suppose \( u \in U \) then, since \( U \cap \text{Cov}(f) \), \( u = f(x) \) for some \( x \) and \( x \in f^{-1}(u) \).

\[ h(u) = g(f^{-1}(u)) \subseteq V \text{ by (1)}. \]

It is clear that \( g(x) \in h \circ f \) (x) and that \( (2) \Rightarrow (1) \). \]

[3] Proposition: \( h \) is singlevalued if and only if \( f(x) = f(y) \)
implies \( g(x) = g(y) \).

Proof: \( h(f(x)) = \{ g(y) \mid f(x) = f(y) \} \).

[4] Theorem: (1) If \( g \) is continuous and \( f^{-1} \) is upper (lower) semicontinuous \( h \) is upper (lower) semicontinuous.

(2) If \( g \) is convex with respect to \( S \) and \( f \) is linear then \( h \) is multivalued convex with respect to \( S(1.86) \)

(3) If \( g \) and \( f \) are linear \( h \) satisfies \( th(x) + h(y) \in h(t \cdot x + y) \) for \( t \in R, x, y \in Y \). (This property is called multivalued linearity).

Proof: (1) Follows by a result of Berge (1969) or directly.

(2), (3) follow from the observation that the inverse of a linear map is multivalued linear and from \( h = g f^{-1} \).

It is a property of multivalued upper semicontinuous mappings \( F \) that \( \{ F(x) \} \) is compact. (Berge (1959)). It is apparent that the previous results could have been rephrased for \( f \) and \( g \) multivalued. In this case the continuity conditions in \([4](1)\)

would ask only for \( g \) to be semicontinuous.

Craven (1972) has proved that \( h \) is continuous when \( g \) is continuous and \( f \) is continuous, open and surjective. He
was only considering maps satisfying \([5]\). The next proposition shows that even in this case \([4]\) (1) generalises his result.

\([5]\) **Proposition:** If \(f : X \to Y\) is an open map then \(f^{-1}\) is lower semicontinuous as a multivalued map from \(Y\) into \(X\).

**Proof:** Suppose there is a neighbourhood \(V \subset X\) such that
\[
f^{-1}(y_0) \cap V \neq \emptyset.
\]
Let \(U = f(V)\). \(U\) is open in \(Y\) since \(f\) is open. Also, since \(f^{-1}(y_0) \cap V \neq \emptyset\), there is some \(x_1 \in V\) with \(f(x_1) = y_0\) and \(y_0 \in U\). If \(y \in U\) then \(y = f(x)\) for some \(x \in V\) since \(U = f(V)\).

For this \(x\) one has
\[x \in f^{-1}(y) \cap V.
\]
Thus for any \(V\) there is a neighbourhood \(U\) of \(y_0\) such that \(f^{-1}(y) \cap V \neq \emptyset\) if \(y \in U\). This is just definition \([1]\) (2).

Combining \([5]\) and \([4]\) (1) one has that \(h\) is lower semicontinuous when \(f\) is open and \(g\) is continuous. This in turn implies that \(h\) is continuous when it is singlevalued and everywhere defined. This does not need Craven's hypothesis that \(f\) is continuous.

As a corollary one has the following linear Farkas lemma.

\([6]\) **Theorem:** Let \(X, Y, Z\) be convex spaces with \(X\) fully complete and \(Y\) separated and barrelled. Let \(S \subset Y\) and \(Q \subset Z\) be cones with \(Q\) pointed. Suppose \(A : X \to Y\) and \(B : X \to Y\) are continuous linear mappings with \(R(A) = Y\). The following are equivalent:

1. \(Ax \in S \Rightarrow Bx \in Q\)
2. \(B = T^*A\) for some continuous linear \(T : Y \to Z\) with \(T(S) \subset Q\).

**Proof:** \(Ax = Ay + A(x - y) = 0 \in S \cap -S\). By (1) \(B(y - x) \in Q \cap -Q = \{0\}\).
Thus the condition of \([3]\) is satisfied and by \([2]\) there is a single valued map \(T\) with \(B = T \cdot A\) and \(T(S) \subset Q\). \([4]\) (3) shows that \(T\) is linear. Finally \(T\) is continuous. This is true because \(A\) is open (the hypothesis of the open mapping theorem are satisfied) and one may apply the preceding remark. \(\Leftarrow\)

This is immediate.

With \(R(A)/S\) replacing \(S\) the theorem can be proven for \(R(A)\) a fully complete subspace since \(A\) will be open onto \(R(A)\). It is also worth noting that if \(A\) is assumed open the theorem holds in general (convex) spaces.

\(\[7\] \)

If \(R(A)\) is not assumed closed there appears to be no satisfactory continuity result in the literature. The best one might hope for is the equivalence of \([6]\) (1) with (2)'

\[B = \lim_{n \to \infty} T_n A, \quad T_n(S) \subset Q, \quad T_n \text{ continuous.}\]

Since the only results I have obtained in this direction are restricted and weaker, they are not included herein. Such results would be useful in optimization.

\(\[8\] \)

The next result gives a convex extension to \([6]\).

**Theorem:** Let \(f : X \to Y\) and \(g : X \to Z\) be differentiable at a point \(x_0\) with \(R(f'(x_0)) = Y\). Suppose further that \(f\) is convex with respect to \(S\) and \(g\) is convex with respect to \(Q\) with \(Q\) pointed. Suppose that \(X\) is fully complete and \(Y\) is barrelled, if

\[ (1) \quad f(x) \geq_s f(x_0) \Rightarrow g(x) \leq_q g(x_0) \]

there is some continuous \(T\) mapping \(Y\) into \(Z\) with \(T(S) \subset Q\) and such that

\[ (2) \quad g(x) + T f(x) \geq_q g(x_0) + T f(x_0). \]
Proof: Since $f$ and $g$ are differentiable and convex

\begin{align*}
(3) \quad f(x) - f(x_0) &\geq g^\prime(x_0)(x-x_0) \\
(4) \quad g(x) - g(x_0) &\geq q g^\prime(x_0)(x-x_0) \quad \text{by } [1.69].
\end{align*}

Using (1), (3) and (4) one sees that

$$f^\prime(x_0)(x-x_0) \in S \text{ implies } g^\prime(x_0)(x-x_0) \in -Q.$$ 

Since $R(f^\prime(x_0)) = Y$ the Farkas Lemma, [6], can be applied to $f^\prime(x_0)$ and $g^\prime(x_0)$ giving $T: Y \to Z$, $T$ continuous with $T(S) \subset Q$ and

\begin{align*}
(5) \quad T f^\prime(x_0) + g^\prime(x_0) &= 0.
\end{align*}

Using (3) and $T(S) \subset Q$ one sees that

\begin{align*}
(7) \quad T(f^\prime(x_0)(x-x_0)) &\leq q T(f(x) - f(x_0)).
\end{align*}

Combining (7) with (4) and (5) one has

$$g(x) - g(x_0) \geq q g^\prime(x_0)(x-x_0) = -T f^\prime(x_0)(x-x_0) \geq q - T(f(x) - f(x_0)) \text{ as desired.}$$

If $f$ and $g$ are linear this reduces to [6] since continuous linear mappings are their own derivatives. Setting $x_0 = 0$

$$g(x) + T f(x) \in Q \setminus -Q = \{0\} \quad \text{as before.}$$

[9] If one wishes to extend the result of [6] to

$$A_1 x \in S_1, A_2 x \in S_2, \ldots, A_n x \in S_n \Rightarrow E x \in Q$$

it is necessary to impose further restrictions on the cones. Ritter (1969,b) has proved the following result which is quoted for future use.

Theorem: Suppose $A_i : X \to Y_i$, $i = 1, \ldots, n$, $B : X \to Z$ are continuous linear mappings between Banach spaces and that $Z$ is reflexive. Suppose also that
(1) \exists \bar{x} \Rightarrow A_i \bar{x} \in S_i^0 \ i = 1, \ldots, n

(2) Q \subset Z is a closed normal cone

then the following are equivalent:

(3) A_i x \in S_i, \ldots, A_n x \in S_n \Rightarrow B x \in Q

(4) There are continuous linear mappings \( T_i : X_i \rightarrow Z \ i = 1, \ldots, n \) with \( T_i (S_i) \subset Q \) and \( B = T_1 A_1 + \ldots + T_n A_n \).

Craven (1972) seems to use this result without proving it or imposing conditions (1) and (2), in that he tries to deduce the result directly from the basic lemma by looking at \( A = (A_1, \ldots, A_n) \).

**Transposition Theorems for convex and linear mappings**

The first result of this section generalises a Transposition theorem proved by Craven and Mond (1973) to multivalued convex functions.

[10] **Theorem:** Let \( X, Y, Z \) be convex spaces. Let \( F : X \rightarrow Y \) be a multivalued convex function with respect to \( S \), a closed convex cone with interior. Let \( h : X \rightarrow Z \) be affine and open and suppose that \( F \) is multivalued lower semicontinuous on \( C \subset X \), a convex set with interior. If there is no solution to

\[
(1) \ h(x) = 0, \ x \in C \text{ and } F(x) \cap -S^0 \neq \emptyset
\]

then

\[
(2) \text{ there is some } p^+ \in S^+, \ q^+ \in Z; \text{ with } (p^+, q^+) \neq 0 \text{ and } p^+ (F(x)) + q^+ (h(x)) \geq o \ \forall x \in C.
\]
Proof: Let \( A = \{ (w, z) \mid \exists x \in C, \ h(x) = z \text{ and } \exists y \in F(x) \ni w > y \} \)

Then (i) \( A \) is convex since \( C \) is convex, \( h \) is affine and \( F \) is multivalued convex.

(ii) Let \( s \in S^0 \). There is some balanced neighbourhood \( N_0 \) with \( s + N_0 \subseteq S^0 \). Let \( x_0 \in C \) and \( z_0 = h(x_0) \) then there is some \( y_0 \in S^0 \) with \( a = y_0 - y_1 \in S^0 \) where \( y_1 \) is any given point in \( F(x_0) \). (This could be done for any \( y_1 \)).

Choose a balanced open set \( N_1 \) with \( a + 2 N_1 \subseteq S^0 \) and another open set \( N_2 \subseteq X \) with \( x_0 + N_2 \subseteq C \). Since \( F \) is lower semicontinuous and \( F(x_0) \cap (N_1 + y_1) \neq \emptyset \) there is some open \( N \subseteq N_2 \) with

\[
F(x) \cap (N_1 + y_1) \neq \emptyset
\]

when \( x \in N \).

Choose \( y \in F(x) \) and \( y \in N_1 + y_1 \) for any \( x \in N + x_0 \).

Let \( y \in N_1 + y_0 \). Then

\[
y - y = (y - y_0) + (y_1 - y) + a \in 2 N_1 + a \subseteq S^0.
\]

Moreover, since \( h \) is affine and open, \( h(N + x_0) \) is open and if \( (y, z) \in (N_1 + y_0, h(N + x_0)) \) one has

\[
z = h(x), \ x \in C \text{ and } y > y \in F(x).
\]

Thus, \( (y, z) \in A \) which implies that \( (y_0, h(x_0)) \in A^0 \).

(iii) Let \( M = \{ (w, o) \mid w \in S \} \)

\( M \) is clearly convex. If there is no solution to (1) then \( M \cap A = \emptyset \).

An application of the Hahn Banach theorem allows one to assert the existence of a linear functional \( a^+ = (p^+, q^+) \ni o \) with

\[
p^+(w) + q^+(z) \geq o \text{ if } (w, z) \in A.
\]
If one lets \( y_n = y + \frac{1}{n} s \) for a fixed \( s \in S^0 \) and any \( y \in P(x) \), \( x \in C \); one has \( (y_n, x) \in A \) and

\[
(iv) \quad p^+(y_n) + q^+(h(x)) \geq 0 \quad \forall x \in C.
\]

Taking limits as \( n \to \infty \)

\[
p^+(y) + q^+(h(x)) \geq 0 \quad \forall x \in C, \ y \in P(x).
\]

Further \( p^+ \in S^+ \), since if \( y \in S^0 \) (ty, h(x)) \( \in A \) when \( t \) is sufficiently large (as in (ii)) and if \( p^+(y) < 0 \) (iv) could not hold.

By [1.53] the continuity assumptions are fulfilled when \( S \) is a normal cone and \( P \) is single valued and continuous at some point of \( C \subset (domf)^0 \). A further generalisation is given below.

[11] **Theorem:** Let \( B \) be a closed convex cone in \( Z \) and suppose

the hypotheses of [10] hold. Then:

1. There is a solution to \( P(x) \cap -S^0 \neq \emptyset \), \( h(x) \in -B \), \( x \in C \).
2. There are \( p^+ \in S^+ \) \( q^+ \in Z^+ \), not both zero, with

\[
p^+(P(x)) + q^+(h(x)) \geq 0 \quad \forall x \in C.
\]

**Proof:** Let \( \hat{A} = \{(w, z) \mid x \in C, z - h(x) \in B, w - y \in S^0 \text{ for some } y \in P(x)\} \). Since \( A \) of [10] is a subset of \( \hat{A} \), \( \hat{A} \) has interior.

\( \hat{A} \) is convex in much the same way as \( A \) was.

Let \( \hat{M} = \{(w, z) \mid w \in -S, z \in -B\} \).

Suppose (1) has no solution then \( \hat{A} \cap \hat{M} = \emptyset \). Again, using the Hahn Banach theorem, there are \( p^+, q^+ \) not both zero with

\[
(3) \quad p^+(w) + q^+(z) \geq p^+(s) + q^+(b)
\]

when \( (w, z) \in A \) and \( s \in -S \) \( b \in -B \). In particular, as before,

\[
p^+(y) + q^+(h(x)) \geq 0 \quad \text{if } x \in C \text{ and } y \in P(x).
\]

\( p^+ \in S^+ \) and \( q^+ \in B^+ \) or (3) is impossible.

This includes [10] as the case \( B = 0 \). The next two results are corollaries to [11].

1. There is some $p^+ \neq 0 \in S^+$, $q^+ \in B^+$ with
   
   $$p^+(F(x)) + q^+(h(x)) \geq 0 \quad \forall x \in X$$
   
   or

2. There is a solution $x_o$ to
   
   $$h(x) \in -B \quad F(x) \cap -S^0 \neq \emptyset.$$  

Proof: $\Leftarrow$ If (2) has no solution there is by [11] a solution to (1). It remains to verify that $p^+ \neq 0$. If $p^+ = 0$ then

$$q^+(h(x)) > 0 \quad \forall x \in X.$$  

Since $h$ is open $R(h) = Y$ and $q^+ = 0$. This forces $p^+$ to be non zero which is a contradiction.

$\Rightarrow$ Conversely if $x_o$ solves (2) and $p^+ \in S^+$, $q^+ \in B^+$ there is some $y_o \in F(x_o) \cap -S^0$ and $h(x_o) \in -B$. Then

$$p^+(y_o) + q^+(h(x_o)) \leq p^+(y_o) < 0$$

and there is no solution to (1).

[13] Corollary: Suppose $C$ is a closed convex cone with interior and that $f = Ax + d$ is continuous and affine with $d \in -S$.

Suppose that $B = 0$, $h$ is linear and there is $x_1$ with

$$f(x_1) \in -S^0.$$  

Then, when [11] (1) has no solution, $p^+$ and $q^+$ of [11] (2) are such that $p^+(d) = 0$, and $p^+ \neq 0$ implies $q^+ \neq 0$.

Proof: Setting $x_o = 0 \in C$ and using

$$p^+(Ax + d) + q^+(h(x)) \geq 0 \quad x \in C$$

one has $p^+(d) \geq 0$. This with $p^+ \in S^+$, $d \in -S$ gives $p^+(d) = 0$.

Since $f(x_1) < 0$, if $p^+ \neq 0$ one has $q^+(h(x_1)) > -p^+(f(x_1)) > 0$.

and $q^+ \neq 0$.  

Remarks: (1) This last result is very much like Lemma 2.3 of Ritter (1969,b). Again there is no reason why $A$ should not be multivalued linear [4] with the condition $f(x_1) \cap S^0 = b$.

(2) These results ([10] - [13]) extend many in Mangasarian (1969).

(3) Craven and Mond (1973) prove [12] with $B = 0$ and $F$ single valued. They invoke an extra, and redundant hypothesis that $h$ have an injective transpose $h^*$. This is implied by the hypothesis that $h$ is open.

Proof: Suppose $h^*(y') = 0 \quad y' \in Y'$.

Since $h^*(y')(x) = h(x)(y') \quad \forall x \in X$

one sees that $h(x)(y') = 0 \quad \forall x \in X$. In any topological vector spaces $h$ open implies $R(h) = Y$ so that $h(x)(y') = 0$ for all $x \in X$ means that $y' \in Y^+ = 0$.

Linear transposition theorems

The convex transposition theorems of the previous section suffer from the fact that when linear problems such as

$Ax > 0, \quad Bx > 0, \quad Cx = 0$

have no solution one can deduce that

$a^+ A + b^+ B + c^+ C = 0 \quad a^+ > 0 \quad b^+ > 0$

but not $a^+ \neq 0$.

The following general linear alternative theorem deals with this problem and is used to derive as corollaries most of Mangasarian's (1969) alternative theorems in a more general setting. The next paragraphs are a necessary preliminary. Dual spaces will from now on always be supposed to be endowed with the strong topology $R(X',X)$ unless it is stated otherwise.
[15] Proposition: (1) If $C$ and $D$ are closed convex cones in a locally convex space $X$
\[
(C \cap D)^+ = C^+ + D^+.
\]
(2) If in addition $X$ is normed and $C \cap D \neq \emptyset$
\[
(C \cap D)^+ = C^+ + D^+.
\]
Proof: This is essentially proved in Ritter (1969,a).
Another condition under which $(C \cap D)^+ = C^+ + D^+$ is given by the following result.

[16] Definition: A set $C \subseteq X$ is said to be radial to $x_0$ ($x_0 \in \text{rad } C$) if for each $x \in X$ there is some $t_0 > 0$ such that $(1-t)x_0 + tx \in C$ when $0 < t < t_0$. A convex set is radial at any interior point.

[17] Proposition: Let $X$ be a barrelled, (sequential), Hausdorff, convex space. Let $C, D \subseteq X$ be closed convex cones such that $X \in \text{rad } (C \cap D)$. Then
\[
(C \cap D)^+ = C^+ + D^+.
\]
Proof: It suffices to show that $C^+ + D^+$ is closed; suppose that $\{b_n^+\}$ is a sequence in $C^+ + D^+$ with $b_n^+ \to b^+$, $b_n^+ = c_n^+ + d_n^+$, $c_n^+ \in C^+$ and $d_n^+ \in D^+$. Since $X \in C \cap D$
\[
b_n^+ (\bar{x}) \geq c_n^+ (\bar{x}) \geq 0.
\]
Also, since $b_n^+ \to b^+$, $\{b_n^+ (\bar{x})\}$ is a bounded set. Suppose $x \in X$. Since $C$ is a convex cone with $\bar{x} \in \text{rad } C$ there is some $t = t(x)$ with $\bar{x} + tx$ and $\bar{x} - tx$ in $C$. This means that
\[
c_n^+ (\bar{x}) \geq t c_n^+ (x) \geq - c_n^+ (\bar{x})
\]
This shows that $E = \{c_n^+\}$ is $\sigma (X', X)$ bounded. The hypothesis of barrelledness means that $E$ is equicontinuous which in turn implies that $F$ is $\sigma (X', X)$ compact. (Robinson and Robinson (1964)).
Thus one has a subsequence (not distinguished notationally) 
with \( c_n^+ \to c^+ \). Since \( \overline{F} \subseteq C^+ \) and \( C^+ \) is weakly closed \( c^+ \in C^+ \).

Similarly \( d_n^+ \to b^+ - c^+ \) which must belong to \( D^+ \). Thus
\[
  b^+ = c^+ + (b^+ - c^+) \subseteq C^+ + D^+
\]
and \( C^+ + D^+ \) is a closed set. 

Ritter's proof of [15] (2) relies on the existence of a point in \( C^0 \cap D \). Cones can exist, in normed spaces, which have no interior but have a radial point. Since a normed space need not be barrelled when it is incomplete one sees that [15] (2) and [17] are not strictly comparable. In Banach spaces, though, [17] will include [15].

[18] The general linear theorem can now be proved. It is stated in normed spaces for simplicity.

**Theorem:** Let \( X_i \) \( i = 0, \ldots, 4 \) be normed spaces with \( A_i \in B[X_i, X_1] \).
Suppose \( X_0 \) and each \( R(A_i) \) is a Banach space and that there are closed convex cones \( S_i \subseteq X_i \) \( i = 1, \ldots, 4 \) with \( S_0^0 \cap R(A_i) \neq \emptyset \) \( i = 2, 3 \) and \( S_4 = 0 \). Suppose \( S_2 \) is pointed and that
\[
  (*) \quad A_2^{-1}(S_2^+)^c + A_3^{-1}(S_3^+)^c + N(A_4)^c \quad \text{is closed.}
\]
Then either there is a solution to
\[
  (1) \quad A_1 x \in S_1^0, \quad A_2 x \in S_2^0, \quad A_3 x \in S_3, \quad A_4 x = 0,
\]
or to
\[
  (2) \quad a_1^+ + A_1 + a_2^+ A_2 + a_3^+ A_3 + a_4^+ A_4 = 0,
\]
where \( a_i^+ \in S_i^+, \ i = 1, \ldots, 4 \) and either (a) \( a_1^+ \neq 0 \) or (b) for any fixed \( s_i^+ \in S_i^+ \), \( a_i^+ \) can be chosen with \( a_i^+ - s_i^+ \in S_i^+ \).

If \( (S_2^+)^0 \neq \emptyset \) the alternative is strict.

**Proof:** Suppose (1) has no solution.
Set \( E_1 = \{ x | A_1 x > 0 \} \); \( E_2 = \{ x | A_2 x > 0, \ A_3 x > 0, \ A_4 x = 0 \} \).

Case (i) \( E_1 = \emptyset \); Using a standard separation argument there is no difficulty in satisfying (2) with \( a_1^+ = a_2^+ = a_3^+ = 0 \).
Case (ii): \( E_2 = \emptyset \); This can be written, for a fixed \( s_2^+ \in S_2^+ \), as
\[ A_2 \, x \geq 0, \quad A_3 \, x \geq 0, \quad A_4 \, x = 0 \quad \Rightarrow \quad s_2^+ \quad A_2 \, x \leq 0. \]
Then \( E_3 = A_2^{-1}(S_2^+) \cap A_3^{-1}(S_3^+) \cap N(A_4) \subset (s_2^+ A_2^+)(-S_2) \). Using (*) and [15] (1) one has
\[ (3) - s_2^+ A_2 \in A_2^{-1}(S_2^+) \quad + \quad A_3^{-1}(S_3^+) \quad + N(A_4) = (E_3)^+. \]
Farkas Lemma [6] can be used, since each \( R(A_i) \) is a Banach space, to derive that \( u_2^+ \in A_2^{-1}(S_2^+) \) implies \( u_1^+ = d_1^+ A_1 \) and \( d_1^+ \in (R(A_1) \cap S_1^+) \). The condition \( R(A_i) \cap S_1^+ \neq \emptyset \) for \( i = 2, 3 \) enables one to use [15] (2) and write
\[ d_1^+ = a_1^+ + b_1^+, \quad a_1^+ \in S_1^+, \quad b_1^+ \in R(A_1) \quad i = 2, 3. \]
while \( d_4^+ \) can be replaced by \( a_4^+ \) by an application of the Hahn-Banach theorem. Clearly \( a_1^+ A_1 = d_1^+ A_1 \quad i = 2, 3, 4 \).
Using (3) one has
\[ (4) - s_2^+ A_2 = u_2^+ + u_3^+ + u_4^+ = a_2^+ A_2 + a_3^+ A_2 + a_4^+ A_4 \]
Setting \( a_1^+ = 0 \), one has a solution to (2)(b).
Case (iii): \( E_1 \neq \emptyset, \quad E_2 \neq \emptyset \).

By assumption \( E_1 \cap E_2 = \emptyset \). \( E_1 \) is clearly convex with interior.
The convexity of \( E_2 \) follows from the pointedness of \( S_2 \). The Hahn-Banach theorem asserts the existence of \( u^+ \in E_2^+ \) and \( E_1^+ \).
It is easy to show that \( E_2^+ = E_2^+ \) and then, as in (4), to write
\[ (5) - u^+ = a_2^+ A_2 + a_3^+ A_3 + a_4^+ A_4. \]
Moreover, since \( u^+ \in E_1^+ \), \( A_1 \, x > 0 \Rightarrow u^+(x) > 0 \). Since there is some \( x_1 \) with \( A_1 x_1 > 0 \) one has in fact that \( A_1 \, x \geq 0 \) implies \( u^+(x) \geq 0 \). (To see this let \( x_\lambda = x + \lambda x_0 \), \( \lambda > 0 \). Then \( A_1(x_\lambda) > 0 \). Hence \( u^+(x_\lambda) \geq 0 \) and \( u^+(x) \geq 0 \). The same argument as before now shows that \( u^+ = a_1^+ A_1 \), \( a_1^+ \in S_1^+ \), and (5) provides a solution to 2(a).
\[ \text{when } (S_2^+)^0 \neq \emptyset \text{ and both (1), (2) have solutions one has} \]

\[ (6) \quad o = a_1^+ A_1 x + a_2^+ A_2 x + a_3^+ A_3 x + a_4^+ A_4 x \geq a_4^+ A_4 x, \quad i = 1, 2. \]

If \( a_1^+ = 0 \) the fact that \( A_1 x > o \) produces a contradiction in (6) while otherwise \( a_2^+ \) can be chosen to belong to \( S_2^+ \circ \) and \( a_2^+ A_2 x > o \) which again contradicts (6).

**Remark:** Banach space hypotheses are only needed for the Parkas lemma while normed conditions are only used for \( (R(A_1))^+ \cap S_1^+ = R(A_1)^+ + S_1^+ \). It is therefore possible to prove [18] with the spaces sequential barrelled and fully complete with the operators with barrelled range. This merely uses [17] instead of [15] (2).

**Proposition:** \((*)\) is closed when \( S_2, S_3 \) are polyhedral cones ([2.19]) and \( A_4^* \) has finite dimensional range.

**Proof:** Since \( S_2 \) is polyhedrally convex

\[ A_2^{-1} (S_2) = \{ x | y_j^+ (A_2 x) \geq o \quad j = 1, \ldots, m \} \]

\[ \{ x | (A_2^* y_j^+) (x) \geq o \quad j = 1, \ldots, m \} \]

which is polyhedral. Similarly \( A_3^{-1} (S_3) \) is. Using the simplest Parkas Lemma \( A_2^{-1} (S_2), A_3^{-1} (S_3)^+ \) are finitely generated.

Since \( N(A_4^*) = R(A_4^*) \) and \( \dim R(A_4^*) \leq \dim D(A_4^*) \)

\[ A_2^{-1} (S_2)^+ + A_3^{-1} (S_3)^+ + N(A_4) \]

is finitely generated and hence closed. Ritter's alternative theorem (1969, b) is included as the case of [18] in which \( X_2, X_3, X_4 \) are finite dimensional Euclidean spaces and \( S_2, S_3 \) are the orthant orderings. The proposition shows that \((*)\) is satisfied since the orthants are polyhedral.
Theorem: (Generalised Gale's Theorem) Let $X, Y$ be normed spaces and let $A \in B [X, Y]$ have closed complete range. Suppose $S \subseteq X$ is a closed convex cone such that $N(A^\perp) + S^+$ is closed.

Then for any $q^+ \in Y'$ either

1. $Ay = 0, \quad q^+(y) = -1, \quad y \in S$

or

2. $A^* y' \leq s^+ + q^+ \quad (A^* \text{ is the adjoint of } A^*)$

has solution but not both.

Proof: $\Rightarrow$ Suppose (1) has no solution then there is no solution to

1. $Ay = 0, \quad -q^+(y) < 0, \quad y \in S$

Applying the theorem of [18] to (1)', and noting that $S^0 \neq \emptyset$ is not needed since the operator $Ly = y$ is surjective, one has

$$-rq^+ + s^+ + q^+ A = 0$$

with $r > 0, s^+ \in S^+, a^+ \in Y'$. This can be rewritten as

$$A^* (r^{-1} a^+) = q^+ - r^{-1} s^+ - q^+.$$  \(\square\) This theorem includes Gale's equality theorem (Mangasarian (1969)) since $S = Y$ is a perfectly good candidate. The closure condition is met by any finite dimensional map and any polyhedral cone.

It seems worth noting that at least for some convex functions an analogue of [18] can be proved. Suppose $f: X \to Y$ is convex with respect to $S_1$ and that $f(0) = 0, f'(0)$ exists and has closed range.

Theorem: With all terms as in 18 and $f$ satisfying the condition above either there is a solution to

1. $f(x) \in S_1^0, A_2 x \in S_2 \subseteq \mathcal{C}_2, A_3 x \in S_3, A_4 x = 0$

or multipliers exist as described in [18] with

2. $-a_1^+ f(x) + a_2^+ A_2 x + a_3^+ A_3 x + a_4^+ A_4 x \geq 0 \quad \forall x \in X$.  


Proof: If there is no solution to (1) there is also no solution to

\[ (1)' \quad -f'(o) x > o, \quad A_2 x \not\subseteq o, \quad A_3 x \not\subseteq o, \quad A_4 x = o. \]

[18] can be applied to this yielding

\[ (3) \quad a_1^+ f'(o) + a_2^+ A_2 + a_3^+ A_3 + a_4^+ A_4 = o \]

Since \( f \) is convex with respect to \( S_1 \) and \( f(o) = o \)

\[ a_1^+ f(x) \geq a_1^+ f'(o)(x) \]

which gives the desired conclusion when substituted in (3).

[23] A generalised form of Steinke's Lemma is an easy consequence of the transposition theorems.

Proposition: Let \( X, Y \) be normed spaces with \( A \in \mathbb{B}[X, Y] \) having closed, complete range. Let \( S \times X \) be a closed convex cone with interior. Then either (1) or (2) is solvable but not both.

\[ (1) \quad A^* y' \in S^+ / \{ o \}^2 \]
\[ (2) \quad A x = o, \quad x \in S^0. \]

Proof: If (2) has no solution then for some \( s^+ \in S^+/\{ o \}^2 \)
\[ s^+ + a^+ A = o \]

or equivalently \( A^*(-a^+) = s^+ \in S^+/\{ o \}^2 \). If (1) and (2) have solutions \( y' \) and \( x \) respectively one has
\[ o = (Ax) \quad (y') = (A^* y')(x) > o. \]

In \( \mathbb{R}^n \) a derivation of Tucker's Theorem (Mangasarian (1969)) is now an easy corollary.
Chapter Four

SUBGRADIENTS, TANGENT CONES
AND LOCAL SUPPORTABILITY
Definition: If \( f: X \to Y \) and \( S \) is a closed convex cone in \( Y \), then the subgradient \( \partial_S f(x) \) at \( x \in X \) is the set of linear continuous mappings \( \overline{Z}: X \to Y \) satisfying

\[
(*) \quad f(y) - f(x) \geq \langle \overline{Z}(y-x), y \rangle \quad \forall y \in X.
\]

When \( \partial_S f(x) \) is not empty, \( f \) is said to be subdifferentiable at \( x \). If there is no ambiguity, \( \partial_S f(x) \) will be written \( \partial f(x) \). \( f \) is said to have supergradient \( \partial^+_S f(x) \) at \( x \) if \( -f \) has subgradient \( \partial^-_S f(x) \).

Proposition: Let \( S \) be any closed convex cone and \( f: X \to Y \). Then:

1. \( \partial_S f(x) \) is a convex set in \( B[X,Y] \) and is closed in the weak operator topology.
2. \( T \subseteq S \Rightarrow \partial_T f \subseteq \partial_S f \).
3. \( u^+ \in S^* \quad \Rightarrow \quad u^+ \partial_S f \subseteq \partial u^+ f \).

Proof: These all follow directly from [1].

The notion of subgradients originates in the study of convex functionals (Rockafellar (1970a), Fenchel and others) and their importance arises largely from the following composite theorem. It is stated in \( R^n \) but retains much of its validity more generally.

Theorem: Let \( f: R^n \to R \) be a (proper) convex function.

1. For \( x \notin \text{dom} f \), \( \partial f(x) = \emptyset \);
   For \( x \in \text{ri}(\text{dom} f) \), \( \partial f(x) \neq \emptyset \).
2. \( x \in (\text{dom} f)^o \iff \partial f(x) \) is closed and bounded and non-empty.
(3) $\partial f(x)$ is everywhere single valued on $\text{dom} f \iff \partial f(x) = \{ f'(x)^T \}$.  

In this chapter a few existence theorems for subgradients of convex functions with respect to $S$ are given. These are followed by some tangent cone properties associated with subgradients.

[3] **Proposition:** If $f = X \to Y$ is convex with respect to $S$, a pointed convex closed cone and $f'(x)$ exists, then $\{ f'(x)^T \} = \partial_S f(x)$. 

**Proof:** Now 

$$ \lim_{t \to 0} \frac{f(x + th) - f(x)}{t} = (f'(x_o)) (h) $$

and since for $T \in \partial_S f(x)$ and $t > 0$

$$ \frac{f(x + th) - f(x)}{t} - T(h) \in S $$

one has $[f'(x) - T](h) \in S$ for all $h \in X$.  

This means that $[f'(x) - T](h) \in S \cap -S = \{ 0 \}$ for all $h$ and $f'(x) = T$.  

From now on $\{ f'(x)^T \}$ and $f'(x)$ are identified.

When $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ the existence of a single valued $\partial f(x)$ is also sufficient for $f'(x)$ to exist and equal $\partial f(x)$. This and most other properties of subgradients rely on relationships between support hyperplanes and convex sets. Since these do not appear to have natural extensions to planes of greater deficiency it seems unlikely that much can be said in general about subgradients of convex functions.
One case in which the situation is radically different is when $Y$ is a real sequence space and $S$ is the co-ordinate cone. In this case $f$ is convex exactly when each co-ordinate is convex and (*) of [1] is satisfied by a function $T = \{ z_i \}$ exactly when each $z_i$ satisfies (*) for the corresponding real valued $f_i$. The situation can then be read off from [2].

Some properties which hold more generally are given below. A few preliminaries are necessary.

[4] **Definition:** If $A \subset R^n$ and $B \subset R^m$ are convex cones then $A$ will be said to be **polygonal with respect** to $B$ (denoted $A \text{p} B$) if there is some $m$ by $n$ matrix $K$ with $x \in A \iff Kx \in B$.

[5] **Proposition:** If $A \text{p} B$ and $K$ is surjective then

$$y^+ \in A^+ \iff y^+ = Kc^+, \; c^+ \in B^+.$$ 

**Proof:** If $y^+ \in A^+$ then $y^+(x) > 0 \forall x \in A$. Thus

$$Kx \in B \implies y^+(x) > 0.$$ 

By the Farkas Lemma ([3.6]) $y^+T = c^+TK$ for some $c^+ \in B^+$.

[6] **Proposition:** (1) $A \text{p} B$, $B \text{p} C \implies A \text{p} C$.

(2) Suppose $A$ and $B$ and pointed and $A \text{p} B$, $B \text{p} A$.

Then, $K$ of definition [4] is invertible.

**Proof:** There are matrices $K$ and $L$ with $K_{m \times n}$, $L_{n \times m}$ and

$$x \in A \iff Kx \in B; \; x \in B \iff Lx \in A.$$ 

Since $A$ is pointed $Kx = 0 \in B^\perp - B \iff x \in A \cap \{ 0 \}.$
Similarly since \( B \) is pointed \( Lx = 0 \implies x = 0 \). This in turn implies that

\[
\text{rank } K = n < m = \text{rank } L \leq n
\]

Thus \( m = n \) and since a square matrix of full rank is invertible, \( K(L) \) is invertible.

**Proposition:** If \( AB \) and \( A \) and \( B \) are related by an invertible \( K \)
then \( \bar{x} \in \partial_A f(x) \iff Kx \in \partial_B Kf(x) \).

**Proof:** \( \bar{x}(h) \succ_A f(x + h) - f(x) \iff K\bar{x}(h) \succ_B Kf(x + h) - Kf(x) \)

since \( AB \). Because \( K \) is invertible, when

\[
\bar{y}(h) \succ_B Kf(x + h) - Kf(x)
\]

then \( \bar{y} = K\bar{x} \) for some \( \bar{x} \in \mathbb{R}^n \) and

\[
\bar{x}(h) \succ_A f(x + h) - f(x).
\]

Since, as was remarked above, the subgradient of a function \( f: \mathbb{R}^m \to \mathbb{R}^n \) with the orthant ordering is completely determined by its co-ordinates' behaviour and subgradients [7] only extends this result to any closed convex cone in \( \mathbb{R}^n \) generated by \( n \) distinct half spaces, i.e., generated by \( n \) linearly independent constraints.

**[8]** **Proposition:** Let \( f: \mathbb{R}^m \to \mathbb{R}^n \) be convex with respect to a cone \( S \) which is specified by \( x \in S \iff Kx \) has each column non-negative for an invertible matrix \( K \in \mathbb{R}^{n^2} \).

Then all the properties of \( \partial f(x) \) asserted in [2] hold for \( \partial_S f(x) \).
Proof: \( x \in \text{ri}(\text{dom} f) \iff Kx \in \text{ri}(\text{dom} Kf) \) since \( K \) is invertible. Similarly \( x \in (\text{dom} f)^\circ \iff Kx \in (\text{dom} Kf)^\circ \).

All the conclusions now follow from the above discussion except for the boundedness of \( \partial_S f(x) \) which follows from

\[ \| \bar{x} \| \leq |K^{-1}| \| K \bar{x} \| \]

and from [7] with \( B = \mathbb{R}^{n+1} \).

[9] Proposition: If all hypotheses are as in [8] and \( \partial_S f(x) \) is single valued then \( \partial_S f(x) = f'(x) \).

Proof: This also follows from the previous discussion.

It is possible to extend the result in [8] so that the cone \( S \) is generated by \( n+1 \) half-spaces.

[10] Proposition: Suppose all the hypotheses of Proposition [8] hold and that \( S_1 = S \cap \{ x \mid \bar{a}(x) \succeq 0 \} \) then \( \partial_{S_1} f(x) \) is nonempty on \( \text{ri}(\text{dom} f) \).

Proof: \( f \) is supposed to be convex with respect to \( S_1 \)

and hence with respect to \( S \). Thus \( \exists Z = (\bar{z}_1, \ldots, \bar{z}_n) = K^{-1}(\bar{y}_1, \ldots, \bar{y}_n) \in \partial_{S} f(x) \) when \( x \in \text{ri}(\text{dom} f) \) and \( (\bar{y}_1, \ldots, \bar{y}_n) \in \partial Kf \).

If the rows of \( K \) are denoted by \( \bar{a}_1, \ldots, \bar{a}_n \) then the \( a_i \) are independent and \( \bar{a} = r_1 \bar{a}_1 + \ldots + r_n \bar{a}_n \) \( r_1, \ldots, r_n \in \mathbb{R} \).

Since, by a theorem of Rockafellar (1970,a),

(1) \( \partial g(x) + \partial h(x) = \partial (g + h)(x) \) when \( x \in \text{ri}(\text{dom} g) \cap \text{ri}(\text{dom} h) \).
this also implies that

(2) \( \partial (g - h)(x) \subseteq \partial g(x) - \partial h(x) \) when \( g, h \), \( g - h \), are convex.

(1) and (2) mean that

(3) \( \partial \overline{af} = \partial (r_1 \overline{a}_1 f + \ldots + r_n \overline{a}_n f) \subseteq r_1 \partial \overline{a}_1 f + \ldots + r_n \partial \overline{a}_n f \)

since \( \overline{af}, \overline{a}_1 f, \ldots, \overline{a}_n f \) are all convex.

By the previous results \( \partial \overline{af}(x) \neq \emptyset \) when \( x \in \mathbf{ri(\text{dom}f)} \).

Let \( \overline{y} \in \partial \overline{af}(x) \). Using (3)

\[
\overline{y} = r_1 \overline{y}_1 + \ldots + r_n \overline{y}_n \quad \overline{y}_i \in \partial \overline{a}_i f
\]

\[
= r_1 K \overline{z}_1 + \ldots + r_n K \overline{z}_n = \overline{a} \overline{z}.
\]

Then \( \overline{a} \overline{z}(h) \leq \overline{af}(x + h) - \overline{af}(x) \)

and from above \( \overline{a}_k \overline{z}(h) \leq \overline{a}_k f(x + h) - \overline{a}_k f(x) \).

Thus \( K_1 \overline{z} \in \partial K_1 f(x) \) where \( K_1 \) is \( \begin{bmatrix} K \\ \overline{a} \end{bmatrix} \)

and \( \overline{z} \in \partial_{S_1} f(x) \).

This argument cannot be extended inductively since there is no guarantee that the vectors chosen from (3) would agree for two different \( \overline{a} \) and \( \overline{b} \). However, as an easy corollary one has:

[11] Proposition: If \( S = \{ x \mid \overline{a}_t(x) \geq 0, \ t \in T \} \) and there are \( \overline{a}_{t_1}, \ldots, \overline{a}_{t_n} \) with

\[
(2) \ \partial (\overline{a}_t f) = \sum_{i=1}^{n} r_i \partial (\overline{a}_i f) \quad t \neq t_1, \ldots, t_n,
\]

then \( \partial_{S} f(x) \neq \emptyset \) if \( x \in \mathbf{ri(\text{dom}f)} \).
Proof: (2) guarantees that the extra functionals in $S$ are irrelevant.

In particular (3) of the proposition in [1] guarantees that for any cone $S_2$ containing $S$ in [8], [10], or [11]

$$\partial_{S_2} f(x) \neq \emptyset \text{ when } x \in \text{dom} f.$$

For the remainder of the Chapter it will be assumed that $f$ has a non trivial subgradient at the relevant points. From the previous discussion when $f$ is convex this is at worst a requirement that $f'(x)$ exists and if often weaker.

Tangent cones and subgradients.

[12] Proposition: Let $f: X \rightarrow Y$. Then $x$ is a strong minimum for $f$ with respect to $S$ if and only if $0 \in \partial_S f(x)$.

Proof: $\Rightarrow f(y) - f(x) \in S \ \forall x \in \text{dom} f$ if $x$ is a minimum.
This means $0 \in \partial_S f(x)$. $\Leftarrow$ If $0 \in \partial_S f(x)$

$$f(y) - f(x) \succ_S 0(y - x) = 0 \ \forall y \in X$$

and $x$ is a strong minimum. [1]

[13] Proposition: Let $f: X \rightarrow Y$ and let $x$ be a strong maximum for $f$ over $A$ with respect to $S$. Then for any $Z \in \partial_S f(x)$

$$Z \left[ wP(A, x) \right] \subseteq S.$$

Proof: Let $\lambda_n(x_n - x) \rightarrow h \in wT(A, x)$. By assumption

$$0 \succ f(x_n) - f(x) \succ Z(x_n - x) \text{ since } x_n \in A.$$
Thus $\lambda_n \bar{Z}(x_n - x) = \bar{Z} \lambda_n (x_n - x) \in -S$ since $\lambda_n \succ 0$.

By definition $\bar{Z}$ is continuous, hence

$$\bar{Z}(h) = \lim \bar{Z} \lambda_n (x_n - x) \in -S,$$

since $S$ is a closed convex cone and thus weakly closed.

[14] **Proposition:** Let $g : X \to Y$, $S$ be a closed convex cone in $Y$ and $\bar{Z} \in \partial_S g(x)$. Then $\bar{Z} [wT(g^{-1}(-S),x)] \subseteq P(-S, g(x))$.

**Proof:** let $h \in wP(g^{-1}(-S),x)$ Then $h = w\lim_{n} h_n$ where $h_n = \lambda_n (x_n - x)$, $g(x_n) \in -S$, $x_n \to x$ and $\lambda_n \succ 0$. Since $g$ is subdifferentiable

$$\lambda_n [g(x_n) - g(x)] \succ \bar{Z}(h_n).$$

For each $n$, the left hand side belongs to $\lambda_n (-S - g(x))$ and thus for some net $\{s_n \}$ in $S$

$$\bar{Z}(h) = w\lim_{n} (\lambda_n [g(x_n) - g(x) - s_n]) \subseteq \bigcup_{\lambda \succ 0} (-S - g(x_0))$$

By [2.12] this last set is $P(-S, g(x))$.

[15] **Proposition:** Let $g : X \to Y$. Suppose $g$ has a supergradient $\partial_S g(x)$ and suppose $g^{-1}(-S)$ is convex. Then if there is $\bar{Z} \in \partial g(x)$ and $h_0 \in X$ with

(1) $\bar{Z}(h_0) + g(x) \in -S^0$,

it follows that

(2) $h \in P(g^{-1}(-S),x)$ whenever $\bar{Z}(h) \in P(-S, g(x))$.

**Proof:** Suppose $\bar{Z}(h) \in P(-S, g(x))$ then there are $\lambda_n \succ 0, s_n \in S$ with $-s_n \to g(x)$ such that
\(\lambda_n(-s_n - \varepsilon(x)) \to \bar{Z}(h)\).

Let \(h_t = h + t h_0\), \(0 < t < 1\). Then

\[\bar{Z}(h_t) + t g(x) < \bar{Z}(h)\]

and for \(n\) sufficiently large

\[\bar{Z}(h_t) + t g(x) < \lambda_n(-s_n - \varepsilon(x)) < -\lambda_n g(x)\).

Since \(\bar{Z} \in \partial g(x)\)

\[g(x + rh_t) - g(x) < -r(\lambda_n + t) g(x), \quad r > 0.\]

For \(r < r_0\) the right hand scalar will be larger than \(-\frac{1}{t}\) and

\[g(x + rh_t) < \frac{1}{r} g(x) \leq 0.\]

Thus \(x + rh_t \in g^{-1}(-S)\) or \(h_t \in r^{-1}(g^{-1}(-S) - x)\) which is contained in \(P(g^{-1}(-S), x)\) since \(g^{-1}(-S)\) is assumed convex.

Letting \(t \to 0\) one has \(h \in P(g^{-1}(-S), x)\).

The condition that \(g^{-1}(-S)\) be convex is satisfied if \(g\) is convex with respect to \(S\). In the terminology of [2.36] [15] says \(K(\bar{Z})\) is contained in \(P(g^{-1}(-S), x)\), except that \(\bar{Z}\) need not belong to \(C\).

The last four paragraphs have partially illustrated the relationships between tangent cones and subgradients. They will be used in combination with various other results to derive necessary conditions for constrained minima. They are rather unsatisfactory, though, since both [13] and [15] require supergradients for minima rather than subgradients. This will be discussed further in the next chapter in the sections on one sided derivatives and stationary point theorems for convex functions.
Local Supportability

[16] **Definition:** A set $A \subseteq X$ is **locally supportable** at $x$ if there is a neighbourhood $N$ of $x$ and a continuous linear function $u^+$ such that $u^+(x) \leq u^+(y)$ $\forall y \in A \cap N$.

By a theorem of Valentine (1964) if $A$ is locally supportable at each boundary point of $A$ and $A$ is connected with interior then $A$ is convex.

In the proofs of [13] to [16] it is apparent that it would suffice for the subgradient inequality to hold locally. This suggests the following generalization which has been investigated for real valued by Bazaraa et al. (1970).

[17] **Definition:** $f: X \to Y$ is **locally supportable from below** at $x$ with respect to $S$ if there is a $\overline{Z} \subseteq B[X,Y]$ and a neighbourhood $N$ of $x$ such that

$$f(y) - f(x) \triangleright_S \overline{Z}(y - x) \quad \forall y \in N.$$  

[18] **Theorem:** (Bazaraa (1971)). For a continuous function $f: \mathbb{R}^n \to \mathbb{R}$, $f$ being locally supportable from below at $x$ by $\overline{Z}$ is equivalent to any of the following

1. $(x, f(x)) \in \text{bd}\overline{[\text{Epif} \cap N]}$ for some neighbourhood $N$ of $(x, f(x))$.
2. $(-\overline{Z}, 1) \in \text{P}^+([\text{Epif} \cap N], (x, f(x)))$ for some neighbourhood $N$ of $(x, f(x))$.
3. $\text{Epif}$ is locally supportable at $(x, f(x))$. 


With the appropriate interior conditions on \( \overline{\text{Epif} \cap N} \), the theorem can be extended to any convex domain space \( X \). When \( f \) is no longer required to be real valued the same problem that occurs for subgradients occurs here, in that the type of equivalence given by the last theorem seems unobtainable. The equivalences do suggest the next two propositions, though.

[19] **Proposition:** If \( X \) is a normed space and \( f: X \rightarrow \mathbb{R} \) satisfies

1. \( \limsup_{y \to x} \frac{f(y) - f(x)}{\|y - x\|} > -\infty \),
2. \( x \) maximizes \( f \) over \( A \),
3. \( (-\overline{z}, 1) \in \text{wP}^+(\text{Epif}, (x, f(x))) \),

then \( -\overline{z} \in \text{wP}^+(A, x) \).

**Proof:** Let \( h \neq 0 \in \text{wT}(A, x) \). As usual \( h_n \rightarrow h \) where \( h_n = \lambda_n (x_n - x) \), \( x_n \in A \), \( x_n \rightarrow x \) and \( \lambda_n \geq 0 \). Using (1) there is a subsequence of \( \{x_n\} \) (which will not be distinguished notationally) with

\[
0 > \frac{\lambda_n f(x_n) - f(x)}{\lambda_n \|x_n - x\|} \geq -M \quad n > n_0.
\]

Since \( h_n \rightarrow h \), \( \lambda_n \|x_n - x\| \) is bounded and \( k_n = \lambda_n (f(x_n) - f(x)) \) must also be bounded and has a convergent subsequence (which again will not be re-labeled) with \( k_n \rightarrow k_0 \).

Then

\[
(h, k) = \text{wlim} \lambda_n (x_n - x, f(x_n) - f(x)).
\]
Thus \((h,k) \in wP(Epi f, (x,f(x)))\). Using (3)

\[ Z(h) \lesssim k. \]

Since \(f(x_n) \lesssim f(x), k \lesssim 0 \) and \(Z(h) \lesssim 0\). Hence

\[-Z \in wT^+(A,x) = wP^+(A,x). \]

Condition (1) is guaranteed by local supportability while condition (3) is very close to [18] (2).

[20] **Proposition:** Let \(X,Y\) be reflexive normed spaces. Suppose \(g:X \to Y\) is continuous and that \(B\) is any set in \(Y\) containing 0. Suppose the following conditions hold.

1. \(\limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} < \infty\)
2. \(\exists Z \in B[X,Y] \) with

\((Z, -I) [wP(Epi_B g, (x,g(x)))] \subset wP(B,g(x)).\)

Then \(Z \in wC\) (in the notation of [2.31]).

**Proof:** Let \(h \in wT(g^{-1}(B), x)\). Then \(h_n = \lambda_n (x_n - x) \to h\) with \(x_n \to x\), \(g(x_n) \in B\) and \(\lambda_n \geq 0\). As in [19] (1) can be used to derive the existence of a subsequence (not relabeled) of \(\lambda_n (g(x_n) - g(x))\) tending weakly to \(k\).

(This uses the reflexivity of \(Y\) and the weak compactness of the unit ball in a reflexive space.)

\((h,k) \in wP(Epi_B g, (x,g(x)))\) since \((x_n, g(x_n)) \in Epi_B g.\)

Since \(g\) is continuous

\(k = \lim wlim \lambda_n (g(x_n) - g(x)) \in wP(B,g(x)).\)

Using (2) \(Z(h) \in k + wP(B,g(x)) \subset wP(B,g(x)).\) Thus

\[ Z(wT(g^{-1}(B), x)) \subset wP(B,g(x)) \]

and since \(Z\) is supposed linear and continuous

\[ Z(wP(g^{-1}(B), x)) \subset wP(B,g(x)) \]

as desired.
This last proposition illustrates the problem associated with (local) supportability of epigraphs when \( f \) is not real valued. It seems necessary to impose rather strong conditions like (2) which will be easy to verify on the line when \( I = 1 \) but are not so convenient in general.

[21] Examples: (1) Let \( C \subseteq \mathbb{R}^n \) be a convex set and let 
\[ f: \mathbb{R}^n \to \mathbb{R} \text{ be defined by } f(x) = 0 \quad x \in C; \quad f(x) = \infty \quad x \notin C. \]
\[ \partial f(x) = \left\{ \begin{array}{l} \{ z \mid z(z - x) \leq 0 \quad \forall z \in C \} = -(C - x)^+. \end{array} \right. \]
(2) \( f(x) = x^3 \) is a real valued differentiable function which is not locally supportable at 0.
(3) The duality map \( J \) of a Banach pace \( X \) into its dual \( X' \) defined by 
\[ J(x) = \left\{ x' \in X' \left| \begin{array}{l} x'(x) = \| x \| \| x' \| \| x' \| = \| x \|^2. \end{array} \right. \right\} \]
is the subgradient of the convex mapping \( g(x) = \frac{1}{2} \| x \|^2 \).

In a Hilbert space \( J \) is just the identity.

[22] The last result of this chapter gives a simple necessary condition for a set \( E \) to be pseudoconvex at point \( x \). ([2.5]).

Proposition: \( E \subseteq X \) is weakly pseudoconvex at \( x \in E \) only if every hyperplane which supports \( E \) locally at \( x \) supports \( E \) globally at \( x \).

Proof: Suppose \( x' \in X' \) is such that for some neighbourhood \( N \) of \( x \) \( x'(y) \geq x'(x) \) for all \( y \in N \cap E \).

Then if
\[ h \in wT(E, x) \]

\[ h_n = \lambda_n (x_n - x) \to h \text{ where } x_n \to x, \lambda_n \not\in E, \lambda_n \geq 0. \]

For \( n \geq n_0 \), \( x_n \not\in N \) and \( x'(x_n - x) \geq 0 \). Since \( x' \) is continuous and linear

\[ x'(h) = \text{wlim} \ x'(h_n) = \text{wlim} \ \lambda_n x'(x_n - x) \geq 0. \]

Thus \( x' \in wT^+(E, x) = wP^+(E, x) \).

Suppose \( E \) is weakly pseudoconvex at \( x \). Then if \( y \in E \)

\[ y - x \in wP(E, x). \]

This together with \( x' \in wP^+(E, x) \) gives

\[ x'(y) \geq x'(x) \quad \forall y \in E. \]
Chapter Five

FIRST ORDER CONDITIONS
This chapter is concerned with first order necessary and sufficient conditions for the optimization problem

\[(P) = \min f(x) \quad \text{subject to } g(x) \in B, \ x \in C\]

where \( f: X \rightarrow Y, \ g: X \rightarrow Z, \ B \subset Y \) is any set as is \( C \subset X \).

Minimization is taken with respect to a closed convex cone \( S \subset Y \) and it will be indicated whether the minimum is weak or strong in each case. \( X, Y, Z \) are Hausdorff locally convex spaces unless otherwise noted.

\[\text{[1]}\]

Notation: \( A \), the set of \text{feasible solutions} of \((P)\), is defined by

\[A = \left\{ x \mid g(x) \in B, \ x \in C \right\}.\]

\( \Delta \) will denote the inverse image of \( B \) under \( g \)

\[\Delta = \left\{ x \mid g(x) \in B \right\}.\]

\( g \) will be called the \text{constraint function}.

\( f \) will be called the \text{objective function}.

Section one: Fritz John Conditions

The first section of this chapter is restricted to some generalizations of first order necessary conditions for \((P)\) in which \( g \) is subject to no 'constraint' qualification. The basic theorems generalize results of Fritz John (1948), Mangasarian and Fromovitz (1967), Nagahisa and Sakawa (1969) and Zlobec and Massam (1973). The sets \( S, B \) will sometimes be supposed to closed convex cones with nonempty interior. Then the problem will be denoted by \((Q)\) and written as

\[(Q) \ \min f(x) \quad g(x) \in -B \ (\text{or } g(x) \leq_{B^0} x \in C).\]
Such theorems are generally called Fritz John type conditions.

[2] Theorem. Suppose \( x \) is a weak (local) minimum with respect to \( S \) for (Q). Suppose \( f, g \) are compactly differentiable at \( x \). Then there exist \( p^+ \in S^+ \), \( q^+ \in B^+ \), not both zero, such that \[ p^+f'(x) + q^+g'(x) \in F^*(c,x) \quad q^+(g'(x)) = 0. \]

Proof. Let \( M = \{ (y,z) \mid \exists h \in T(c,x) \text{ with } (f'(x)(h) \leq y, \quad (g'(x))(h) \leq -z \} \)
\[ N = \{ (y,z) \mid y \leq -3, \quad z \leq -B \}. \]

\( M \) and \( N \) are closed convex sets and \( N^\circ \neq \emptyset \).

Suppose that \( M \cap N^\circ \neq \emptyset \). There then exists \( h \in T(c,x) \) with \( (f'(x))(h) \leq -S^\circ \) and \( (g'(x))(h) + g(x) \in -B^\circ \).

Let \( h_n = \lambda_n (x_n - x) \to h \) where \( x_n \in C, \quad x_n \to x, \quad \lambda_n \geq 0 \). By the definition of the compact derivative
\[ \lambda_n [f(x_n) - f(x)] = \frac{f(x + \lambda_n^{-1} h_n) - f(x) - (f'(x))(h)}{\lambda_n^{-1}} \in S^\circ. \]

Thus, for \( n \geq n_0 \), \( \lambda_n [f(x_n) - f(x)] \leq -S^\circ \). Since \( S^\circ \) is a cone
\[ f(x_n) \leq_M f(x) \quad \text{(2)} \]

Moreover, similarly
\[ \lambda_n [g(x_n) - g(x)] + g(x) \to (g'(x))(h) + g(x) \in -B^\circ. \]
\[ g(x_n) \leq \lambda_n^{-1} (-B^\circ - F(x)) \leq -B^\circ. \]

Since \( g(x) \in -B, \quad (\lambda_n - 1) g(x) \in -B \) for \( n \geq n_2 \) (Since \( h \neq 0 \),
\( \lambda_n \) can be assumed convergent to \( \infty \)) and
\[ g(x_n) \in \lambda_n^{-1}(-B^\circ - B) \subseteq -B^\circ. (4) \]

Examining (2) and (4) one sees that for \( n \geq n_3 \)
\[ f(x_n) < f(x), \quad g(x_n) \leq_B, \quad x_n \in C \]

which contradicts the weak minimality of \( x \) for (Q).

This means that \( M \cap N^\circ = \emptyset \) so that the Hahn Banach theorem
can be invoked to assert the existence of \((p^+, q^+)\) \(\forall \in (Y', Z')\) with
\[ p^+(y) + q^+(z) \triangleright p^+(s) + q^+(b) \quad (y, z) \in K, s \in -S^0, b \in -B^0. \]
In particular \(p^+ \in S^+, q^+ \in B^+\) and
\[(5) \quad p^+((f'(x))(h)) + q^+((g'(x))(h)) + q^+(g(x)) \triangleright 0 \quad \forall h \in T(C, x) \]
Letting \(h = 0\) one sees that \(q^+(g(x)) \triangleright 0\) which in conjunction with \(g(x) \in -B, q^+ \in B^+\) gives \(q^+(g(x)) = 0\). Finally, since \(q^+(g(x)) = 0\) and \(T^+(C, x) = P^+(C, x)\), (5) gives the desired conclusion.

The proof of the following theorem corresponds closely to that of [2]. Note that \(X\) is implicitly a sequential space ([2.43]).

[3] **Theorem:** Suppose in addition to the hypotheses in [2], \(f\) and \(g\) are supposed boundedly differentiable at \(x\) with \(f'(x)\) and \(g'(x)\) completely continuous. Then there exist \(p^+ \in S^+, q^+ \in B^+\) not both zero with
\[ p^+(f'(x)) + q^+(g'(x)) \in wP^+(C, x) \quad q^+(g(x)) = 0. \]

**Proof:** The relevant lines (1), (3) follow from the fact that \(\{h_n\}\) is now a weakly convergent net and the argument in [2.26] and [2.41].

[4] **Remarks:** (1) Theorem [2] was proved in Banach spaces for real valued \(f\) by Nagahisa and Sakawa, and was sketched, with an incorrect statement, by Zlobec and Massam (1973) for convex spaces and real valued \(f\).

(2) In return for strengthened differentiability assumptions [3] gives a stronger necessary condition since \(wP^+(C, x) \subset P^+(C, x)\) and they need not be equal.

The next results are concerned with the specific case in which \(C\) is the null set of a mapping \(h\). That is \(h: X \rightarrow \mathbb{W}\) and
\[ C = \left\{ x \mid h(x) = 0 \right\} = N(h). \]

[5] **Theorem:** Let \( x \) be a weak local minimum (Q) and suppose that \( h: X \to W \) is compactly differentiable at \( x \) with \( X \) fully complete and \( W \) barrelled. Suppose further that \( h \)' satisfies the following regularity condition

1. \( P(h'(x)) \) is closed in \( W \).
2. If \( R(h'(x)) = \bar{W} \), then \( N(h'(x)) = P(N(h), x) \).

Then there are \( p^+ \in S^+, q^+ \in B^+, w^+ \in W' \) not all zero with

\[ p^+(f'(x)) + q^+(g'(x)) + w^+(h'(x)) = 0 \quad q^+(g(x)) = 0. \]

Moreover, in case (2) both \( w^+ \) and \( (p^+, q^+) \) can be assumed nonzero.

**Proof:** By assumption \( R(h'(x)) \) is closed. If the range is not \( \bar{W} \) then there is (by Hahn-Banach) some \( w^+ \in W' \setminus \{ 0 \} \) with \( w^+((h'(x))(y)) = 0 \) \( \forall y \in X \). In this case \( p^+ = q^+ = 0 \) and the given \( w^+ \) suffice in the conclusion. Otherwise let \( p^+, q^+ \) be as in [2] and define

\[ K(y) = p^+((f'(x))(y)) + q^+((g'(x))(y)) \]

Using [2] and (2)

\[ y \in P(C, x) = N(h'(x)) \Rightarrow K(y) \geq 0. \]

All the hypotheses of the Farkas Lemma [3, 6] are satisfied and one can deduce that

\[ K(y) + w^+((h'(x))(y)) = 0 \quad \forall y \in X, w^+ \neq 0 \]

which gives the desired conclusion.

[6] **Theorem:** Suppose in [5] \( f, g \) are boundedly differentiable and \( f'(x), g'(x) \) are completely continuous; the regularity condition (2) can be replaced by

(2)' If \( R(h'(x)) \neq \bar{W} \), then \( N(h'(x)) = wP(N(h), x) \).


[7] Definition: A continuous linear map B: X → Z between Banach spaces is said to be adequate (Craven (1970)) if

1. \( R(B) \) is closed in \( Z \),
2. If \( B(e) = Z \) then there is a continuous projection \( q \) of X onto \( N(B) \).

[8] Proposition: If \( h: X \rightarrow W \) is continuously Fréchet differentiable at \( x \) and \( h'(x) \) is adequate then \( h \) satisfies the regularity condition of Theorem [5].

Proof: (1) of [7] is the same as (1) of [5]. Suppose that (2) of [7] holds. The image then satisfies the hypotheses of Halkin's "correction" theorem [2,37] with \( T = (1 - q)X \). Using [2,39]

\[
N(h'(x)) = P(N(h),x).
\]

Note that if \( W \) is finite dimensional \( h'(x) \) need only be assumed to exist since one may then apply [2,38].

[9] Proposition: If \( h: X \rightarrow W \) is continuous and affine then

\[
P(N(h),x) = N(h'(x)).
\]

Proof: This follows from the fact that \( h'(x) \) exists and \( h'(x)(u) = h(u) - h(o) \), \( x,u \in X \).

This means that when \( h'(x)(u) = o, h(u) = h(o) \) and

\[
h(x + tu) = h(x) + t(h(u) - h(o)) = h(x)
\]

Thus when \( x \in N(h) \) and \( h'(x)(u) = o, u \in P(N(h),x) \) since
\[ u = \lim_{n} (x + \frac{1}{n} u - x). \] The opposite containment is standard.

10. Theorem: (Craven and Mond (1973)). This is \[ 5 \] with \( Y = R, X \) and \( W \) Banach spaces, \( f'(x) \) and \( g'(x) \) Fréchet and \( h \) continuous, affine and open or continuously differentiable with \( h'(x) \) adequate.

Proof: This follows from \[ 5, 8 \] and \[ 9 \].

11. The next results lists some other properties of adequate mappings which can be of use.

Proposition: (1) In \[ 7 \] if \( R(B) \) is finite dimensional or \( X \) is a Hilbert space then a continuous projection \( q: X \to N(B) \) exists.

(2) If \( h \) satisfies the conditions of \[ 8 \] then when \( h'(x) \) is surjective there is a bijection \( T \) of a neighbourhood \( N \) of \( o \) in \( h'(x)^{-1}(o) \) onto a neighbourhood \( U \) of \( x \subseteq N(h) \) with \( T \) and \( T^{-1} \) Fréchet differentiable at \( o \) and \( x \) respectively.

Proof: (1) follows from the fact that projections in Hilbert space are continuous. (2) is proved in Craven (1973, submitted).

12. A generalization of \[ 5 \] which utilizes a condition similar to the Cuiignard constraint condition (\[ 16 \]) is given by the next result.

Theorem: Suppose everything is as in \[ 5 \] with the provision that (2) is replaced by (2')

(2') \( R(h'(x)) = W \) implies \( N(h'(x)) \cap G = P(N(h), x) \)

for some closed convex cone \( G(\subseteq X) \) with interior.

Then there are \( p^+, q^+, w^+ \) as in \[ 5 \] such that

\[ p^+(f'(x)) + q^+(g'(x)) + w^+(h'(x)) \in G^+, q^+(g(x)) = o. \]

Proof: In the terminology of \[ 5 \] one has (when \( R(h'(x)) = W \))

\[ y \in G, (h'(x))(y) = o \Rightarrow k(y) > o. \]
Thus there is no solution to

\[(h'(x))(y) = 0, K(y) < 0, \ y \in G.\]

The transposition theorem \([3,10]\) applies since \(h'(x)\) is open and

\[\left[ r p^T(r'(x)) + r q^T(g'(x)) + w^T(h'(x)) \right] (y) > 0 \ \forall y \in G\]

for \(w^+ \in W^*, r > 0.\) This is the desired result.

When \(B^o = \emptyset\) in \((Q)\) Craven (1973, in preparation) has proved, using the concept of adequacy, an extension to \([11]\). Since this condition on \(g\) is a constraint condition generalizations will be given in the section on Kuhn-Tucker type results.

**Converse Duality**

The following theorem generalizes results of Craven and Mond (1971) and Mangasarian (1969) using the result of \([5]\). The proof method is taken from Craven and Mond.

**Theorem:** Suppose \(X\) and \(Z'\) are reflexive fully complete spaces and \(X, Z\) are barrelled spaces. Suppose \(f: X \to R\) and \(g: X \to Z\) are twice compactly differentiable and that \(B \subset Z\) is a closed convex cone with \((B^+)^o \neq \emptyset\). Let (\(D_1\)) (\(D_2\)) denote the programs

\[
(D_1) \min f(x) \text{ subject to } g(x) \in -B,
\]

\[
(D_2) \max f(x) + q^T(g(x)) \text{ subject to } q^+ \in B^*, f'(x) + q^T(g'(x)) = 0.
\]

Suppose \((x_0, q_0^+)\) is optimal for (\(D_2\)) and that

\[X^* = [f'''(x_0) + q_0^+ g'''(x_0)]^*\]

has zero kernel.

Then if (i) \(f\) is pseudoconvex and (ii) \(q_0^* g\) is quasiconvex or \(\Delta = g^{-1}(-B)\) is pseudoconvex at \(x_0\), \(x_0\) is optimal for \(D_1\), and the optimal values agree.

**Proof:** Suppose \((x_0, q_0^+)\) is optimal for (\(D_2\)). Applying \([5]\) to (\(D_2\)) one has \(u > 0, v \in X'' = X, q \in (B^+)^* = B\), not all zero, and...
(1) \( u \left[ f'(x_o) + q_o^+ g'(x_o) \right] + \mu^*(v) = 0 \)
(2) \( u g(x_o) + (g'(x_o))(v) + q = 0 \)
(3) \( q_o^+(q) = 0 \).

These are just the necessary conditions of [5] rewritten. Note that [5] can be applied to \( D_o \) since \( B^+ \) has interior and \( f'(x_o) + q_o^+ g'(x_o) \) satisfies the regularity condition of [5].

(Craven submitted claims that the condition on \( M \) is sufficient to prove adequacy of \( f'(x_o) + q_o^+ g'(x_o) \); if this is not so then the condition needs to be added.)

Since \( (x_o, q_o^+) \) is feasible for \( D_o \), \( f'(x_o) + q_o^+(g'(x_o)) = 0 \). (1) says then that \( M^*(v) = 0 \) which by hypothesis means \( v = 0 \).

If \( u = o \) (2), in turn, forces \( q = 0 \). Since all 3 cannot be zero \( u \neq o \). (2) again implies that \( g(x_o) = 1/u, q \in B \). (3) gives \( q_o^+(g(x_o)) = 0 \).

(4) \( f'(x_o))(x - x_o) = -(q_o^+ g'(x_o))(x - x_o) \).

By hypothesis either \( q_o^+ g \) is quasiconvex, which with \( q_o^+(g(x_o)) = 0 \), \( q_o^+ g(x) \) when \( g(x) \in -B \), gives

(5) \( f'(x_o))(x - x_o) \geq 0 \quad \forall x \in \Delta \);

or \( \Delta \) is pseudoconvex at \( x_o \). This means that

\( (g'(x_o))(\Delta - x_o) \subseteq \mathbb{R}(-B, g(x_o)) \) which with \( q_o^+ \in B^+ \) and \( q_o^+(g(x_o)) = 0 \) gives (5) again. Thus since \( f \) is pseudoconvex and (5) holds one has

\( f(x) \geq f(x_o) \) if \( x \in \Delta \) (\( g(x) \in -B \)).

Because \( x_o \) is feasible for \( D_o \), \( x_o \) is optimal. Moreover, the extreme values agree because

\[ f(x_o) + q_o^+(g(x_o)) = f(x_o). \]

[14] Remarks: (1) \( q_o^+ g \) can be pseudoconvex without \( \Delta \) being pseudoconvex at \( x_o \) and conversely. Convexity guarantees both conditions.
(2) Craven and Mond's result (1974) is given with the hypotheses that \( f \) and \( g \) are convex.

(3) It is clearly possible to use Kuhn-Tucker type results to prove [13].

Section Two: Real valued objective functions

Kuhn and Tucker (1951) were the first to use the notion of a constraint qualification on \( g \) to guarantee the existence of a multiplier for the problem (\( M \)).

\[
\begin{align*}
(M) \quad \min f(x) & \quad g_i(x) \geq o, \ldots, g_n(x) \geq o, h_{n+1}(x) = o, \ldots, \\
h_p(x) = o
\end{align*}
\]

where all the functions are real valued and are assumed to have continuous first partials at any optimal point.

[15] Definition: Kuhn-Tucker constraint condition. Let \( x \in \mathbb{R}^n \) satisfy the constraints of (\( M \)). The constraint qualification is said to hold at \( x \), if for any \( y \neq o \) with

\[
\begin{align*}
(g'_i(x)(y) \geq o & \quad \forall i \in \bar{\{i \mid g_i(x) = o\}} \\
(h'_j(x))(y) = o & \quad j = n + 1, \ldots, p
\end{align*}
\]

\( y \) is tangent to an arc \( \alpha(\Theta) \) differentiable at \( x \) and contained in the constraint region. That is \( \alpha(o) = x, \alpha'(o) = y \) and \( \alpha(\Theta) \) satisfies the constraints for \( \Theta < \Theta_1 \).

Kuhn and Tucker proved that if \( x \) satisfied this constraint condition

\[
f'(x) - \sum_{i=1}^{n} u_i g'_i(x) + \sum_{i=n+1}^{p} z_i h'_i(x) = o
\]

for some \( u_i \geq o \quad i = 1, \ldots, n \) with \( u_i g_i(x) = o \).
In other words, in the terminology of [2] the multiplier associated with $f$ could be assumed positive. Varaiya (1967), Guignard (1969), Zlobec (1970) and others have considered constraint qualifications for the problem (P) which are given in tangent cone terms and which generalize Kuhn-Tucker's condition. When $f$ is no longer real valued certain problems arise in the attempt to produce an analogue of the real results. The results in this section assume that $f$ is real valued and give analogous results to those of Guignard and Zlobec for weak cones.

[16] **Theorem:** Suppose $x$ is a minimum for (P) with $f, g$ boundedly differentiable and $f$ real valued. Let

$$\mathcal{WH} = \{ h \in X' \mid h = u^+ g (x), u^+ \in \mathcal{wP} (B, g(x)) \}$$

$$\mathcal{WK} = \{ y \mid (g'(x))(y) \in \mathcal{wP} (B, g(x)) \}.$$  

Suppose $\mathcal{WH}$ is closed and $G$ is a closed convex cone such that $\mathcal{WK} \cap G \subset \mathcal{wF} (A, x)$ and $\mathcal{WK}^+ + G^+$ is closed, then there is some $u^+ \in \mathcal{wP} (B, g(x))$ with

$$f'(x) - u^+ (g'(x)) \in G^+.$$  

The condition is also sufficient if

1. $G$ is a closed convex cone with $A - x \subset G$.
2. $A$ is weakly pseudoconvex with respect to $\Delta$ at $x$.
3. $f$ is pseudoconvex over $A$ at $x$.

**Proof:** Guignard proved the necessary condition for strong tangent cones and Fréchet derivatives. Propositions [2.26] and [2.35] provide the only alterations necessary in the proof. The method will be shown later when a similar result is proved for more general objective functions. The sufficiency condition will also be proved more generally later. (2) is in fact more general than Guignard's condition which was $A$ or $\Delta$ is pseudoconvex at $x$. ([2.5])
If the closure conditions are not assumed in [16] the following asymptotic version holds.

[17] Theorem: In the terminology of chapter 2 suppose $x \in A$ and

1. $G \subset X$ is a closed convex cone with $E \in wE$ and $\text{weak} \left( \mathbb{E} \right) \cap G \subset \text{weak} \left( A, x \right)$

2. $D \in \text{weak}^+(A, x)$.

Then $D \in \text{weak}^+(B, g(x))E + G^+$.

Proof: $x \in \text{weak} \left( E \right) \cap G \Rightarrow D(x) \succ 0$ by (1) (2). Thus $E(x) \in \text{weak} \left( B, g(x) \right), x \in G \Rightarrow D(x) \succ 0$.

This is the same as $E(x) \in \text{weak}^+(B, g(x))^+, x \in (G^+)^+ \Rightarrow D(x) \succ 0$.

By a proposition of Zlobec & Massam (1973) this last line is equivalent to the conclusion. In fact it relies on a separation argument which is independent of the nature of the closed convex cones.

[18] Corollary: If $x$ is a minimum for (P) where $f, g$ are boundedly differentiable with $f$ real valued, then if $\text{weak} \left( g'(x) \right) \cap G \subset \text{weak} \left( A, x \right)$

$f'(x) \in \text{weak}^+(B, g(x))g'(x) + G^+$.

Proof: By [2.26] $f'(x) \in \text{weak} \left( A, x \right)$ while by [2.33] $g'(x) \in wE$.

[19] Proposition: Suppose that the sufficient conditions (1) (2) (3) of [16] hold and that for some closed convex cone $G$

$f'(x) \in \text{weak}^+(B, g(x))g'(x) + G^+$.

Then $x$ is a minimum for (P).

Proof: This is similar to an argument in Zlobec (1971). Again it will be shown more generally later. It seems worth emphasising that conditions for $\text{weak}^+ + G^+$ to be closed are given
Section Three: Conditions for weak minima

If $x$ is a weak minimum with respect to $S$ it is too much to hope that a direct analogue of [16] or [17] should hold. The next generalization is of some help, however.

[20] **Theorem:** Suppose $x$ is a weak minimum for (F) and suppose $f, g$ are boundedly differentiable at $x$ with $f'(x)$ completely continuous. Then, if $w^+(g'(x)) \preceq G \subseteq w^+(A, x)$ for some closed convex cone $G$, there is $p^+ \in S^+ / \{ 0 \}$ with

$$p^+(f'(x)) \in \underbrace{w^+(B, g(x))g'(x)}_{G^+} + G^+.$$ 

**Proof:** By proposition [2.26] there is a suitable $p^+$ with $p^+(f'(x)) \in w^+(A, x)$. The proof proceeds as in [18].

[21] Equally, if the closure conditions are met as in [16] one can actually assert that $p^+(f'(x)) - u^+(g'(x)) \in G^+, p^+ \in S^+ / \{ 0 \}$, $u^+ \in w^+(B, g(x_0))$.

[22] It was noted in [2.30] that functions $f$ with weak minima can exist for which there is no equivalent real problem. The sufficiency condition of [19] can be rephrased to exclude this possibility.

**Theorem:** Suppose $x \in A$ and that for some $p^+ \in S^+ / \{ 0 \}$ and some closed convex cone $G$ one has

1. $p^+(f'(x)) \in \underbrace{w^+(B, g(x))g'(x)}_{G^+} + G^+$
2. $A - x \in G$
3. $A$ is pseudoconvex with respect to $\Delta$ at $x$. 
(4) $u^+f$ is pseudoconvex at $x$.

Then

$$(P') \min (p^+f)(x) \text{ subject to } g(x) \in B, x \in C$$

has a minimum at $x$.

Proof: This is just [19] for $p^+f$. 

[23] Remarks: (1) [21], [22] in conjunction give that if $(P)$ has a weak minimum at $x$ that $(P')$ has a minimum at $x$ when the appropriate conditions are met.

(2) [20], [21], [22] could equally well have been phrased for strong tangent cones and compact derivatives without $f'(x)$ in [20] being completely continuous.


Theorem: Suppose in [21] that (1) $K(g'(x)) = F(\Delta, x)$ and (2) $P(\Delta, x) \cap F(C, x) = F(A, x)$ then $\exists p^+ \in S^+ / \{0\}$, $u^+ \in P^+(B, g(x))$ with

$$p^+(f'(x)) - u^+(g'(x)) \in P^+(C, x)$$

Proof: This is just a special case of [21].

Conditions for (1), (2) to hold were given [2.18] and [2.39].

[25] Remarks: When $-B$ is a closed convex cone then [24] (1) is just a regularity condition on $g$ similar to the one imposed on $h$ in [5]. The proof method of [5] could be applied to [24] to obtain a generalization of Craven's (1973, in preparation) announced result which was used in Craven (1973, submitted) for cones $B$ with $B^o = \emptyset$. 
Section Four: Conditions for strong minima

In this section analogous results to the Kuhn-Tucker necessary condition are derived for (P) when x is assumed to be a strong minimum for (P) with respect to S. In this case the multipliers are no longer linear functionals but are continuous linear operators.

[26] Definition: $T \in B[X,Y]$ is said to be positive with respect to two convex cones $K \subset X$, $S \subset Y$ if $T(K) \subset S$.

[27] Definition: The set of all positive mappings of K into S will be called the maximum cone and denoted by $K^S$. When $S = R^+$ $K^S$ is denoted $K^+$. When $S = R^{n+}$, $K^S$ is denoted $K^{n+}$.

In the notation above $g'(x)(F(\Delta,x)) \subset F(B,g(x))$ which was denoted $g'(x) \in E$ can be rewritten as $g'(x) \in F(\Delta,x) \cap F(B,g(x))$.

[26] Positive mappings and maximum cones have been studied by Ritter (1969, a, b). The next result of Ritter's is central to the generalized Cauchy condition.

Theorem: (Ritter (1969,a)) Let X be a normed space and Y a reflexive Banach space. Suppose $S \neq \emptyset$ is a closed convex normal cone in Y and that $K_1, K_2, K_3 = K_2 \cap K_2$ are closed convex cones in X. Suppose that either

1. $K_3^0 \neq \emptyset$ or 2. (i) $K_3$ has interior relative to $K_3 - K_3$
   (ii) $K_3$ has a point interior to $K_2$ or $K_1$
   (iii) $\forall y' \in Y' \ y'(S) = 0$ implies $y' = 0$ (S is called full).

Then

$$K_1^S + K_2^S = (K_1 \cap K_2)^S.$$
Definition: If $H \subseteq B[X,Y]$, $S \subseteq Y$ are convex cones, define $(H)^{S}$ by
\[
(H)^{S} = \{ x \in X \mid T(x) \in S \quad \forall T \in H \}.
\]

Proposition: If $S \neq \{ 0 \}$ is a closed pointed convex cone then $\overline{K} = (K^{s})^{S}$.

Proof: $\Rightarrow$ Let $x \in \overline{K}$, then there is a sequence (net) $\{ x_{n} \} \subseteq K$ with $x_{n} \to x$. Then $T(x_{n}) \subseteq S$ and since $T$ is continuous and $S$ is closed $T(x) \subseteq S$ for any $T \in K^{s}$. Thus $\overline{K} \subseteq (K^{s})^{S}$. $\Leftarrow$ Suppose $\overline{K} \not\subseteq (K^{s})^{S}$. Let $u^{+}$ be a continuous linear functional which is nonnegative on $\overline{K}$ but has $u^{+}(\overline{x}) = -1$. Let $T(y) = u^{+}(y)$ where $y \in S / \{ 0 \}$. Then
\[
T(k) = u^{+}(k) \in S \quad \forall k \in \overline{K}
\]
and $T \in (K^{s})^{S}$. However, $T(\overline{x}) = -s \in -S / \{ 0 \}$ and since $S$ is pointed $T(\overline{x}) \notin S$. Thus $\overline{x} \not\in (K^{s})^{S}$ and $(K^{s})^{S} \not\subseteq \overline{K}$. $\blacksquare$

This generalizes the standard result for closed convex cones that $(C^{+})^{+} = C$.

Proposition: Suppose $H = \{ Tg'(x) \in B[X,Y] \mid T \in F(B,g(x))^{S} \}$; then $H^{S} \subseteq K(g'g(x))$.

Proof: Let $\overline{y} \in H^{S}$. Then $(Tg'(x))(\overline{y}) \in S \quad \forall T \in F(B,g(x))^{S}$.

Suppose $(g'(x))(\overline{y}) \notin F(B,g(x))$. Then there is some $u^{+} \in F^{+}(B,g(x))$ with $u^{+}((g'(x))(\overline{y})) = -1$.

Let $s \in S / \{ 0 \}$ and let $T(y) = u^{+}(y)$. $T$ belongs to $F^{+}(B,g(x))^{S}$ and $T((g'(x))(\overline{y}) \notin S$ which is a contradiction. Thus $(g'(x))(\overline{y}) \in F(B,g(x))$ for any $\overline{y} \in H^{S}$ which is $H^{S} \subseteq K(g'(x))$. $\blacksquare$
This result can be improved if one imposes extra conditions on $g$ and $P(B, g(x_o))$.

**Theorem:** Suppose $X, Y, Z$ are Banach spaces with $Y$ reflexive.

Suppose that $S \neq 0 \subseteq Y$ is a closed convex normal cone and that $g'(x_o)$ has closed range. Suppose also that

1. $R(g'(x_o)) = Z$ or $R(g'(x_o)) \neq Z$ and
2. $R(g'(x_o)) \cap P^o(P, g(x_o)) \neq \emptyset$

(ii) $S$ is a full cone.

Then $K^S \subseteq H$.

**Proof:** Suppose $T \in K^S$. Then

$$g'(x_o)(h) \in P(B, g(x_o)) \Rightarrow T(h) \in S.$$

The Farkas Lemma [3.5] can be applied since $R(g'(x_o))$ is closed and

$$T = T_0 g'(x_o); \quad T_0 \in (P(B, g(x_o)) \cap R(g'(x_o)))^S.$$

The hypotheses guarantee that

$$T_0 \in P(B, g(x_o))^S + R(g'(x_o))^S,$$

using [28], so that

$$T_0 = T_1 + T_2; \quad T_1 \in P(B, g(x_o))^S$$

$$T_2 \in R(g'(x_o))^1$$

so that

$$T_0 = T_1 g'(x_o) \in H.$$

In particular $(H^S)^S \subseteq K^S \subseteq H$ and $H = \overline{H}$. This theorem gives conditions which exclude Zlobec's (1970) example in which $R(g'(x_o))$ is closed but $H$ is not.
The following is a generalization of Guignard's theorem in the necessary direction.

**Theorem:** Let \( X, Z \) be normed spaces with \( Y \) a reflexive Banach space. Let \( S \subset Y \) be a closed, convex normal cone. Suppose that \( f, g \) are Fréchet differentiable at \( x \) and that there is a closed convex cone \( G \subset X \) with \( G \cap K \subset P(A, x) \) and suppose that

1. \((G \cap K)'' \neq 0\)
2. \(G \cap K\) has interior relative to \((G \cap K)''\)
3. Some point in \( G \cap K \) is interior to \( G \) or to \( K \).
4. \( S \) is full.

Then a necessary condition for \( x \) to be a strong minimum with respect to \( S \) is

\[
f'(x) \in (H^S)^S + G^S.
\]

**Proof:** By [2.24] \( f'(x) \in P(A, x)^S \). By the theorem in [2.25] and (1) or (2)

\[
P(A, x)^S \subset (G \cap K)^S = G^S + K^S.
\]

By [31] \( H^S \subset K \) and it is clear from definition [28] that \( K^S \subset (G^S)^S \). Collecting results one has

\[
f'(x) \in G^S + (H^S)^S.
\]

**Corollary:**

1. If \((H^S)^S = \bar{H}\), which is true at least for \((Y, S) = (R^n, R^n_+), \) or if \( T = g'(x) \) is invertible, then \( f'(x) \in \bar{H} + G^S \).

2. If \((H^S)^S = \bar{H} = H\) then for some \( T \in P(B, g(x))^S \)

\[
f'(x) - T(g'(x)) \in G^S.
\]

**Note:** The result in [33](2) can be proved directly from the Farkas theorem of [3.9] if the appropriate interior conditions are satisfied. These are, perhaps not surprisingly, stronger than...
those in [32].

**Theorem:** (Generalized Guignard) Suppose $X,Y,Z$ are Banach spaces with $Y$ reflexive. Suppose $S + 0 \subseteq Y$ is a closed convex normal cone and that $f,g$ are Fréchet differentiable with $R(g'(x)) = Z$.

Suppose that for some closed convex cone $G \subseteq X$

$$G \cap K \subseteq F(A,x)$$

and suppose that there is some $x \in G^O$ with $(g'(x))(\bar{x}) \in F(B,g(x))^O$.

Then a necessary condition for $x$ to be a strong local minimum for $(F)$ is

$$f'(x) - T(g'(x)) \in G^S, \quad T \in F(B,g(x))^S.$$  

**Proof:** The conditions of the theorem are sufficient to apply the Farkas theorem of [3.9] to obtain from

$$h \in G, \quad (g'(x))(h) \in F(B,g(x)) \Rightarrow (f'(x))(h) \in S$$

that

$$f'(x) = T(g'(x)) + T_2, \quad T \in F(B,g(x))^S, \quad T_2 \in G^S.$$  

[35] **Corollary:** If $G$ can be chosen to be $X$, $F(B,g(x))$ need not have interior and $S$ need not be normal.

**Proof:** In this case one can apply the Farkas Lemma [3.6] to

$$(g'(x))(h) \in F(B,g(x)) \Rightarrow (f'(x))(h) \in S$$

and proceed as before. 

The conditions of [34] are clearly stronger than those of [32] since when $\bar{x} \cap G^O$ and $(g'(x))(\bar{x})$ belongs to $F(\bar{B},g(x))$

$\bar{x}$ must belong to $K^O \cap G^O \subseteq (G \cap K)^O$ and [32] (1) is satisfied.

[36] **Remarks:** (1) [32] could have been phrased for weak tangent cones with the appropriate strengthening of the differentiability hypotheses.
(2) In [16] $H^+ + G^+$ closed was required this was guaranteed for $H^+ + G^+$ in [32] by the interior conditions of [28] applied to $F(A,x)$.

(3) If $S$ is closed but not normal one can derive

$$f'(x) \in \overline{G^+ + (H^+)^+}$$

as can be seen by inspecting Ritter's proof of [28].

[37] **Theorem:** (Generalized Sufficiency Condition)

Suppose that $f, g$ are boundedly differentiable in convex spaces and that for some point $x$ and some closed convex cone $G$

(1) $A - x \subseteq G$

(2) $A$ is weakly pseudoconvex with respect to $\Delta$ at $x$

(3) $f$ is pseudoconvex at $x$ over $A$ with respect to a closed convex cone $S \subseteq Y$.

Then if

(4) $f'(x) \in \overline{wF(B, g(x)) + G^+}$

$x$ is a strong minimum for $(P)$ with respect to $S$.

**Proof:** Using (4) there are nets $\{G_n\} \subseteq G^+$ and $\{T_n\} \subseteq \overline{wF(B, g(x))}$ with

(5) $(f'(x))(y - x) = \lim \left[ T_n (g'(x))(y - x) + G_n(y - x) \right]$ 

Let $y \in A$, then by (1) $y - x \in G$ and thus (6) $G_n(y - x) \in S$.

Now, by (2), $A - x \subseteq wF(\Delta, x)$. Since $g$ is boundedly differentiable $g'(x) \in wE([2.33])$ and $(g'(x))(y - x) \in wF(B, g(x))$ so that

(7) $T_n (g'(x))(y - x) \in S$. Substituting (6) and (7) in (5)

(8) $(f'(x))(y - x) = \lim S_n \in \overline{S} = S$.

Since $f$ is assumed pseudoconvex with respect to $S$ at $x$ over $A$ (8) yields

$$f(y) - f(x) \in S \quad \forall y \in A$$

and $x$ is a strong minimum for $(P)$. $\square$
This theorem clearly remains true if all weak pseudotangent cone relations are replaced by strong ones. It is then only necessary for \( g \) to be compactly differentiable as one uses \([2.32]\) instead of \([2.33]\).

When \( Y = \mathbb{R}^n \) a necessary condition can be derived for \((P)\) directly from the Guignard type result proved for real valued maps in \([16]\). This is phrased for strong tangent cones.

**Theorem:** Suppose in \((P)\) that \( Y = \mathbb{R}^n \) and that \( x \) is a strong minimum with respect to a pointed cone \( S \subset \mathbb{R}^n \). Suppose further that \( g \) satisfies the Guignard constraint condition for some convex cone \( G \). Then a necessary condition for \( x \) to be a strong minimum is

\[
    f'(x) - M g'(x) \in K^{-1}(G^n^+) \tag{35}
\]

where \( M = K^{-1} T \) with \( K \) an invertible \( nxn \) matrix with rows in \( S^+ \) and \( T = (u_1^+, \ldots, u_n^+)^T \), \( u_i^+ \in P^+(\mathbb{E}, g(x)) \).

**Proof:** Since \( x \) is a minimum for \( f \) over \( A \)

\[
f'(x) \in P(A, x)^S
\]

Since \( S \) is pointed in \( \mathbb{R}^n \), \( S^+ \) has interior. Choose \( y_1^+, \ldots, y_n^+ \) linearly independent in \( S^+ \); then

\[
y_i^+ f'(x) \in P^+(A, x) \quad i = 1, \ldots, n.
\]

Applying the necessary condition of \([16]\) to each \( y_i^+ f'(x) \) one derives \( u_i^+ \in P^+(\mathbb{E}, g(x)) \) with

\[
    (1) \quad y_i^+(f'(x)) - u_i^+(g'(x)) \in G^+ \quad i = 1, \ldots, n
\]

which can be written as

\[
    K(f'(x)) - T(g'(x)) \in G^n^+
\]

or, since \( K^{-1} \) exists, as

\[
    f'(x) - M(g'(x)) \in K^{-1}(G^n^+). \quad 1
\]
Corollary: If $S = \mathbb{R}^{n^+}$, $K$ can be chosen as $I$ and one derives
\[ f'(x) - Tg'(x) \in \mathbb{R}^{n^+}, \quad T \in \mathbb{R}^{(B,g(x))^{n^+}}. \]
In particular one obtains for $(M_1)$
\[ (M_1) \min f(x) \text{ subject to } g_i(x) \geq 0, \quad x \in \mathbb{R}^{k^+} \]
where $f: \mathbb{R}^k \rightarrow \mathbb{R}^m$ and $g_i: \mathbb{R}^k \rightarrow \mathbb{R}$ are supposed continuously differentiable. Minimization is with respect to $S = \mathbb{R}^{n^+}$. Denote
\[ g = (g_1, \ldots, g_m)^T. \]

Theorem: A necessary condition for $x$ to be a strong minimum with respect to $\mathbb{R}^{n^+}$ is that when $x$ satisfies the Kuhn-Tucker constraint condition of [39] one has
\[ f'(x) - Tg'(x) \in \mathbb{R}^{(k^+, x)^{n^+}}, \quad T \in \mathbb{R}((B, g(x))^{n^+}). \]
When $f$ is pseudoconvex with respect to $\mathbb{R}^{n^+}$ at $x$ this is also sufficient if $A - x < F(A, x)$. 

Proof: The Kuhn-Tucker constraint qualification gives $G = \mathbb{R}^{(k^+, x)}$. The closure conditions of [16] are met since all the cones involved are polyhedral. Note that since the spaces are finite dimensional weak and strong tangent cones coincide. Thus this necessary condition is just a special case of [39]. Sufficiency follows from [37] since conditions (1) and (2) hold because of the choice of $G$ and because $\mathbb{R}^{k^+}$ is convex.

The last result of this section can be used to recast Fritz John type results for weak minima in operator form.

Theorem: Suppose $G$ is a closed convex cone in $Y$ and that
\[ p^+ \in S^+, u^+ \in F^+((B, g(x))) \text{ exist with} \]
\[ p^+(f'(x)) - u^+(g'(x)) \in G^+. \]
Then there exist $T_1 \in S^s$, $T_2 \in F(B,g(x))^s$, $T_3 \in G^s$ with
\[ T_1(f'(x)) - T_2(g'(x)) = T_3 \quad (T_1, T_2) \neq 0. \]

**Proof:** Define $T_1(y) = sp^+(y)$, $T_2(y) = su^+(y)$, $T_3 = T_1(f'(x)) - T_2(g'(x))$ where $s \in S / \{0\}$.

Section Five: One Sided Derivatives

One drawback to Zlobec's notion of asymptotic consistency (see [17]) is that it is necessary for the functions involved to be linear and continuous. There are many functions which are not differentiable but which have an associated non-linear variation for which optimization results can be framed.

[42] **Definition:** Let $f: X \to Y$ with $X, Y$ vector spaces. The one sided (Gateaux) derivative of $f$ at $x$ denoted $d^+f(x;)$ is defined by
\[
d^+f(x;h) = \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}.
\]

[43] **Proposition:** Suppose $f: X \to Y$ is convex with respect to a closed convex cone $S$ and has a one sided derivative at $x$. Then
\[
d^+f(x;h) = \inf_{t > 0} \frac{f(x + th) - f(x)}{t}
\]
and $d^+f(x;h)$ is convex and positively homogeneous as a function of $h$.

**Proof:** The convex inequality can be written as
\[
(1) \quad f(x + th) - f(x) \geq_s f(x + rh) - f(x) \quad t > r,
\]
which is the force of the first conclusion. Moreover, denoting $d^+_t[f(x + th) - f(x)]$ by $g_t(h)$ it is clear that $g_t$ is convex as a function of $h$. Thus for $0 < r < 1$ and $t > 0$
(2) \( r_{g_t}(h) + (1 - r) g_t(\hat{k}) \geq S g_t(rh + (1 - r) \hat{k}) \).

Taking limits one has the convexity of \( d^+ f(x; h) \) since \( S \) is assumed closed.

Note that this result includes a type of subgradient inequality.

\[ f(x + h) - f(x) \geq_S d^+ f(x; h). \]

[44] It is also apparent that the following proposition holds.

**Proposition**: If \( f: X \rightarrow Y \) is quasiconvex with respect to a closed \( S \) and has a one sided derivative then

\[ d^+ f(x; h) \in -S \; \text{when} \; f(x + h) - f(x) \in -S. \]

[45] As a partial generalization of [43] one has:

**Proposition**: Suppose \( g: X \rightarrow Y \) is \((P)\) strictly quasiconvex or quasiconvex with respect to \( S \). Then when \( d^+ g(x, h) \) exists it satisfies

\[ d^+ g(x; rh + (1 - r) \hat{k}) \leq d^+ g(x; \hat{k}) \]

whenever \( d^+ g(x; h) < d^+ g(x; \hat{k}) \) and \( 0 < r < 1 \).

**Proof**: If \( d^+ g(x; h) - d^+ g(x; \hat{k}) \in -S^0 \) there is some \( t_o \) such that for \( t < t_o \)

\[ g(x + th) - g(x) < g(x + tk) - g(x) \]

which gives \( g(x + th) < g(x + tk) \) \( t < t_o \).

Using either of the convexity hypotheses

\[ g(x + rth + (1 - r)tk) \leq g(x + tk) \; \text{for} \; 0 < r < 1 \]

which in turn gives

\[ d^+ g(x; rh + (1 - r)\hat{k}) \leq d^+ g(x; \hat{k}) \]

The simplest example of a function with a convex one sided...
The derivative is given by \( f'(r) = |r| \) for which \( d^+f(0; h) = |h| \).

The next results give some tangent cone relationships for one-sided derivatives. For simplicity the spaces are taken to be convex throughout.

**Proposition**: Let \( f: X \to Y \) have a one-sided derivative at \( x \) which is uppersemicontinuous with respect to \( S \). Suppose \( x \) is a strong (local) minimum for \( f \) over \( A \subset X \). Then when \( A \) is convex

\[
d^+f(x; h) \in S \quad \forall h \in P(A, x).
\]

**Proof**: Since \( A \) is convex and \( x \) is a minimum over \( A \)

\[
f(x + t(y - x)) \preceq f(x) \quad \forall y \in A
\]

and

\[
d^+f(x; y - x) \succ 0 \quad \forall y \in A.
\]

Since \( d^+f(x; h) \) is positively homogeneous in \( h \)

\[
(1) \quad d^+f(x; t(y - x)) \succ 0 \quad t \succ 0, \quad \forall y \in A.
\]

Let \( h \in P(A, x) \). Then there is a net \( h_n \to h \) with \( h_n = t_n(x_n - x) \), \( t_n \succ 0, \quad x_n \in A \). Using (1) and the uppersemicontinuity of \( d^+f(x; h) \), \( 0 \leq \lim d^+f(x; h_n) \leq d^+f(x, h) \)
since \( S \) is closed. Thus one is done.

**Proposition**: Suppose that in the hypotheses of [46] that rather than \( A \) being convex \( f \) is supposed to be quasiconcave with respect to \( S \). Then

\[
d^+f(x; h) \in S \quad \forall h \in T(A, x).
\]

**Proof**: Let \( h_n = \lambda_n (x_n - x) \to h, \quad x_n \to x, \quad x_n \in A, \quad \lambda_n \succ 0 \).

Then \( f(x + t(x_n - x)) \preceq f(x) \) for \( 0 \leq t \leq 1 \) since \( f(x_n) \preceq f(x) \).

Proceeding as before one derives
\[ d^*f(x; \lambda_n(x_n - x)) = d^*f(x; h_n) \in S. \]

Since \( f \) is uppersemicontinuous and \( S \) is closed \( d^*f(x; h) \in S \).

[48] Corollary: Suppose in \([47]\) that \( d^*f(x; h) \) is concave in \( h \) with respect to \( S \). Then
\[ d^*f(x; h) \in S \quad \forall h \in F(A, x). \]

Proof: Since \( d^*f(x; h) \) is concave and \([47]\) holds
\[ d^*f(x; \tilde{h}) \in S \quad \forall \tilde{h} \in \overline{T(A, x)}. \]
Since \( d^*f(x; h) \) is uppersemicontinuous in \( h \)
\[ d^*f(x; \tilde{h}) \in S \quad \forall \tilde{h} \in \overline{T(A, x)} = F(A, x). \]

[49] Proposition: Suppose \( d^*g(x; h) \) exists and is convex in \( h \) with respect to a closed, convex cone \( B \) with interior and
\[ (1) \quad \exists \tilde{h} \in x \text{ with } d^*g(x; \tilde{h}) + g(x) < 0. \]

Then
\[ \{ h \mid d^*g(x; h) \in F(-B, g(x)) \} \subset F(\Delta, x) \quad (\Delta = g^{-1}(-B)). \]

Proof: Let \( h_t = th + (1 - t)\tilde{h} \quad 0 < t < 1 \). Suppose
\[ d^*g(x; h) \in F(-B, g(x)). \]
There is \( \{ y_n \} \subset -B \) with \( y_n \to g(x) \) and
\[ \lambda_n(y_n - g(x)) \to d^*g(x; h). \]
Since \( d^*g(x; h) \) is convex
\[ d^*g(x; h_t) + (1 - t)g(x) < t^*g(x; h) + (1 - t)d^*g(x; \tilde{h}) +
(1 - t)g(x) < t^*g(x; h). \]
Thus for \( n > n_0 \)
\[ d^*g(x; h_t) + (1 - t)g(x) < \lambda_n(y_n - g(x)) \leq -\lambda_n g(x). \]
Using the definition of the derivative for \( r < r_0 \)
\[ g(x + rh_t) - g(x) \leq (-\lambda_n + t - 1)g(x). \]

Rearranging one has
\[ g(x + rh_t) < g(x) + r(t - (1 + \lambda_n))g(x). \]
Fixing \( n \) at \( n_0 \), if \( r \) is small enough \( r(t - (1 + \lambda_n)) > \frac{1}{2} \) and
\[ g(x + rh_t) \preceq \bar{g}(x) \preceq \circ \]

This means that \( x + rh_t \in g^{-1}(-\mathcal{B}) = \Delta \) and \( h_t \in T(\Delta, x) \).

Letting \( t \to 1 \quad h_t \to h \in T(\Delta, x) \subset \mathcal{F}(\Delta, x) \).

---

**[50] Corollary:** Suppose \( g \) is actually compactly differentiable in \([L9]\) then \( \mathcal{K} = \mathcal{F}(\Delta, x) \).

**Proof:** In this case \( (g'(x))(h) = d^+g(x;h) \) so that \([L9]\) shows that \( g'(x)^{-1}(\mathcal{F}(-\mathcal{B}, g(x))) \subset \mathcal{F}(\Delta, x) \).

The reverse containment was given in \([2.32]\) so that \( g'(x)^{-1}(\mathcal{F}(-\mathcal{B}, g(x))) = \mathcal{F}(g^{-1}(-\mathcal{B}), x) \).

---

With these preliminaries one is in a position to prove a necessary condition which generalizes a problem of Luenberger's (1970).

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**[51] Theorem:** Let \( f: X \to \mathbb{R} \), \( g: X \to Y \), \( B \subset Y \), \( C \subset X \) and let \((P_3')\) be \( \min f(x) \) s.t. \( g(x) \in -\mathcal{B}, x \in C \). Suppose one has

1. \( \mathcal{F}(-\mathcal{B}, g(x))^\circ \neq \emptyset \),
2. \( d^+f(x;h) \) is convex in \( h \),
3. \( d^+g(x;h) \) is convex in \( h \) with respect \( -\mathcal{F}(-\mathcal{B}, g(x)) \),
4. \( h \in \mathcal{F}(\Delta, x) \) when \( d^+g(x;h) \in \mathcal{F}(-\mathcal{B}, g(x)) \),
5. \( A \) is convex or \( f \) is quasiconcave,
6. \( d^+f(x;h) \) is uppersemicontinuous in \( h \).

Then if \( G \) is a closed convex cone with \( G \cap \mathcal{F}(\Delta, x) \subset \mathcal{F}(\Delta, X) \) and there is some \( \tilde{h} \in G \) with \( (7) \ d^+g(x;\tilde{h}) \in -\mathcal{F}(-\mathcal{B}, g(x))^\circ \) a necessary condition for \( x \) to be a minimum for \((P_3')\) is

\[ d^+g(x;h) - u^+(d^+g(x;h)) \preceq 0 \quad \forall h \in G \]

for some \( u^+ \in \mathcal{P}^+(-\mathcal{B}, g(x)) \).

**Proof:** Let \( E = \{ (r, z) \mid d^+f(x;h) \preceq r, d^+g(x;h) - z \in \mathcal{F}(-\mathcal{B}, g(x)), \quad h \in G \} \)
and $F = \left\{(x,z) \mid r \leq 0, \ z \in P(-B,g(x))\right\}$.

Suppose $E \cap F^\circ \neq \emptyset$. Then there is some $h \in G$ with

$$d^+f(x;h) \leq 0, \ d^+g(x;h) \in P(-B,g(x))^\circ.$$  

By (4) $h \in P(\Delta,x) \cap G \subseteq P(A,x)$. Then using (5), (6) and [45] or [46]

$$d^+f(x,h) \not\subseteq 0$$

which is impossible. Thus $E \cap F^\circ = \emptyset$. Moreoer, $E$ is convex since $G$ is convex and (using (2), (3)) $d^+f(x;h)$ and $d^+g(x;h)$ are convex. There is, by the separation theorem for convex sets, a linear functional $z^+ = (\overline{F},u^+) = 0$ with $\overline{F} \supseteq 0$, $-u^+ \in P^+(-B,g(x))$ such that

$$\overline{F}(d^+f(x;h)) + u^+(d^+g(x;h)) \supseteq 0 \quad \forall h \in G.$$  

If $\overline{F} = 0$ then $u^+(d^+g(x;h)) \supseteq 0 \quad \forall h \in G$ which contradicts (7).

$\overline{F}$ can be taken to be $1$ and the theorem is established.

[52] **Corollary:** With $(P_3)$ as in [51] suppose $B$ is a closed convex cone with interior, that $C = X$, and that

(1)' $d^+f(x;h)$ is convex and uppersemi-continuous in $h$,

(2)' Either $f$ is quasiconcave or $\Delta$ is convex,

(3)' $d^+g(x;h)$ is convex with respect to $B$.

Then if $h$ exists with $d^+g(x;\overline{h}) + g(x) \in -B^\circ$ and $x$ is minimal for $(P_3)$ one has

$$u^+(g(x)) = 0, \ d^+f(x;h) + u^+d^+g(x;h) \supseteq 0 \quad \forall h \in G \text{ for some } u^+ \in B^+.$$  

**Proof:** Since $B \subseteq B + g(x) \subseteq -P(-B,g(x)), P(-B,g(x))^\circ \not\subseteq \emptyset$.

(3) is implied by (3)' since $B \subseteq P(B,g(x))$.

(3)' and [49] imply (4) while $G \cap P(\Delta,x) \subseteq P(A,x)$ is satisfied by $G = X$ since $C = X$. Finally $d^+g(x;\overline{h}) + g(x) < 0$ is sufficient to exclude $\overline{F} = 0$ in the proof of [51]. [51] thus gives the
desired result because for $B$ a convex cone

\[ u^+ \in -\mathcal{P}(-B, g(x)) \text{ implies } u^+ \subseteq B^+ \text{ and } u^+(g(x)) = 0. \]

Remarks: (1) If $X = \mathbb{R}$; $B = \{ \frac{1}{n} \} \cup \{ 0 \}$ and $g(x) = 0$ is an example in which $-B^0 = \lambda$ but $\mathcal{P}(-B, 0) = \lambda$ so that condition (1) of [51] is weaker than $B^0 \neq \lambda$.

(2) If $f$ is differentiable (2) of [52] or (5) of [51] can be dropped.

(3) Iuunberger (1969) states [52] without (2) and $d^+ f(x; h)$ uppersemicontinuous in $h$. This can be established in a direct argument but not from [51].

One also has a sufficiency condition for one sided derivatives similar to [37].

Definition: $f: X \rightarrow Y$ is pseudoconvex with respect to $S$ at $x$ over $A$ if

\[ d^+ f(x; y - x) \succ_S 0 \quad \text{implies } f(y) \succ_S f(x) \quad \forall y \in A. \]

Theorem: Suppose there are closed convex cones $G \subset X$; $B \subset Z$ with:

(1) $A \prec_G G$.

(2) $f$ is pseudoconvex over $A$ at $x$ with respect to $S$.

(3) There is some $u^+ \in B^+$ with $u^+(g(x)) = 0$ and

\[ d^+ f(x; h) + u^+(d^+ g(x; h)) \succ 0 \quad \forall h \in G. \]

(4) $u^+g$ is quasiconvex.

Then $x$ is a strong minimum with respect to $S$ for

\[ \min f(y) \text{ subject to } g(y) \preceq_B 0 \quad y \in C. \]

Proof: Let $y \in A$. Using (1) and (3)

\[ d^+ f(x; y - x) + u^+(d^+ g(x; y - x)) \succ 0. \]

If $g(y) \preceq 0$ then $u^+(g(y)) \preceq u^+(g(x)) = 0$. (4) and [54] then
yield
\[
(5) \quad u^+(d^+g(x,y - x)) = d^+u^+(g; y - x) \leq 0.
\]
Combining (4) and (5) produces \( d^+f(\bar{x}; y - x) \geq 0 \) \( \forall y \in A \). Since \( f \) is pseudoconvex \( \bar{x} \) is a strong minimum.

Section Six: Convex Optimization Theorems

Let \((P_{\bar{A}})\) denote
\[
(P_{\bar{A}}) \min f(x) \text{ subject to } g(x) \leq b, \quad h(x) = 0, \quad x \in C \text{ where}
\]
\( f: X \to \mathbb{R}, \quad g: X \to \mathbb{R}, \quad h: X \to \mathbb{R} \) and \( B \) is a closed convex cone with
interior in \( Z \). \( C \) is a convex set in \( X \) a reflexive Banach space.
The functions will all be supposed to satisfy various convexity assumptions. The fundamental utility of subgradients arises from
the following theorem which generalizes some results of Rockafellar's.

[56] **Theorem:** Suppose \( \text{dom} f^0 \neq \emptyset \) and \( f \) is continuous and convex on
\( C = \text{dom} f \). Suppose \( g \) is convex with respect to \( B \), \( g \) is continuous
on \( C \), \( h \) is affine, continuous and open. Suppose that
\[
(1) \quad \exists \ x_1 \in C \text{ with } g(x_1) < 0, \quad h(x_1) = 0,
\]
\[
(2) \quad \exists \ x_2 \in C^0 \text{ with } h(x_2) = 0.
\]
Then a necessary condition for \( \bar{x} \) to be optimal for \((P_{\bar{A}})\) is;
\[
o \in \partial f(\bar{x}) + \partial (u^+g(\bar{x})) + w^+(\partial h(\bar{x})); u^+(g(\bar{x})) = 0
\]
for some \( u^+ \in B^+, \ w^+ \in W^* \).

**Proof:** The hypotheses guarantee an equivalent unconstrained
problem of minimizing
\[
\Psi(x) = f(x) + u^+(g(x)) + w^+(h(x))
\]
for some \( w^+ \in W^*, \ u^+ \in B^+ \) with \( u^+(g(\bar{x})) = 0 \). This is proved in
[5.17].
By the definition of \( \psi(x) \) one has
\[
o \in \partial \psi(x) = \partial (f(x) + u^+(g(x)) + w^+(h(x))).
\]

\( f, u^+g \) and \( w^+h \) are real valued convex mappings which are (lower semi) continuous on \( C^0 \). By a result of Rockafellar's (1966) or from the general theory of maximal monotone mappings
\[
o \in \partial f(x) + \partial u^+(g(x)) + \partial w^+(h(x)) = \partial \psi(x).
\]
Moreover, \( h \) is affine and continuous and thus differentiable. So
\[
\partial (w^+h(x)) = (w^+h)'(x) = w^+(h'(x)) = w^+ \partial h(x).
\]
Combining (3) and (4) gives the result. 

\[57\] Corollary: (1) When \( X, Z, W \) are finite dimensional spaces the continuity results are met automatically if \( C^0 \subseteq (\text{dom} f)^0 \cap (\text{dom} g)^0 \)
\[
(2) \text{when } B = \mathbb{R}^n, \text{and } Z = \mathbb{R}^n \quad \partial u^+(g(x)) \text{ can be replaced by } u^+(\partial_B g(x)).
\]
Proof: (1) is just [1.53] while (2) follows from the definition of \( \partial_B g \).

\[58\] Corollary: If \( B \) is pointed and \( g'(x) \) exists \( u^+(\partial_B g) \) can be used to replace \( \partial (u^+g) \).
Proof: In this case (4) of [56] holds for \( u^+ \) and \( g \).

Corollary [57] (2) is the basic theorem of Rockafellar (1970). There appears to be a general problem in the extension of [56] since it seems that \( \partial (u^+g) \) and \( u^+(\partial g) \) need not in general be equal.

The tangent cone relationships proved in chapter two and in [4.13], [4.15] can be used to derive some more general results about \( (F_4) \).
Theorem: Suppose \( f \) is either differentiable or uppersemicontinuous and concave. Suppose that \( g \) is concave with respect to \( B \) and there is some \( \bar{z} \in \partial g(\bar{x}) \) with
\[
(1) \quad \bar{z}(h) \in P(-B, g(\bar{x})) \Rightarrow h \in P(\Delta, \bar{x}).
\]
\[
(2) \quad \text{For some closed convex cone } G \text{ with } G \cap P(\Delta, \bar{x}) \subseteq P(A, \bar{x})
\]
\[
\bar{z}(\bar{h}) \in P^0(-B, g(\bar{x})) \text{ for some } \bar{h} \in G.
\]

A necessary condition for \( \bar{x} \) to be a minimum for \( (P_4) \) is:
\[
u^+\bar{z} \in \partial f(\bar{x}) \text{ for some } u^+ \in P^+(-B, g(x)).
\]

Proof: By [2.24] \( f'(\bar{x}) \in P^+(A, \bar{x}) \). By [4.13] \( \partial f(\bar{x}) \in P^+(A, \bar{x}) \).

Let \( \bar{y} \in \partial f(\bar{x}) \) or let \( \bar{y} = f'(\bar{x}) \); one can then apply the argument of [51] to \( \bar{y} \), \( \bar{z} \) in place of \( d^+f, d^+g \) to obtain
\[
\bar{y} + u^+\bar{z} = 0 \quad u^+ \in P^+(-B, g(\bar{x})).
\]

Corollary: Suppose \( B^0 \neq B \), \( C = X \) and for some \( \bar{z} \in \partial g(\bar{x}) \)
\[
\bar{z}(\bar{h}) + g(\bar{x}) < 0.
\]

Then the necessary condition in [59] holds.

Proof: This follows from [59] and [4.15] much as [52] follows from [51] and [50].

[59] and [60] give results which are essentially backwards as they give necessary conditions for concave functions' minima and not maxima. The next theorem rectifies this for the constraint function.

Theorem: Suppose in \( (P_4) \) \( f \) is uppersemicontinuous and concave, \( h = 0 \), \( g \) is convex with respect to \( B \) and has a subgradient \( \bar{z} \in \partial g(\bar{x}) \). Then a necessary condition for \( \bar{x} \) to be a minimum in \( (P_4) \) is
\[ \partial f(\bar{x}) \subset P^*(-B, g(\bar{x}))Z + G^* \]

where \( G \) is any closed convex cone with \( \text{K} \cap G \subset P(A, \bar{x}) \).

(\( \text{K} \) denotes \( \{h \mid \bar{Z}(h) \in P(-B, g(\bar{x}))\} \).

**Proof:** Let \( \bar{y} \in \partial f(\bar{x}) \) then by [4.13] \( \bar{y} \in P^+(A, \bar{x}) \).

Since \( \bar{z} \in \partial g(\bar{x}) \) [4.14] guarantees that

\[ Z(P(g^{-1}(-B), \bar{x})) \subset P(-B, g(\bar{x})) \]

One can apply Zlubec's result (which is [17] with strong cones) to deduce that

\[ \bar{y} \in P^*(-B, g(\bar{x}))Z + G^* \]

Note that the assumptions of \( f \) guarantee that \( \partial f(\bar{x}) \neq \emptyset \).
Chapter Six

MULTIPLIER THEOREMS AND MINIMAX THEOREMS
Multiplier Theorems and Minimax Theorems

This chapter is concerned with multiplier theorems for convex and quasiconvex programmes in locally convex spaces. The first section is concerned with various generalizations of a theorem proved by Luenberger (1968) concerning the existence of simpler, but constrained equivalent problems for

\[(P_5) \quad \text{weakmin}_s f(x) \text{ subject to } g(x) \in -B, \ x \in C.\]

where \(f: X \to Y\) is strongly quasiconvex w.r.t. \(S\), a closed convex with interior, \(g: X \to Z\) is convex w.r.t. \(B\), a closed convex cone with interior, and \(C \subset X\) is a convex set.

[1] **Theorem:** Suppose (1) \(f\) is upper semicontinuous on lines w.r.t. \(S\)

\[(2) \quad \exists \ x_1 \in C \text{ with } g(x_1) \in -B^0.\]

If \(x_0\) is a weak minimum for \((P_5)\) there is some \(u^+ \leq B^+ / \mathbb{F}^0\) with \((P_5)\) equivalent to \((P_5')\)

\[(P_5') \quad \text{weakmin}_s f(x) \text{ subject to } u^+(g(x)) < 0, x \in C.\]

**Proof:** Let \(E = \{y | f(x) < u_0 \ ; \ g(x)-y \in -B, x \in C, \}^0\)

where \(u_0 = f(x_0)\).

(i) \(E\) is convex.

Let \(y_1, y_2 \in E\). Then there are \(x_1, x_2 \in C\) with

\[f(x_1) < u_0; f(x_2) < u_0; \ g(x_1) < y_1; \ g(x_2) < y_2.\]

Since \(C\) and \(g\) are convex \(x_\lambda = \lambda x_1 + (1-\lambda) x_2 \in C\) and

\[g(x_\lambda) \leq \lambda g(x_1) + (1-\lambda) g(x_2) \leq \lambda y_1 + (1-\lambda) y_2.\]

Since \(f\) is strongly quasiconvex w.r.t. \(S\), proposition \([18]\) of chapter one gives \(f(x_0) < u_0\) since \(f(x_1) < u_0, f(x_2) < u_0\).

Thus \(\lambda y_1 + (1-\lambda) y_2 \in E\) and \(E\) is convex.

(ii) By construction \(E \cap -B = \emptyset\). Since \(B^0 \neq \emptyset\) the Hahn-Banach Theorem is applicable and there is some non zero \(u^+ \in B^+\) with
\( u^+(y) > 0 \quad \forall y \in E. \)

(iii) Suppose that \( y_2 \notin E \) with \( u^+(y_2) = 0 \). By the second hypothesis \( x_1 \in C \) with \( g(x_1) \notin -B^0 \). Since \( y_2 \notin E \) there is \( x_2 \in C \) with \( g(x_2) \not\preceq y_2 \). Thus for \( 0 < \lambda < 1 \)

\[ u^+(g(\lambda x_1 + (1-\lambda)x_2)) \leq u^+(g(x_1)) + (1-\lambda) u^+(g(x_2)) < 0. \]

Since \( \lambda x_1 + (1-\lambda)x_2 \notin C \), \( g(\lambda x_1 + (1-\lambda)x_2) \notin E \) and

\[ f(\lambda x_1 + (1-\lambda)x_2) \not\preceq -S^0 + u_0. \]

Suppose that \( f(x_2) \in -S^0 + u_0 \). By the first hypothesis

\[ \lim_{\lambda \to 0} f(\lambda x_1 + (1-\lambda)x_2) - f(x_2) \in -S. \]

Thus one would have

\[ \lim_{\lambda \to 0} f(\lambda x_1 + (1-\lambda)x_2) \in -S + (-S^0) + u_0 \subset -S^0 + u_0. \]

Since \( -S^0 + u_0 \) is an open set this would imply that

\[ f(\lambda x_1 + (1-\lambda)x_2) \notin -S^0 + u_0 \quad \text{for } \lambda < \lambda_0 \]

which is impossible. Thus \( f(x_2) \notin -S^0 + u_0 \) and since \( x_2 \) was arbitrary \( y_2 \notin E \). Hence, one has \( y \notin E \) when \( u^+(y) \leq 0 \).

From this one derives that when \( x \in C \) and \( u^+(g(x)) \leq 0 \)

\[ f(x) \not\preceq -S^0 + f(x_0). \]

By the definition of weak minimization and the feasibility of \( x_0 \) one has that \( x_0 \) is a weak minimum for \((P_5')\).

[2] As Iiuenberger notes one need not have complementary slackness, \( u^+(g(x_0)) = 0 \), in \((P_5')\). This is shown by

\[ f(r_1, r_2) = \begin{cases} \ 0 & r_1 + r_2 > 0 \text{ and } g(r_1, r_2) = (r_1, r_2) \\ \neq & r_1 + r_2 \leq 0 \end{cases} \]

for which \((P_5')\) has a minimum of 1 at \((-1, -1)\) among other points.

Setting \( u^+ = (1, 1) \in \mathbb{R}^2 \) one sees that \( u^+ \) satisfies \((P_5')\) but \( u^+(g(x_0)) = -2 \).
There are various results about the slackness of $u^+(g(x_o))$.

The next proposition generalises one of Luenberger's.


1. $\exists \ x_2 \in C$ with $g(x_2) \in -B^o$, $f(x_2) - f(x_o) \in -bS$

or

2. $u^+(g(x_o)) = 0$ for some nonzero $u^+$ satisfying $(P_{5}')$.

Proof: Let $A = \{ y \mid \exists x \in C, y - g(x) \in B^o, \ u_o - f(x) \in S \}^2$

(i) Clearly $A$ is convex.

(ii) If $A \cap B^o \neq \emptyset$ there is some $\overline{x} \in C$ with $f(\overline{x}) \leq u_o$, $g(\overline{x}) < 0$

and if (1) does not hold $f(\overline{x}) - f(x_o)$ cannot belong to

$-S / -S^o = bdS$. This means that $f(\overline{x}) \in -S^o + u_o$ and

$g(\overline{x}) < 0$, $\overline{x} \in C$ which contradicts the definition of $x_o$. Thus

$A \cap B^o = \emptyset$. There is, therefore, a separating hyperplane with

$u^+ \in B^o / \{0\}^2$ and $u^+(A) \geq 0$. It is simple to prove that this implies

that $u^+(E) \geq 0$ and to proceed as in theorem [1] to show $u^+$ satisfies $(P_{5}')$.

Moreover, $u^+ \in B^+ \text{ so that } u^+(g(x_o)) \leq 0$. Since $g(x_o) \in A$

$u^+(g(x_o)) \geq 0$ and must be zero.

[4] Proposition: Suppose that $f$ satisfies the following condition:

1. $x \neq y$ and $f(x) \leq g^f(y)$ implies $f(\lambda x + (1-\lambda)y) \leq g^f(y)$ $0 < \lambda < 1$:

then if the conditions of theorem [1] hold either $g(x_o) \in -S^o$ or

$u^+(g(x_o)) = 0$.

Proof: By [3], if $u^+(g(x_o)) \neq 0$ one has some $x_1 \in C$ with

$g(x_1) < 0$ and $f(x_1) - f(x_o) \in -bS$. This means $f(x_1) \leq f(x_o)$.

If $x_1 \neq x_o$ one has by property (1) that $f(\lambda x_1 + (1-\lambda)x_o) < f(x_o)$ $0 < \lambda < 1$.

Since $\lambda x_1 + (1-\lambda)x_o \in C$ and $g(\lambda x_1 + (1-\lambda)x_o) < 0$ this contradicts the
the minimality of $x_0$.

Note that if $f$ is strictly strongly quasiconvex w.r.t. $S$, $f$ is strongly quasiconvex and satisfies (1).

Continuity conditions can be imposed on $g$ to insure that for some $u^+$ (possibly 0) one does have complementary slackness.

[5] Proposition: Suppose in [4] that $g$ is fully uppersemicontinuous with respect to $F$ and that $x_0 \in C^0$ then either $x_0$ is a global minimum and $u^+(g(x_0)) = 0$ for $u^+ = 0$ or $u^+(g(x_0)) = 0$ and $u^+$ can be taken nonzero.

Proof: By [4] $u^+(g(x_0)) \neq 0 \Rightarrow g(x_0) \notin -C^0$.

Since $x_0$ is assumed to lie in $C^0$ and since $g$ is fully upper semi-continuous there is a neighbourhood $N$ of $x_0$ in $C$ with $g(N) \subset -C$.

Thus

$$f(x) - f(x_0) \notin -C^0 \quad \forall x \in N.$$ 

Since property (1) of [4] is stronger than $(P)$ strict quasiconvexity, proposition [83] of chapter one implies that $x_0$ is a weak global minimum. In this case $(P_0)$ is equivalent to $(P_0')$ with $u^+ = 0$.

These results exclude Luenberger’s example which satisfies all the conditions except the strict quasiconvexity of $f$.

Equality Constraints

[6] Definition: $h \colon X \to W$ is subaffine if $A(a) = \{ x \mid h(x) - a \}$ is convex $\forall a \in W$.

[7] Proposition: $h$ is subaffine if $h$ is any of the following:

(1) affine.
(2) maximal monotone from \( X \) to \( X' \).

(3) quasi-convex and-concave with respect to a pointed cone \( S \).

Proof: (1) is immediate. (2) is a standard result of monotone operator theory. (3) Suppose \( h \) is quasi-convex and-concave.

If \( a \not\in R(h) \) \( A(a) = \emptyset \). If \( a \in R(h) \) then \( a = h(b) \) and

\[
\{ x \mid h(x) = h(b) \} = \{ x \mid h(x) \leq h(b) \} \cap \{ x \mid h(x) > h(b) \} = \emptyset
\]

because \( S \) is pointed.

Since the last two sets are convex \( A(a) \) is convex. \( \square \)

[8] Proposition: Suppose that \( h: X \to W \) is subaffine and that in \([1] \) \( (P_5) \) is replaced by

\[
(P_6) \text{ weakmin}_S f(x) \text{ s.t. } g(x) \leq 0, \ h(x) = 0, x \in C.
\]

Then there is some \( u^+ \in \mathbb{B}^+ / \mathcal{O}_S \) with \( (P_6) \) equivalent to

\[
(P_6') \text{ weakmin}_S f(x) \text{ s.t. } u^+(g(x)) \leq 0, h(x) = 0, x \in C.
\]

Proof: Let \( C'' = C \cap \{ x \mid h(x) = 0 \} \) which is convex and apply \([1] \) to \( (P_5) \) with \( C \) replaced by \( C'' \). \( \square \)

The next result extends \([1] \) to include a finite dimensional affine constraint.

[9] Theorem: Suppose in \([8] \) that \( h: X \to \mathbb{R}^n \) is actually affine and that the following hold:

1. \( \exists x_1 \in C \text{ s.t. } h(x_1) = 0, g(x_1) < 0. \)
2. \( \exists x_2 \in C^0 \text{ with } f(x_2) < u_0 h(x_2) = 0 \text{ and with } h \text{ open at } x_2. \)
3. \( f \) is actually fully upper semicontinuous with respect to \( S. \)

Then there is some \( u^+ \in \mathbb{B}^+ / \mathcal{O}_S \) and \( z^+ \in \mathbb{R}^n \) with \( (P_6) \) equivalent to \( (P_6'). \)

\[
(P_6') \text{ weakmin}_S f(x) \text{ s.t. } u^+(g(x)) + z^+(h(x)) \leq 0, x \in C.
\]
Proof: By [8] \((P_6')\) is equivalent to \((P_6'')\).

Let \(E_1 = \{(x,z) \mid f(x) < u_0, u^+(g(x)) < x, h(x) = z, x \in C\}^c\)
where \(u^+ \in \mathbb{R}^n / \{0\}\) is as in [8].

As in [11], since \(h\) is affine, \(E_1\) is convex.

Set \(P_1 = \{(r,0) \mid r < 0, o \in \mathbb{R}^n\}\). By [8] \(E_1 \cap P_1 = \emptyset\) and one can apply Hahn-Banach since the sets are finite dimensional. This produces \(\bar{r} > 0, z^+ \in \mathbb{R}^n, (r,z) \neq 0\)

\[\bar{r} u^+(g(x)) + z^+(h(x)) > 0 \text{ if } x \in C, f(x) < u_0.\]

Suppose \(\bar{r} = 0\) then

\[z^+(h(x)) > 0 \text{ when } x \in C, f(x) < u_0.\]

By (2) there is an \(x_2 \in C\) with \(f(x_2) < u_0\) and \(h(x_2) = 0\) and with \(h(N)\) a neighbourhood in \(\mathbb{R}^n\) for some \(N\) with \(x_2 + N \subset C\). By (3) there is a neighbourhood \(N_1\) of \(x_2\) with \(f(N_1) < u_0\). Combining these facts one sees that \(z^+(U) > 0\) for \(U = h(N)\) a neighbourhood of \(C\). This implies that \(z^+ = 0\) which is impossible.

Without loss of generality suppose \(r = 1\). One sees that \(u^+(g(x)) + z^+(h(x)) < 0\) when \(x \in C, f(x) < u_0\). One can now proceed as in [11] to derive the promised equivalence. This uses \(u^+ \neq 0\) and (1) in place of \(g(x) < 0\).

The argument of [8] and [9] does not extend to infinite dimensional affine constraints because separation can not be guaranteed. The next result gives an infinite dimensional variation on [9].

[10] Theorem: Suppose in \((P_6')\) that the following conditions hold:

(1) \(h\) is open; \(g\) is continuous and \(f\) is fully upper semi-
continuous. (on \(C\)).

(2) \(C^c \neq \emptyset\) and if \(C \neq X\), \(h\) is continuous.
(3) \( \exists x_2 \in C^0 \) with \( f(x_2) < u_0, h(x_2) = 0 \).

(4) \( \exists x_1 \in C^0 \) with \( g(x_1) < 0, h(x_1) = 0 \).

Then \( u^+ \in B^+/\{0\} \), \( z^+ \in W^+ \) with \((P_6')\) equivalent to \((P_6')\).

**Proof:** Let \( A = \{ (y,w) | f(x) < u_0, x \in C^0, g(x) < y, h(x) = w \} \)

\( = \{ (y,w) | g(x) < y, h(x) = w, x \in C^0 \cap f^{-1}(u_0 - S^0) \} \).

The full upper semicontinuity of \( f \) guarantees that

\( \bigcup = C^0 \cap f^{-1}(u_0 - S^0) \) is an open convex set. By the construction of \( A \) there is no \( x \) in \( \bigcup \) with \( h(x) = 0 \) and \( g(x) \in -B^0 \).

This means that the transposition theorem of \([5,10]\) is applicable and \( u^+ \in B^+, z^+ \in W^+ \) exist (not both zero) with

\( (5) u^+(g(x)) + z^+(h(x)) > 0 \) if \( x \in C^0, f(x) < u_0 \)

Suppose \( C \neq X \) and \( \overline{x} \in \overline{C} \). Since \( h \) and \( g \) are continuous and \( f \) is upper semicontinuous,

\( u^+(g(\overline{x})) + z^+(h(\overline{x})) \) is still non negative.

As in \([9]\) since \( h \) is open \( u^+ \neq 0 \). Now suppose \( x_2 \in C \) and \( u^+(g(x_2)) + z^+(h(x_2)) = 0 \). Let \( x_2 = \lambda x_1 + (1 - \lambda) x_2 \),

then \( u^+(g(x_2)) + z^+(h(x_2)) < 0 \) \( 0 < \lambda < 1 \) and thus \( f(x_2) \not\leq u_0 \)

by \( (5) \). Since \( f \) is upper semicontinuous \( f(x_2) \not\leq u_0 \) and one has \( x \in C \), \( u^+(g(x)) + z^+(h(x)) < 0 \) implies \( f(x) \not\leq u_0 \) which is the desired statement.

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[11] It is clear that \( x_0 \) is in turn minimal for \((P_6''')\)

\[ \min_s f(x) \text{ s.t. } u^+(g(x)) < 0, z^+(h(x)) = 0, x \in C \]

since if \( f(\overline{x}) < f(x_0) \) and \( \overline{x} \) is feasible for \((P_6''')\) it is feasible for \((P_6^{''})\) which is a contradiction.

[12] The results of the previous section could have been phrased for multivalued convex constraints; in particular the following generalization of \([10]\) holds.
Theorem: Suppose that $g$ in [10] is replaced by a lower semicontinuous multivalued convex mapping $G$ and that (4) is replaced by

$$(4') \exists x_1 \in C^0 \text{ with } h(x_1) = 0, G(x_1) \cap -B^o \neq \emptyset.$$ 

Then $(Q'_6)$ weakmin$_g f(x)$ s.t. $G(x) \cap -B \neq \emptyset$, $h(x) = 0$ is equivalent to

$$(Q'_6) \text{ weakmin}_g f(x) \text{ s.t. } u^+ G(x) + z^+ h(x) \cap R^+ \neq \emptyset$$

for some $u^+ \in B^+/\{0^+_1\}, z^+ \in W'$.

Proof: The transposition theorem [3,10] is still applicable to

$$A_1 = \{(y,w) \mid x \in X, f(x) < u, h(x) = w, [y-G(x)] \cap B^o \neq \emptyset\}.$$ 

The argument is derived in exactly the same fashion as in [10] using the multivalued convexity rather than convexity to show that $u^+(y) + z^+(w) = 0$ implies $(y,w) \not\in A_1$.

It was noted in proposition [94] of chapter one that a real valued multivalued convex function with compact images has a single valued restriction. When $G$ is upper semicontinuous $G(x)$ is compact (Chapter 3) and thus $u^+(G(x))$ is compact and there is a single valued $k(x) \subseteq u^+(G(x))$. This means that $(Q'_6)$ is equivalent to $(Q''_6)$ weakmin$_g f(x)$ s.t. $k(x) + z^+(h(x)) \leq 0$.

Equivalent convex and quasiconvex constraints.

In connection with $(P_5)$ Luenberger noted that if $g$ was only assumed quasiconvex (with respect to the orthant in $R^n$) that there would generally be an equivalent convex constraint. The discussion below makes this more precise.

[13] Proposition: Let $Y$ be any real sequence space with index $I$.
Suppose $g:X \to Y$ is strongly quasiconvex w.r.t. the coordinate
ordering then each coordinate mapping \( g_i(x) \) is quasiconvex.

**Proof:** \( g(x) \leq z \iff g_i(x) \leq z_i \quad \forall i \in I. \)

Suppose \( g_i(x) < z_i \) and \( g_j(y) < z_j \) \( z_i, z_j \in R. \)

Then \( g_i(x) \leq \max(g_i(x), g_j(y)) = z_i \quad i \neq j, \)
similarly \( g_j(y) \leq z_j. \)

Let \( z = \sum_{i=1}^I z_i \). By construction one has \( g(x) \leq z \)
and \( g(y) \leq z \). Since \( g \) is strongly quasiconvex
\( g(\lambda x + (1-\lambda)y) \leq z \), \( 0 \leq \lambda \leq 1 \). From this it is immediate
that \( g_i(\lambda x + (1-\lambda)y) \leq z_i \), \( 0 \leq \lambda \leq 1 \), and each coordinate
is quasiconvex. This argument required that all the \( g_i \) are
finite together or infinite together. \]

[4] **Proposition:** Let \( g : X \to R \) be quasiconvex with \( S(0) = \{ x | g(x) \leq 0 \} \)
a closed bounded set with nonempty interior. There is an equiva-
lent convex constraint \( f \) with \( f(x_0) < 0 \) for some \( x_0 \) and with \( f \)
finite everywhere.

**Proof:** Without loss of generality let \( x_0 = 0 \in S^0(0). \)

Let \( S_1 = (S(0), 0) \subset (X, R) \) and let \( s_1 = (0, 1) \in (X, R). \) Let \( C \)
be the cone of lines through points in \( S_1 \) with vertex \( s_1. \)

1. **C is a closed convex set:** \( C = \{ (s, \lambda s) | s \in S(0), \lambda \geq 0 \}. \)

From the expression \( C \) is obviously convex. \( C \) is closed because
\( S(0) \) is closed and bounded. Suppose \( c_n = (\lambda_n s_n, \lambda_n s_n - 1) \to c_0. \)
One has \( \lambda_n \to 0 \), \( \lambda_n s_n \to b_0. \) If \( \lambda_0 \neq 0 \) then actually
\( s_n \to b_0/\lambda_0 \in S(0) \) while if \( \lambda_0 = 0 \), \( \lambda_n s_n \to 0 \) since \( S(0) \) is
bounded. In either case \( c_0 \in C. \)

2. **Let** \( f(x) = \min \{ r | (x, r) \in C \}. \)

\( f \) is convex. Since if \( f(x_1) = r_1 \quad f(x_2) = r_2 \)
\((x_1, r_1) \in C_1, (x_2, r_2) \in C \) so that \((\lambda x_1 + (1-\lambda)x_2, \lambda r_1 + (1-\lambda)r_2) \in C\)
and \(\lambda f(x_1) + (1-\lambda)f(x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)\).

(3) \(f(o) = -1\).

(4) If \(f(x) \leq 0\), \(\exists r \leq 0\) with \((x, r) \in C\) since \(C\) is closed.

Thus \((x, r) = (ts, t-1)\), \(0 \leq t \leq 1\) and \(s \in S(o)\).
Since \(o \in S(o)\), \(ts = ts + (1-t)o \in S(o)\) and \(g(t) = g(ts) \leq 0\).

(5) If \(g(x) \leq 0\), \((x, o) \in C\) and \(f(x) \leq 0\).

(6) \(f\) is finite everywhere. By construction \(o \in S(o)\) and \(N(o) \subset S(o)\);
then \(x \in N(o) \Rightarrow (tx, t-1) \in C\) \(\forall t > 0\).

Let \(y \in X\). For \(o < d < d_0\), \(dy \in N(o)\) and \((tdy, t-1) \in C\).

In particular \((y, d^{-1}, 1) \in C\) and \(f(y) \leq d^{-1} - 1\).

The construction could be performed without \(S\) bounded. It would appear, however, that (4) need not hold in that case.

The next result guarantees the existence of equivalent convex constraints for a class of quasiconvex constraints.

[15] Theorem: Let \(Y\) be a sequence space indexed by \(I\). Let \(Y\) have the coordinate ordering \(\theta\), and let \(G: X \to Y\) be strongly quasiconvex w.r.t. \(G\). Suppose for each \(i\) that

(1) \(\sum x_i | g_i(x) \leq 0\) is closed and bounded with a common interior point \(a_0\).

(2) \(|g_i(x)| < \infty \iff |g_j(x)| < \infty\).

Then there is a convex mapping \(F: X \to Y\) with

\(\sum x_i | F(x) \leq 0\) = \(\sum x_i | g(x) \leq 0\).

with \(F\) everywhere finite and with \(F(a_0) < 0\).

Proof: \(G(x) \leq 0 = x \cap \bigcap \sum x_i | g_i(x) \leq 0\). By (2) and the strong quasiconvexity of \(G\) one has, using [15], that each \(g_i\) is quasiconvex. [14] and (1) now imply the existence of \(F\) convex.
everywhere defined with
$\{ x | f_1(x) \leq 0 \} = \{ x | g_1(x) \leq 0 \} ~ \text{and with} ~ f_1(a_0) < 0.$

Setting $\overline{P}(x) = \frac{\epsilon}{\xi} f_1(x)^2$ one has the desired constraint.

[16] **Remarks:** (1) Condition (2) implies no loss of generality since if $A = \bigcap \{ x \mid g_1(x) \leq 0 \}$ one can redefine $\overline{g}_1(x)$ by

$\overline{g}_1(x) = \sum_{x \in A}^\infty \overline{g}_1(x) \quad \forall x \not\in A$

$\overline{G}(x)$ then satisfies (2) and $\{ x \mid G(x) \leq z \} = \{ x \mid \overline{G}(x) \leq z \} \quad \forall z$

(2) $\{ x \mid G(x) \leq 0 \}$ can be closed without $\{ x \mid g_1(x) \leq 0 \}$ closed

as is seen by $G(x) = (g_1(x), g_2(x))$ with $g_2(x) = |x - \frac{1}{2}|$ and

$\overline{g}_1(x) = \sum_{0}^\infty \{ |x| \leq 1 \} \cdot \{ |x| < 1 \}$

The closedness of each component level set

would be guaranteed by the semicontinuity of $G$.

At least in the cases covered by [15] equivalent convex constraints exist. This does not mean necessarily that $u^+ F$ and $u^+ G$ are equivalent.

**Convex Multiplier Theorems**

By requiring in $(P_5)$ that $f$ be convex w.r.t. $S$ one returns to the standard problem of convex programming. Moreover, it is clear that any convex $f$ is upper semicontinuous on lines w.r.t. $S$ so that [7] still applies to any convex function.

The first result of this section gives a multiplier theorem for infinite dimensional affine constraints.

[17] **Theorem:** Let $X, W, Z$ be convex spaces and $B \subseteq Y$ a closed convex cone with interior. Suppose the following hold:

$(1)$ $C \subseteq X$ is convex with interior,
(2) $f: C \subseteq X \rightarrow R$ is convex and continuous on $C$,

(3) $g: C \subseteq X \rightarrow Z$ is convex w.r.t. $B$ and continuous on $C$,

(4) $h: X \rightarrow W$ is open and affine and is continuous if $C \neq X$,

(5) $\exists x_1 \in C$ with $g(x_1) \in -B^0$, $h(x_1) = 0$,

(6) $\exists x_2 \in C^0$ with $h(x_2) = 0$.

Then

$$\min \{ f(x) : g(x) \in -B, h(x) = 0, x \in C \}$$

is equivalent to:

$$\min \{ f(x) + u^+(g(x)) + z^+(h(x)) : x \in C; u^+ \in B^*, z^+ \in W^* \}.$$ 

Moreover, if the infimum $u_o$ is achieved in (P7) at $x_o$ then

$$u^+(g(x_o)) = 0.$$ 

**Proof:** Let $f(x) = (f(x)) - u_o$, $g(x)$ and $S = R^+ \times B$.

The Transposition theorem [3.10] can be applied to $f, h$ and $S$.

There is no solution to $h(x) = 0, f(x) \in -S^0$ and $x \in C^0$ since $f(x) \in -S^0$ $\iff$ $f(x) \prec u_o$ and $g(x) \in -B^0$.

Thus there is $\overline{u} = (\overline{u}, u^+) \in (R^+, B^+)$, $z^+ \in W^*$ with $(u^+, z^+) \neq 0$ and

(7) $\overline{f}(f(x)) + u^+(g(x)) + z^+(h(x)) \geq \overline{f}(u_o) \ \forall x \in C^0$.

If $f$ and $u^+$ are 0 then $z^+(h(x)) \geq 0 \ \forall x \in C^0$. Since $h$ is open and by (5) $z^+(N_{x_2}) \geq 0$ where $N_{x_2} = h(N(x_2))$ is a neighbourhood of 0 and $N(x_1) \subseteq C$. This implies $z^+ = 0$ which is impossible.

If $\overline{f} = 0$ and $u^+ \neq 0$ (5) gives some $x_1 \in C$ with $u^+(g(x_1)) + z^+(h(x_1)) = u^+(g(x_1)) < 0$ in contradiction with (7).

Thus $\overline{f} \neq 0$ and w.l.o.g. $\overline{f} = 1$.

Suppose $\{ x_r \}$ is a feasible net with $f(x_r) \rightarrow u_o$. (7) gives

$$f(x_r) + u^+(g(x_r)) + z^+(h(x_r)) \geq u_o.$$ 

Moreover, since $x_r$ is feasible $g(x_r) \in -B, h(x_r) = 0$ and
\[(8) \ f(x_0) \geq \ f(x_0) + u^+(g(x_0)) + z^+(h(x_0)) \geq u_0\]

and \(\inf_{x \in C} f(x) + u^+(g(x)) + z^+(h(x)) = u_0.\)

If the infimum is achieved at \(x_0\), (8) shows that \(u^+(g(x_0)) = 0.\)

[18] **Remark:** If in \((P_7)\) the affine constraint is finite dimensional
the theorem can be proved without most of the continuity condi-
tions analogously to the quasiconvex theorem [9] but using the
next theorem rather than [1].

[19] **Theorem:** Let \(f: C \times X \to R\) be convex and let \(G: C \times X \to Z\) be con-

vex as a multivalued map with respect to a closed convex cone \(B\)

with interior and let \(C \subset X\) be convex. Let \((P_8)\) denote:

\[(P_8) \ u_0 = \inf f(x) \ s.t. \ G(x) \cap -B \neq \emptyset, \ x \in C.\]

If there is some \(x_1\) such that \(G(x_1) \cap -B^{0} \neq \emptyset\) \((P_8)\) is equiva-

lent to:

\[\inf_{x \in C} f(x) + u^+(g(x)) = u_0.\]

If the infimum is attained there is some \(y_0 \in G(x_0)\) with

\(u^+(y_0) = 0.\)

**Proof:** Let \(A = \{(x, z) \mid f(x) \leq r, \ [z - G(x)] \cap B \neq \emptyset, \ x \in C\}^C\)

(1) \(A\) is convex. Let \((x_1, z_1), (x_2, z_2) \in A\). There are \(x_1, x_2 \in C\)

with (i) \(f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \leq \lambda x_1 + (1-\lambda) x_2\)

(ii) \(\lambda x = \lambda x_1 + (1-\lambda) x_2 \in C\) (iii) \([z_1 - G(x_1)] \cap B \neq \emptyset\) so there

are \(y_1, y_2\) with \(y_1 \in G(x_1)\) and \(z_1 \geq y_1.\) Since \(G\) is m.v. con-

vex there is some \(y_\lambda\) in \(G(x_\lambda)\) with \(y_\lambda \leq \lambda y_1 + (1-\lambda)y_2\)

and \([\alpha x_1 + (1-\lambda) x_2 - G(\lambda x_1 + (1-\lambda) x_2)] \cap B \neq \emptyset.\) This establishes

that \(A\) is convex.

(2) By construction there is no \((r, z)\) in \(A\) with \(r < u_0\) and
\( z \in -B^0 \). Again the Hahn Banach theorem guarantees an \( \overline{r} > 0 \) and \( u^+ \in B^+ \) with:

\[
\overline{r}(x) + u^+(z) \geq \overline{r}(u^o) \quad \text{if } (x,z) \in \Lambda.
\]

Since in particular for any \( z \in G(x) \) \((f(x),z) \in \Lambda\) when \( x \in C \) one has

\[
\overline{r}(x) + u^+(z) \geq \overline{r}(u^o) \quad \forall x \in C.
\]

If \( \overline{r} = 0 \) then \( u^+(G(x)) \geq 0 \) \( \forall x \in C \) which contradicts \( x \in C \) and \( g(x_0) \cap -B^0 \neq \emptyset \). Thus \( \overline{r} > 0 \) and can be taken to be 1.

Then

\[
\inf_{x \in C} f(x) + u^+(G(x)) > u^o
\]

and they are in fact equal since there is some net \( \{ x^i \}_i \subset C \) with \( f(x^i) \to u^o \) and \( g(x^i) \cap -B \neq \emptyset \). Let \( y^i \in -B_{G}(x^i) \) then

\[
f(x^i) \geq f(x^i) + u^+(y^i) > u^o.
\]

and the infima are the same. If the infimum is attained at \( x_0 \), this means \( u^+(G(x_0)) \geq 0 \). For some \( y_0 \in G(x_0) \), however, \( y_0 \not\in -B \) and thus \( u^+(y_0) = 0 \).

It is apparent that [17] could also have been phrased with a multivalued constraint. The proof of multiplier theorems with equality constraints could be managed by applying [19] to

\[
\inf f(x) \quad \text{s.t.} \quad u^+(G(x)) + s^+(h(x)) \leq 0, \quad x \in C
\]

under the conditions of [9] or [10]. To see this it is only necessary to note that since \( u^+ \neq 0 \) the point \( x_1 \) of [7], [10] satisfies \( u^+ g(x_1) + s^+ h(x_1) \leq 0 \) and to remark that \( u^+ g^+ s^+ h \) is convex. One recovers complementary slackness in [19]. Finally, the conditions (2) of [9] and (3) of [10] are only necessary to guarantee the constraint \( g \) is active. If \( g \) is not active the result is still true but is not deducible from [9] or [10].
Luenberger (1969) claims in a problem that the theorem of remark [16] (and so analogously [9]) holds without using in condition (2) that $x \in C^0$. This seems unlikely.

[20] Remarks: The continuity conditions on $f$ and $g$ can be weakened in [17] by applying [19] (which is in the single valued case just the standard result) to (P) deducing the existence of $u^+ \in B^+$ with $u^+(g(x_0)) = 0$ and $u_0 = \inf f(x) + u^+(g(x))$ s.t. $h(x) = 0$, $x \in C$. The transposition theorem in [17] is then applied to $\bar{F}(x) = (f(x) + u^+(g(x)) - u_0)$ and $S = R$. The statement of [17] could then be weakened by replacing (2),(3) by the condition that $f + u^+g$ is continuous on $C$. This has the drawback of using in the hypotheses the multiplier whose existence is desired.

A simple minimization theorem for problems with non-affine equality constraints in proved below. It seems worth including because it involves several previously discussed concepts.

[21] Theorem: Suppose $f:X \to Y$ is fully upper semicontinuous and (P) strictly quasiconvex with respect to $S$. Suppose that $h:X \to Z$ is a Fréchet differentiable map between Banach spaces with $R(h'(\bar{x})) = Z$ and with $h$ continuously differentiable in a neighbourhood of $\bar{x}$. Suppose $\bar{x}$ is weakly minimal for

$$(P_8) \quad \min f(x) \text{ subject to } h(x) = 0$$

Then $\bar{x}$ is also minimal for

$$(P_8') \quad \min f(x) \text{ subject to } (h'(\bar{x}))(x - \bar{x}) = 0.$$ 

Proof: Since $f$ is (P) strictly quasiconvex and fully upper semicontinuous [2,25] shows that $\bar{x}$ is minimal for $f$ over $\bar{x} + T(N(h), \bar{x}) = \bar{x} + \tilde{N}(h'(\bar{x}))$. 


where the equivalence follows from the discussion of regularity in [2.40]. Since \( y \in N(h'(x)) \) exactly when \( h'(x)(x - x) = 0 \) the theorem is established.

[22] **Corollary:** If \( f \) is actually real valued continuous and convex then \( x \) is a minimum of \( f(x) + z^+(h'(x))(x - x) \) for some \( z \in Z' \).

**Proof:** The transposition theorem [3.10] guarantees, since \( h'(x) \) is surjective and hence open, that for some \( z \in Z', r > 0 \)

\[
    r(f(x) - f(x)) + z^+(h'(x))(x - x) > 0 \quad \forall x \in X.
\]

\( r \) can be taken non zero by [3.12]. Since \( x \) is feasible and

\[
    f(x) + z^+(h'(x))(x - x) > f(x)
\]

the result is proved.

**Convex Programs with \( f: X \to Y \)**

The preceding theorems on convex multipliers can be adapted to prove results when \( f: X \to Y \) is convex w.r.t. \( S \) and \( x_0 \) is a weak minimum for (P) with respect to \( S \).

[23] **Proposition:** Let \( f: X \to Y \) be convex w.r.t. \( S \), \( S^0 \neq \emptyset \) and \( A \) be a convex set. If \( x_0 \) is a weak minimum for \( f \) over \( A \) there is some \( s^+ \in S^+ / T^0 \) with \( x \) minimal for \( s^+(f(x)) \) over \( A \). \( s^+f \) is clearly still convex.

**Proof:** Let \( E = \{ x \in x + f(x_0), x \in A \} \). \( E \) is a convex set and is separated from \( - S^0 \) because \( x_0 \) is a weak minimum. The Hahn-Banach Theorem then guarantees some non zero \( s^+ \) with

\[
    s^+(z) > 0 \quad s^+(-s) \quad \forall z \in E, \quad s \in S
\]

which means that \( s^+ \in S^+ / T^0 \) and \( s^+(f(x)) > s^+(f(x_0)) \quad \forall x \in X \).
[24] **Theorem:** Suppose \( x_0 \) is a weak minimum with respect to \( S \) for

\[
(P_0) \quad \min f(x) \quad \text{s.t.} \quad g(x) \in -B, \quad x \in C
\]

with \( f:X \to Y \) convex w.r.t. \( S \), \( g:X \to Z \) convex w.r.t. \( B \), \( B^0 \neq \emptyset \),
and \( C \subset X \) a convex set. Suppose there is some \( x_1 \in C \) with

\[ g(x_1) \in -B^0. \]

Then there is some \( T \in B^0 \) with \( Tg(x_1) = 0 \) such that \( x_0 \) is a weak minimum for

\[
f(x) + Tg(x) \quad \text{s.t.} \quad x \in C.
\]

**Proof:** [23] and [19] give in conjunction the existence of

\[ u^+ \in B^+, \quad s^+ \in S^0/\{0\} \]

with \( u^+(g(x_0)) = 0 \) and

1. \( s^+(f(x_0)) \leq s^+(f(x)) + u^+(g(x)) \) if \( x \in C \).

Let \( s \in S^0 \) then \( s^+(s) > 0 \) since \( s^+ \neq 0 \).

Define \( T \) by \( T(x) = s^+(s)^{-1}s u^+(x) \). Clearly \( T(B) \subset S \) and \( T \) is continuous. Hence \( T \in B^0 \). Equally clearly \( T(g(x_0)) = 0 \).

Suppose now that there is some \( x \in C \) with

2. \( f(x) + T(g(x)) \leq f(x_0) + T(g(x_0)) \).

Then

3. \( s^+(f(x) + T(g(x)) < s^+(f(x_0)) + s^+(T(g(x_0))) = s^+(f(x_0)) \)

but \( s^+T = u^+ \) and (3) contradicts (1). Thus \( T \) satisfies the claims of the statement.

It is worth remarking that the same trick could be performed in chapter four to rewrite the Fritz John Theorems on weak minimization in operator form.

**Strong minima:**

Turning now to strong minima, one faces much more of a problem in obtaining multiplier theorems. The only easy avenue appears to lie in application of the generalized Kuhn-Tucker results of the last chapter. One case in which direct theorems can be
proved does exist and this is dealt with first. The problem is still \((P_0)\) of \([23]\).

**Proposition:** \(f: X \to Y\) has a strong minimum (with respect to \(S\) a closed convex cone) over \(A\) at \(x_0\) if and only if \(s^+f\) has a minimum at \(x_0\) for all \(s^+ \in S^+\).

**Proof:** \(f(x) \geq s f(x_0) \quad \forall x \in A \quad \Rightarrow \quad s^+(f(x)) \geq s^+(f(x_0)) \quad \forall x \in A\). Conversely if \(s^+(f(x)) \geq s^+(f(x_0)) \quad \forall x \in A, s^+ \in S^+\) then

\[
(f(x) - f(x_0) \leq (s^+)^+ \quad \forall x \in A
\]

which since \(S\) is closed gives \(f(x) \geq f(x_0) \quad \forall x \in A\).

**Theorem:** Let \(f: X \to \mathbb{R}^n\) be convex w.r.t. a closed convex pointed cone \(S\). Let \(g: X \to \mathbb{R}\) be convex w.r.t. \(B\), a closed cone with interior. Let \(C \subseteq X\) be convex and suppose some \(x_1 \in C\) has \(g(x_1) < 0\). Let \(S_1 \supseteq S\) be defined by

\[
x \in S_1 \iff Kx \in \mathbb{R}^{n^+}
\]

where \(K\) is an invertible matrix with rows in \(S^+\). Suppose \(x_0\) is a strong minimum with respect to \(S\) for \((P_0)\). Then \(x_0\) is a strong minimum with respect to \(S_1\) for

\[
f(x) + T_0(g(x)), \quad x \in C
\]

where \(T_0 \in S_1^\perp\) and \(T_0(g(x_0)) = 0\).

**Proof:** Since \(S \subseteq \mathbb{R}^n\) \(S\) is pointed \(\iff (S^+)^0 \neq \emptyset\). Thus there is at least one such matrix \(K\) and cone \(S_1\). By \([25]\) \(x_0\) is minimal for

\[
(1) \quad \min s_i^+(f(x)) \text{ subject to } g(x) \in -B, x \in C, s_i^+ \in S^+, i = 1, \ldots, n.
\]

[19] can be applied to \((1)\) since \(s_i^+f\) is convex. One derives that \(x_0\) is minimal for

\[
(2) \quad s_i^+(f(x)) + u_i^+(g(x)) \text{ s.t. } x \in C; u_i^+ \in B^+, u_i^+(g(x_0)) = 0
\]
for $i = 1, \ldots, n$. Letting $s_1^+, \ldots, s_n^+$ be the rows of $K$ and setting $T = (u_1^+, \ldots, u_n^+)^T$ one has

$$K(f(x) - f(x_0)) + Tg(x) \in R^{n^+} \quad \forall x \in C.$$

Since $K^{-1}$ exists this can be written as

$$K \left[ f(x) - K^{-1}Tg(x) - f(x_0) \right] \in R^{n^+} \quad \forall x \in C.$$

But this is just the definition of

$$f(x) - K^{-1}Tg(x) \geq s_i \quad f(x_0) \quad \forall x \in C.$$

Setting $T_0 = K^{-1}T; \quad T_0(g(x_0)) = 0$ and $T_0 \in B_{s_i^+}^1$.

In the case that $S$ is completely determined by $n$ linearly independent constraints the cones $S_1$ and $S$ agree and the unconstrained problem is equivalent to the initial problem. This is certainly true if $S = R^{n^+}$ and in that situation $K$ can be chosen to be the identity.

If in [26] the minimum was only required to be weak one could require the existence of $(s_1^+, \ldots, s_n^+)$ linearly independent in $S^+$ with $x_0$ minimal for $s_i^+f(x)$ s.t. $g(x) \in -E, x \in C$.

It is apparent, from the last theorem that this is equivalent to $x_0$ being a strong minimum over $S_1$.

The preceding discussion is therefore incomplete unless $S = S_1$. More general results can be derived from the differential conditions of the last chapter:

[27] Theorem: Let $X, Y, Z$ be Banach Spaces with $Y$ reflexive and $S \subset Y$ a closed convex normal cone. Suppose $B \subset Y$ is a closed convex cone with interior and that $f:X \rightarrow Y$ is convex on $C$ w.r.t. $S$ and $g:X \rightarrow Z$ is convex on $C$ w.r.t. $B$. Suppose $f$ and $g$ are differentiable at $x_0$ with $g'(x_0)$ surjective. Suppose there is some $x_1 \in C^0$ with $x_1 \in g^{-1}(-B)^0$. 
Then if \( s_0 \) is a strong minimum for \( f \) over \( g^{-1}(-B) \cap C \) w.r.t. \( S \) there is some \( T_0 \in B[Z,Y] \) with \( T_0(B) \subset S \) and \( T_0(g(x_0)) = 0 \) such that \( x_0 \) is a strong minimum w.r.t. \( S \) for
\[
f(x) + T_0g(x) \quad \forall x \in C.
\]

**Proof:** The conditions guarantee that
\[
\text{(1)} \quad P(c,x_0) \cap P(A,x_0) = P(A,x_0). \quad [2.13].
\]
\[
\text{(2)} \quad g'(x_0)^{-1}(P(B,g(x_0))) = P(A,x_0) \quad [5.50].
\]

Thus the generalized Guignard condition of [34] of the last chapter gives \( T_0 \) with the requisite properties such that
\[
\text{(3)} \quad f'(x_0) + T_0(g'(x_0)) \in P(c,x_0)^S.
\]

\( T_0(P(-B,g(x_0)))C - S \) implies \( T_0(B) \subset S \) and \( T_0(g(x_0)) = 0 \) since \( B \) is a convex set and \( S \) is pointed (normal).

Let \( x \in C \), since \( C \) is convex \( x - x_0 \in P(C,x_0) \).

(3) gives
\[
(f'(x_0))(x - x_0) + T_0(g(x_0))(x - x_0) \in S
\]

Because \( f \) and \( g \) are convex one has
\[
\text{(4)} \quad (f'(x_0))(x - x_0) \preceq_S f(x) - f(x_0)
\]
\[
\text{(5)} \quad (g'(x_0))(x - x_0) \preceq_B g(x) - g(x_0).
\]

Substituting these in the previous expression one derives
\[
f(x) - f(x_0) + T_0(g(x) - g(x_0)) \geq_S 0
\]
and since \( T_0(g(x_0)) = 0 \) this is the desired result.

---

[28] **Definition:** In keeping with the terminology for the real valued case \( f(x) + T(g(x)) \) is called a Lagrangian and is denoted \( L(T,x) \).

It is a mapping from \( B(\mathbb{R}^n, \mathbb{R}^m) \times X \to Y \).

It is immediate that whenever a multiplier \( T_0 \) exists, with \( T_0 \in B^S \), \( T_0g(x_0) = 0 \), \((T_0,x_0)\) is a saddle point of the Lagrangian over \( B^S \times C \).
[29] Proposition: The existence of $T_o \in B^g$ with $T_o(g(x_o)) = 0$

such that $f(x_o) = \min_C f(x) + T_o g(x)$ is equivalent to $(T_o, x_o)$

being a saddle point of $L(T, x)$ over $B^g \times C$.

$S$ is supposed to be a pointed cone.

Proof: Suppose $f(x_o) \leq s f(x) + T_o(g(x))$.

Since $T_o(g(x_o)) = 0$ this gives

$L(T_o x_o) \leq L(T_o x)$

Since $T_o g(x_o) = 0$ and $g(x_o) \in -B$ one has, for any $T \in B^g$,

$L(g(x_o)) \in -S \leq T_o(g(x_o))$ and so

$L(T, x_o) \leq L(T_o x_o)$.

Conversely if $(T_o, x_o)$ is a saddle point over $B^g \times C$

$f(x_o) + T g(x_o) \leq f(x_o) + T_o g(x_o) \leq f(x) + T_o g(x)$

from which it is clear that $T_o(g(x_o)) \in S \cap -S = \{0\}$ and $x_o$

is thus minimal over $C$.

[30] Sensitivity:

Theorem: Consider the problems

strong $\min_g f(x)$ s.t. $g(x) \leq z_i, x \in C \quad i = 1, 2$ .

Suppose that multipliers $T_i$ exist for each problem

then

$T_2(z_1 - z_2) \leq s f(x_1) - f(x_2) \leq s T_1(z_2 - z_1)$.

Proof: The multiplier $T_1$ gives

$f(x_1) = f(x_1) + T_1(g(x_1) - z_1) \leq f(x) + T_1(g(x) - z_1) \quad \forall x \in C$.

Setting $x = x_2$ one has

$f(x_1) - f(x_2) \leq T_1(g(x_2) - z_1) \leq T_1(z_2 - z_1)$

since $g(x_2) \leq B^g_2$ and $T(B) \leq S$.

The same argument can be applied to $T_2, x_2$, and produces

the other inequality.
The primal function

[31] Suppose that for \( z \) in some neighbourhood \( N \) of \( 0 \) the problem

\[
\text{strong min}_s f(x) \quad \text{s.t.} \quad g(x) \leq z, \quad x \in C
\]

has solution. Let \( \Pi(z) \) denote this solution. \( \Pi(z) \) is called the \textit{primal function} associated with the given problem. \( S \) is assumed pointed throughout.

[32] \textbf{Proposition:} Suppose \( g: X \rightarrow Z \) is convex w.r.t. \( B \) and \( C \) is convex.

\begin{enumerate}
\item If \( f \) is convex w.r.t. \( S \), \( \Pi \) is convex w.r.t. \( S \) on \( N \).
\item If \( f \) is strongly quasiconvex w.r.t. \( S \), \( \Pi \) is also strongly quasiconvex w.r.t. \( S \) on \( N \).
\end{enumerate}

\textbf{Proof:} Suppose \( f \) is convex. Let \( z_1, z_2 \in N \). Let

\[ f(x_1) = \Pi(z_1); \quad f(x_2) = \Pi(z_2). \]

Then \( tx_1 + (1-t)x_2 \in C, \quad 0 \leq t \leq 1 \)

Since \( f \) is convex \( f(tx_1 + (1-t)x_2) \leq_s t\Pi(z_1) + (1-t)\Pi(z_2). \)

Since \( g(x_1) \leq z_1, g(x_2) \leq z_2 \) and \( g \) is convex w.r.t. \( B \)

\[ g(tx_1 + (1-t)x_2) \leq g(tx_1 + (1-t)x_2). \]

Thus \( tx_1 + (1-t)x_2 \) is a potential solution for \( \Pi(tx_1 + (1-t)x_2) \). Since it is supposed that the strong minimum exists \( (tx_1 + (1-t)x_2 \in N) \)

\[ \Pi(tx_1 + (1-t)x_2) \leq_s f(tx_1 + (1-t)x_2) \leq_s t\Pi(z_1) + (1-t)\Pi(z_2). \]

(2) Suppose \( f \) is strongly quasiconvex. With \( x_1, x_2 \) as before one has, if \( \Pi(z_1) \leq y, \quad \Pi(z_2) \leq y \), that

\[ f(tx_1 + (1-t)x_2) \leq y. \]

As in (1) \( tx_1 + (1-t)x_2 \) is feasible for the problem associated with \( \Pi(tx_1 + (1-t)x_2) \) and this means

\[ \Pi(tx_1 + (1-t)x_2) \leq y. \]

It is clear that if \( z_1 - z_2 \in B \) then \( \Pi(z_2) - \Pi(z_1) \in S \)

(for \( z_1, z_2 \in N \)).
**Definition:** The dual function \( \Phi \) of \( \overline{\Pi} \) defined on \( B^g \) is given by \( \Psi(T) = \text{strongmin}_{x \in C} (f(x) + Tg(x)) \) where as was the case with \( \overline{\Pi} \) the minimum is supposed to exist on some non trivial set \( X = \text{dom} \overline{\Pi} \). Let \( \nabla \) denote \( \text{dom} \overline{\Pi} \).

**Proposition:** If \( f \) is convex then \( \Psi \) is concave and
\[
\Phi(T) = \min_{z \in \nabla} \left[ \overline{\Pi} (z) + T(z) \right] \text{ if it exists.}
\]

The domain of definition may well not be a convex set. It is assumed throughout that definitions of convexity (concavity) are modified to take this into account.

**Proof:** (1) It is simple to verify that \( \Phi \) is concave. Then if \( T \in B^g \) and \( z \in \nabla \)
\[
\Phi(T) = \min_{x \in C} \left[ f(x) + Tg(x) \right].
\]
If \( g(x) \leq z \) then \( Tg(x) \leq T(z) \) and
\[
\Phi(T) \leq \overline{\Pi}(z) + T(z).
\]
Also, if \( z_1 = g(x_1) \)
\[
f(x_1) + Tg(x_1) \geq \overline{\Pi}(z_1) + T(z_1).
\]
This with the definition gives
\[
\Psi(T) \geq \min_{z \in \nabla} \overline{\Pi}(z) + T(z) \geq \min_{z \in \nabla} \Psi(T).
\]

**Theorem:** (Generalized Lagrange duality)

Suppose the conditions of \([27]\) are met and that \( f(x_0) = \overline{\Pi}(0) \).

Then if \( \Psi(T) \) exists on \( K \)
\[
\overline{\Pi}(0) = \max_{T \in B^g \cap K} \Psi(T) = \Psi(T_0)
\]
and \( T_0(g(x_0)) = 0 \).
Proof: \( \Phi(T) = \min_{C} \left[ f(x) + T(g(x)) \right] \) exists on \( K \). Let 

\( T \in B^g \), then if \( g(x) \in E \quad T(g(x)) \in S \) 

and \( \Phi(T) \leq \inf_{s} f(x) \quad \forall x \in C \) with \( g(x) \in E \). 

Thus \( \Phi(T) \leq f(x_o) \).

The result of [24] says that \( \Phi(T_o) = f(x_o) \) and that 

\( T_o(g(x_o)) = 0 \). This concludes the theorem.

The major limitations on this extension are that the domains of definition of \( \Phi, \Pi \) need not be convex even when the conditions of [27] are satisfied. Thus one may well be maximizing over non convex sets.

It is also difficult to verify in general when \( \Phi(T) \) exists even if one knows \( \Phi(T_o) \) exists. The result, however, does indicate that the duality can be extended.

The result of [35] can be reworded as

\[
\min_{C} \Pi(z) = \max_{T \in B^h K} \Phi(T).
\]

Minimax Theorems

It seems natural while examining the various extensions of programming theory from real valued to more general objective functions to investigate minimax theorems. It turns out that the Sion minimax theorem has an entirely adequate extension with the essential and limiting proviso, unlike the real valued case, that both the minimax and maximin have to be assumed to exist.

The proof is derived from Browder's (1963) investigation of fixed points of multivalued mappings.
[36] Theorem: Let $K_1, \ldots, K_n$ be compact convex sets in topological vector spaces $E_1, \ldots, E_n$. Let $f_j: \bigcap_{i=1}^n K_i = K \to Y$ a convex space with a closed convex cone $B$ with nonempty interior. Suppose $\hat{K}_j = \bigcap_{j \notin \hat{i}} K_i$ and 

(1) For each $j = 1, \ldots, n$ $f_j(x_j, \hat{x}_j)$ is fully lower semicontinuous with respect to $B$ on $\hat{K}_j$ for fixed $x_j$ in $K_j$.

(2) $f_j(x_j, \hat{x}_j)$ is strongly quasiconcave with respect to $B$ on $K_j$ for fixed $\hat{x}_j$ in $\hat{K}_j$.

(3) Let $a_1, \ldots, a_n \in Y$. Suppose for each $j$ and $\hat{x}_j \in \hat{K}_j$ there is $y_j \in K_j$ with $f_j(y_j, \hat{x}_j) - a_j \in B^\circ$.

Then there is $u \in K$ with $f_j(u) - a_j \in B^\circ$ $j = 1, \ldots, n$.

Proof: Let $S_j = \{u \in K, f_j(u) - a_j \in B^\circ \}$ then $S_j(x_j) = \{\hat{x}_j \in \hat{K}_j | \hat{x}_j \in \hat{K}_j, f_j(x_j, \hat{x}_j) - a_j \in B^\circ \}$ is open by (1) and $S_j(\hat{x}_j) = \{x_j | x_j \in K_j, f_j(x_j, \hat{x}_j) - a_j \in B^\circ \}$ is convex by (2) and proposition [18] of chapter one.

By (3) $S_j(\hat{x}_j) \neq \emptyset$ if $\hat{x}_j \in \hat{K}_j$.

The sets $S_j, S_j(x_j), S_j(\hat{x}_j)$ then satisfy all the conditions of Theorem 11 of Browder (1968) (which was constructed to prove the case $Y = R$) and the theorem allows one to deduce that some $u \in K$ exists with

$$u \in \bigcap_{j=1}^n S_j$$

This $u$ satisfies the claim of the theorem.

[37] Theorem: Suppose $f$ maps $K_1 \times K_2$ into $Y$ where $K_1$ and $K_2$ are nonempty compact convex sets in separated topological vector spaces $E_1$ and $E_2$. Let $Y$ be a convex space and $B \subset Y$ a pointed
closed convex cone with interior. Suppose that for fixed $y_2$ in $K_2$ $f(x, y_2)$ is fully lower semicontinuous w.r.t. $B$ and strongly quasiconvex on $K_1$. Suppose, similarly, that for fixed $x_1$ in $K_1$ $f(x_1, y)$ is fully upper semicontinuous and strongly quasiconcave on $K_2$. If both minimization and maximization are supposed to be strong with respect to $B$ and

$$\min_{K_1} \max_{K_2} f(x, y) = A_1$$

$$\max_{K_2} \min_{K_1} f(x, y) = A_2$$

then $A_1 = A_2$.

(It is implicit in the hypotheses that the minimizations and maximizations are well defined).

**Proof:** $f(x, y) \leq \max_{y \in K_2} f(x, y)$ and thus

$$\min_{x \in K_1} f(x, y) \leq \min_{x \in K_1, y \in K_2} \max_{x \in K_1} f(x, y) = A_1$$

This clearly implies that if the strong max. of the left hand side exists that $A_2 \leq A_1$.

Set $f_1(x, y) = -f(x, y); f_2(x, y) = f(x, y)$.

Then $f_1, f_2$ satisfy conditions (1) (2) of [36].

Let $a \in B^0$ then

$$\min_{x \in K_1} \max_{y \in K_2} f(x, y(x)) = A_1 = \min_{x \in K_1} f(x, y(x)).$$

For each $x \in K_1 = K_2$, $f(x, y(x)) \geq A_1$ and $f_2(x, y(x)) > A_1 - a$.

Similarly for each $y \in K_2$, $f(x(y), y) < A_2 + a$ and $f_1(x(y), y) > -A_2 - a$.

These points $x(y), y(x)$ satisfy condition (3) of [36].
Since all the conditions are satisfied one has some point
\( u = (x, y) \) with \( f_1(x, y) > -A_2 - a \)
\( f_2(x, y) > A_1 - a, \)
or \( f(x, y) < A_2 + a \quad A_1 - a < f(x, y). \)

Since \( a \in B^c \) is arbitrary this gives
\( A_1 < f(x, y) < A_2 \)
which with \( A_2 \leq A_1 \) and \( B \cap -B = 0 \) implies that \( A_1 = A_2 \).

It is apparent from the theorem that it is too much to hope that the existence of \( A_1 \) is sufficient for \( A_1 = A_2 \) or vice versa. This is clarified by the following example.

Example: Let \( A \) be the matrix
\[
\begin{bmatrix}
[0, 0] & [1, 0] \\
[0, 0] & [0, 1]
\end{bmatrix}
\]
with entries \( a_{ij} \)
in \( R^2 \). Let \( f(x, y) = \sum_{i,j=1}^{2} x_i a_{ij} y_j = x^T A y \) and let
\[
K_1 = \left\{ x \mid x = (x_1, x_2), \quad x_1 + x_2 = 1, \quad 0 \leq x_2 \leq 1 \right\}
\]
\[
K_2 = \left\{ y \mid y = (y_1, y_2), \quad y_1 + y_2 = 1, \quad 0 \leq y_1 \leq 1 \right\}
\]
(a) \( \max \min_{y \in K_2} f(x, y) = \max_{x \in K_1} (0, 0) = (0, 0) \)
(b) \( \min_{x \in K_1} \max_{y \in K_2} f(x, y) = \min_{x \in K_1} \max_{y \in K_2} (y_1, y_2) \)

but this interior maximum does not exist strongly since all the points \( (y_1, 1 - y_1) \) are incomparable. It is clear that all the other conditions of theorem [37] are satisfied since \( f \) is continuous and linear in each variable. In a sense \((0, 0)\) is still a minimax since \( x_2 = 0 \) in (b) is minimal for all \((x_2 y_1, x_2 y_2)\).

Despite this type of drawback it seems worth phrasing the following generalisation of the Von Neumann minimax theorem.
Theorem: Suppose $A$ is a $m \times n$ matrix with entries in a topological vector space $Y$ with a pointed closed convex cone with interior $B \subset Y$.

Let $K_1 = \{ x \mid x = (x_1, x_2, \ldots, x_m), \sum x_i = 1, x_i \geq 0 \}$
and let $K_2 = \{ y \mid y = (y_1, \ldots, y_n), \sum y_j = 1, y_j \geq 0 \}$

Suppose that (1) $\max_{i} a_{ij}$ exists strongly w.r.t. $B$ for each $j$ and (2) $\min_{j} a_{ij}$ exists strongly for each $i$.

Then

$$\min_{x \in K_1} \max_{y \in K_2} x^T y = \max_{y \in K_2} \min_{x \in K_1} x^T y$$

if all the strong optimizations are well defined.

Proof: All the conditions of [37] are met since $x^T y$ is linear and continuous.

The next proposition shows that condition (1) and (2) of [39] are essential.

Proposition: Let $a_1, \ldots, a_n$ be elements of $Y$. A necessary and sufficient condition for

1. strong $\max_{1 \leq j \leq n} \sum_{i=1}^{n} \lambda_i a_i$ to exist, where $C_n = \{ \lambda \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \}$

is for

2. strong $\max_{1 \leq i \leq n} a_i$ to exist.

Proof: The proof proceeds by induction. Suppose $n$ is the smallest natural number such that a set $\{a_1, \ldots, a_n\}$ exists satisfying (1) but not (2). Let $\lambda$ be optimal in (1). Then
\[ \sum a_1 + \ldots + \lambda_{n-1} a_{n-1} + \lambda_n a_n \geq \sum_{i=1}^{n} \lambda_i a_i \quad \forall \lambda \in C_n. \]

This can be rewritten as
\[ \sum_{i=1}^{n-1} \lambda_i (a_i - a_n) \geq \sum_{i=1}^{n-1} \lambda_i (a_i - a_n) \quad \lambda \in C_n. \]

Since \( \lambda_n = 1 \) is impossible, as it would imply \( a_n \) is maximal for (2), one has \( \sum_{i=1}^{n-1} \lambda_i = L > 0 \).

Now in particular (3) holds for any \( \lambda \in C_n \) with \( \sum_{i=1}^{n-1} \lambda_i = L \leq 1 \) or equivalently
\[ \sum_{i=1}^{n-1} L^{-1} \lambda_i (a_i - a_n) \geq \sum_{i=1}^{n-1} \lambda_i (a_i - a_n) \quad \lambda \in C_{n-1}. \]

This means that \( (L^{-1} \lambda_1, \ldots, L^{-1} \lambda_{n-1}) \) is optimal for (1) with the set \( \{a_1 - a_n, \ldots, a_{n-1} - a_n\} \) over \( C_{n-1} \). The induction hypothesis then implies that this set has a largest member with respect to \( B \); that is
\[ (a_j - a_n) - (a_i - a_n) \in B \quad i = 1, \ldots, n-1 \]
and equivalently \( a_j \geq a_i \quad i = 1, \ldots, n-1 \). There is no loss in assuming that \( j = 1 \). One then has that the maximum in (1) must be \( \alpha a_1 + (1-\alpha)a_n \) for some \( 0 \leq \alpha \leq 1 \). The values 0 and 1 can be excluded since \( a_n \) are assumed incomparable. Setting \( \alpha = \alpha + \varepsilon \) where \( \varepsilon > 0 \) is chosen such that \( 0 < \alpha = \alpha + \varepsilon < 1 \) one can deduce from (3) that
\[ \alpha a_1 + (1-\alpha) a_n \geq (\alpha + \varepsilon) a_1 + (1 - (\alpha + \varepsilon)) a_n \]
which gives \( \varepsilon a_n \geq \varepsilon a_1 \) a contradiction. Thus no such set exists and (1) implies (2). The converse is immediate. \( \Box \)
This condition (which is always met in a total ordering) is thus necessary in [39] to even begin looking for a minimax.

It is clear that if there is a saddle point
\[ a_{i_o j_o} \] with \[ a_{i_j o} > a_{i_o j_o} > a_{i_o j} \]
\[ a_{j_o} \leq A_j \]
\[ i_o A \leq i A \]
then it is certainly a minimax. (Here \( A_j, i_A \) denote rows and columns respectively)

Relating Saddle points and Minimax points one has for pointed cones.

[41] Proposition: If the min max \( f(x, y) \) and max min \( f(x, y) \) are defined any minimax point is a saddle point and vice versa.

Proof: If \((x_o, y_o)\) is a saddle point for \( f \) over \( C \times D \)
\[ f(x_o, y) \leq_B f(x_o, y_o) \leq_B f(x, y_o) \]
\[ x \in C, y \in D. \]

\[ \min_C \max_D f(x, y) \leq \max_B f(x, y) = f(x_o, y_o) = \min_C f(x, y_o) \leq \max_D f(x, y) \]

and since the other inequality always hold \((x_o, y_o)\) is a minimax.

The converse is clear. \( \square \)
Chapter Seven

SECOND ORDER CONDITIONS
Second Order Necessary and Sufficient Conditions

When the generalized Kuhn-Tucker conditions are not also sufficient it is possible to have the Lagrangian stationary at a point which is not optimal for the associated problem. In this case further information concerning the nature of optimal points can be extracted by examining the second derivative of the Lagrangian - assuming that it exists.

Second order necessary conditions have been derived by McCormick (1967) and others for the problem \( (P) \) \( \min f(x) \) subject to
\[
\begin{align*}
    g_i(x) &\leq 0 & i = 1, \ldots, n \\
    h_j(x) &= 0 & j = n + 1, \ldots, p
\end{align*}
\]
where all the functions concerned are real valued and defined on \( \mathbb{R}^m \). McCormick introduces a second order constraint condition which he uses in conjunction with the Kuhn-Tucker constraint qualification to derive his necessary condition. The first theorem of this section generalizes this result to a qualification which can be used in conjunction with the Guignard constraint qualification. McCormick's condition is then derived as a special case.

[1] Theorem: Suppose \( f: X \to \mathbb{R}, g: X \to \mathbb{Z} \) are twice compactly differentiable at a point \( x_0 \) which is optimal in a sequential, barrelled space \( X \).

\( (P) \) \( \min f(x) \) s.t. \( g(x) \in B, x \in C \).

Suppose further that the following conditions are met:
(1) The Guignard Necessary condition holds at $x_o$. That is for some closed convex cone $G$ such that $G \cap A \subset P(A, x_o)$ and some $u^+ \in P^+(B, g(x_o))$

$$f'(x_o) - u^+ g(x_o) \in G^+.$$

(2) If $h \in (G \cap K) \cap - (G \cap K)$ then for some nets $\{\lambda_n\} \subset A$, $\lambda_n \to o$ with $x_n \to x_o$ and $h_n = \lambda_n (x_n - x_o) \to h$ there exists a net $\{k_n\}$ with $k_n \in - P(A, x_n)$ and such that $k_n \to h$ and

$$\lim_n \lambda_n (k_n - h) = z \in - G.$$

(3) For this net $\{x_n\}$, $P(B, g(x_n)) \subset P(B, g(x_o))$ if $n \geq n_0$.

(4) $\lim_n (f'(x_n)) (\lambda_n k_n) \to o$.

Then when $h \in G \cap -G$ and $(g(x_o))(h) \in P(B, g(x_o)) \cap - P(B, g(x_o))$

one has

$$(f''(x_o))((h, h)) - u^+(g''(x_o))((h, h)) \to o.$$

**Proof:** $(g''(x_o)) ((h, h)) = \lim_n (g'(x_n) - g'(x_o))(h)$

where $x_n = \lambda_n^{-1} h_n + x_o \to h$ as in (2).

$$\lim_n (g'(x_n) - g'(x_o))(h) = \lim_n \lambda_n^{-1} \left[ (g'(x_o) + \lambda_n^{-1} h_n) (k_n) - (g'(x_o))(h) \right]$$

$$- \lim_n (g'(x_n))(\lambda_n (k_n - h))$$

with $k_n$ as guaranteed by (2).

Thus $(g''(x_o))((h, h)) = \lim_n \lambda_n^{-1} \left[ (g'(x_n))(k_n) - (g'(x_o))(h) \right] - (g'(x_o))(z)$

where the final limit can be taken since $\lambda_n (k_n - h) \to z$ and since $g'(x_n) \to g'(x_o)$.

Now $(g'(x_o))(h) \in P(B, g(x_o))$ by hypothesis and by (2)

$k_n \in - P(A, x_n) \subset - P(A, x_o)$. Using proposition [2.32] one has

$(g'(x_n))(-k_n) \in P(B, g(x_n))$. For $n \geq n_0$ one derives using (3) that

$-(g'(x_n))(k_n) \in P(B, g(x_o))$. 

Since \( P(B, g(x_0)) \) is a closed convex cone and \( (g'(x_0))(h) \) is assumed in \( P(B, g(x_0)) \cap -P(B, g(x_0)) \)
\[
\lambda_n [(g'(x_n))(k_n) - (g'(x_0))(h)] \in -P(B, g(x_0)) \quad n \searrow n_0.
\]
This in turn gives
\[
(5) \quad (g''(x_0))((h,h)) + (g'(x_0))(z) \in -P(B, g(x_0)).
\]
In the same way one derives
\[
(f''(x_0))((h,h)) + (f'(x_0))(z) = \lim_n \lambda_n [(f'(x_n))(k_n) - f'(x_0))(h)].
\]
Since \( h \in G \cap -G \) and \( f'(x_0) - u^+g(x_0) \in G^+ \),
\[
(f'(x_0))(h) - u^+(g(x_0))(h) = 0. \quad \text{Moreover, since } u^+ \in P^+(B, g(x_0)) \text{ and } h \in K \cap -K, \quad u^+(g(x_0))(h) = 0 \text{ which means that } (f'(x_0))(h) = 0.
\]
Condition (4) then gives
\[
(6) \quad (f''(x_0))((h,h)) + (f'(x_0))(z) \succ 0.
\]
Collecting (5) and (6) one has since \( u^+ \in P^+(B, g(x_0)) \)
\[
(7) \quad (f''(x_0))((h,h)) = u^+(g'(x_0))((h,h))
\]
\[
\lambda_n [(f'(x_0))(z) - u^+(g'(x_0))(z)].
\]
Since \( f'(x_0) - u^+g'(x_0) \in G^+ \) and \( z \in -G \) (7) becomes
\[
(f''(x_0))((h,h)) - u^+(g''(x_0))(h,h) \succ 0 \text{ if } h \in G \cap -G \text{ and } \quad (g'(x_0))(h) \in P(B, g(x_0)) \cap -P(B, g(x_0)).
\]
An equality constraint can be incorporated in [1] if \( G \) is specified more closely. This mirrors the situation for regularity conditions in the first order Fritz John conditions.

[2] Theorem: Suppose \( h: X \to W \) is twice differentiable with \( h'(x_0) \) surjective. Suppose \( X \) is fully complete and \( W \) is barrelled.
Suppose that in Theorem [1] \( G = N(h) \) and that
\[
G = P(N(h), x_0) = h'(x_0)^{-1}C. \quad \text{Then if } (h'(x_0))(h) = 0 \text{ and } \quad (g'(x_0))(h) \in P(B, g(x_0)) \cap -P(B, g(x_0)) \text{ one has}
\( \left( f''(x_o) \right)((h,h)) - u^+ \left( g''(x_o) \right)((h,h)) + z^+ \left( h''(x_o) \right)((h,h)) = 0 \)

with \( z^+ \in W^c \) and \( u^+ \) as above.

**Proof:** The Guignard condition (1) \([1]\) now becomes \( h'(x_o)(h) = 0 \)

implies \( f'(x_o)(h) - u^+ \left( g'(x_o) \right)(h) = 0 \). Since \( R(h'(x_o)) = W^c \)

the Farkas Lemma \([3,6]\) can be used to derive \( z^+ \in W^c \) with

\( f'(x_o)(x) - u^+ \left( g'(x_o) \right)(x) + z^+ \left( h'(x_o) \right)(x) = 0, \forall x \in X. \)

As in \([1]\) one can also show

\( \lim \frac{k_n}{m} \left[ \left( h'(x_n) \right)(k_n) - \left( h'(x_o), h \right) \right] = \left( h''(x_o) \right)(h,h) + \left( h'(x_o) \right)(z). \)

By hypothesis \( h'(x_o)(h) = 0 \) and since \( k_n \in -F(A, x_n) \)

\( k_n \in -F(N(h), x_n), \ k_n = \lim \frac{m}{m} \left( x_{nm} - x_n \right) \) with

\( h(x_{nm}) = 0 = h(x_n) \) and \( \lim \frac{m}{m} \left[ h(x_{nm}) - h(x_n) \right] = 0. \)

This last limit is \( -\left( h'(x_n) \right)(k_n) \) so that (2) gives

\( \left( h''(x_o) \right)(h,h) + \left( h'(x_o) \right)(z) = 0. \)

Multiplying (2) by \( z^+ \) and adding it to the equation (7)

of \([1]\) one derives that

\( \left( f''(x_o) \right)(h,h) - u^+ \left( g''(x_o) \right)((h,h)) + z^+ \left( h''(x_o) \right)((h,h)) \)

\( f'(x_o)(z) - u^+ \left( g'(x_o) \right)(z) + z^+ \left( h'(x_o) \right)(z). \)

The right hand side is 0 using (1) which gives the conclusion.

Note that \( z \in -G \) is *not* necessary in this formulation.

It will now be demonstrated that McCormick's condition is

subsumed by \([1]\).

---

**Definition:** Let \( x_o \) be a point satisfying the constraints of \((P_i)\)

and assume \( g_1, \ldots, g_n, h_{n+1}, \ldots, h_p \) are twice differentiable

continuously at \( x_o \). The **second order qualification** (McCormick)

holds at \( x_o \) if the following is true. Let \( y \) be any vector such

that \( \left( g_i'(x_o) \right)(y) = 0 \) for all \( i \in B_o = \bigcap \{ i | g_i \left( x_o \right) = 0 \} \)

and such that
\((h_j'(x_0'))(y) = 0\) \(j = n + 1, \ldots, p\). Then \(y = \alpha'(o)\) where \(\alpha(o)\) is a twice continuously differentiable arc \((e > o)\) along which
\(g_i'(\alpha(o)) = 0\) if \(i \in B_0\) and \(h_j'(\alpha(o)) = 0\) if \(e < e_0\) and with \(\alpha(o) = x_o\).

[4] Theorem: (McCormick) if \(f, g_i, h_j\) \(i = 1, \ldots, n\) and \(j = n + 1, \ldots, p\) are twice differentiable at \(x_o\) and the Kuhn-Tucker and Second order constraint conditions hold at \(x_o\) then a necessary condition for \(x_o\) to be a minimum for \((P_1)\) is that there exist \(u^+ = (u_1^+, \ldots, u_n^+), z^+ = (z_{n+1}^+, \ldots, z_p^+)\) such that \(u_i^+ > o\) and \(u_i^+ g_i'(x_o) = 0\) with

\[f'(x_o) + \sum_{i=1}^n u_i^+ g_i'(x_o) + \sum_{j=n+1}^p z_j^+ h_j'(x_o) = 0\]

and such that for any \(y\) with \((g_i'(x_o))(y) = 0\) \(\forall i\) such that \(g_i(x_o) = 0\) and with \((h_j'(x_o))(y) = 0\), it follows that

\[\left( f'(x_o) + \sum_{i=1}^n u_i^+(g_i''(x_o)) + \sum_{j=n+1}^p z_j^+(h_j''(x_o)) \right)(y, y) > 0 \]

Proof: Set \(C = \mathbb{R}^n, B = \{x|x = (x_1, \ldots, x_n, o_{n+1}, \ldots, o_p) x_i < o i = 1, \ldots, n\}\)

and set \(G = (g_1, \ldots, g_n h_{n+1}, \ldots, h_p)\). It will now be shown that the conditions of [1] hold.

1) The Kuhn-Tucker constraint condition being satisfied by \(\delta\) can be rewritten as: \((g'(x_o))(y) \in P(B, g(x_o))\) implies
\(y = \lim n(\delta'(\frac{1}{n}) - \delta'(0))\) which implies(since \(\delta'(\frac{1}{n}) \in A\) that \(y \in P(A, x_o) = K\) and that \(G\) can be taken as \(R^n\) in [1](2).

2) The second order condition can be written as:
if \((g'(x_o))(y) \in P(B, g(x_o)) \cap P(\mathbb{R}, g(x_o))\) then \(y = \alpha'(o)\)
with \(\alpha(o)\) satisfying the second order qualification. Set
\[ x_n = \alpha\left(\frac{1}{n}\right) \text{ and } \lambda_n = n \text{ then } h = \alpha'(o) = \lim_n (x_n - x_o) \text{ and } \alpha''(o) = z = \lim_n \left[ \alpha\left(\frac{1}{n}\right) - \alpha'(o) \right]. \]

Set \( k_n = \alpha\left(\frac{1}{n}\right) \). Then \( k_n = \lim_m \left[ \alpha\left(\frac{1}{n} - \frac{1}{m}\right) - \alpha\left(\frac{1}{n}\right) \right]. \)

Since \( \alpha(o) \) is contained in the constraint region \( \alpha\left(\frac{1}{n}\right) \) and \( \alpha\left(\frac{1}{n} - \frac{1}{m}\right) \in A \) and \( k_n \in - F(A, x_n) \).

Moreover, from the second order constraint condition one has

\[ F(B, g(x_n)) < F(B, g(x_o)) \quad n > n_o \]

as can be seen by examining components.

Since continuously differentiable mappings from \( \mathbb{R}^m \) to \( \mathbb{R} \) are Fréchet differentiable it only remains to verify (4). This is proved as a separate proposition. I

[5] Proposition: Let \( \alpha \) be a twice continuously differentiable arc

satisfying the second order qualification with \( \alpha'(o) = h \quad \alpha'(o) = z. \)

Setting \( \alpha\left(\frac{1}{n}\right) = x_n, \quad \alpha\left(\frac{1}{n}\right) = k_n \) and \( \lambda_n = n \) [1](4) is satisfied.

Proof: \( (f''(x_o))((h,h)) + (f'(x_o))(z) = \lim_n \lambda_n (f'(x_n) - f'(x_o))(h). \)

Since \( h \) satisfies the constraint condition the above discussion shows that \( (g'(x_o))(h) \in F(B, g(x_o)) \cap -F(B, g(x_o)) \). Because the Kuhn-Tucker condition holds

\[ (f'(x_o))(h) = u^*(g'(x_o))(h) = 0. \]

Let \( \hat{g}(e) = f(\alpha(e)). \) Then \( \hat{g}: \mathbb{R} \to \mathbb{R} \) and has a local minimum at 0 since \( \alpha(e) \in A \) for \( e < e_o \). (See [6] (1).)

Thus

\[ \hat{g}'(e) = (f'(\alpha(e)))(\alpha'(e)) \text{ and } \hat{g}'(o) = (f'(x_o))(h) = 0 \]

while

\[ \hat{g}''(e) = (f'(\alpha(e)))(\alpha''(e)) \bigg|_o + (f''(\alpha(e)))(\alpha'(e), \alpha'(e)) \bigg|_o = (f'(x_o))(z) + (f''(x_o))(h,h) \]

which must be non negative since 0 is a minimum and \( \hat{g}'(o) = 0. \)
Remarks: (1) It is not clear from the statement of the second order constraint condition that the promised arc lies inside the constraint region. An inspection of the definition shows that this follows from the continuity of the finite number of constraints $g_1, \ldots, g_n$. This observation is essential in the proof of the proposition in [5].

(2) McCormick shows that if the set $E$
$$E = \left\{ g_i \mid i \in \{ i \mid g_i(x_0) = 0 \} \cup \{ h_j \mid j = n + 1, \ldots, p \} \right\}$$
is independent the second order condition holds. He gives examples to show that the two conditions are not strictly comparable.

[7] Suppose now that in (P) the objective function $f$ is assumed to map $X$ into $Y$ and that $x_0$ is a strong minimum with respect to a pointed $S$ for (P). The following extension of [1] holds.

Theorem: Suppose that in the statement of [1] the following alterations are made.

(1)' For the given cone $G$ there is some $T \in \mathcal{P}(B,g(x_0))^{-S}$ with
$$\left( f'(x_0) + Tg'(x_0) \right)(h) \in S \quad \forall h \in G.$$

(4)' $\lim (f'(x_n))(\lambda_n x_n) \in S$. Then a necessary condition for $x_0$ to be a strong minimum for (P) is

$$\left( f''(x_0) + Tg''(x_0) \right)((h,h)) \in S \quad \forall h \in G \cap -G \text{ with } (g'(x_0))(h) \in \mathcal{P}(B,g(x_0)) \cap -\mathcal{P}(B,g(x_0)).$$

Proof: With these changes the proof in [1] can be mirrored exactly.

Remarks: It would also be possible to phrase a version of [1] dealing with weak minima. This could be done by considering $a^+ f'(x_0)$ rather than $f'(x_0)$ and by using the same trick as in [2].
of the last chapter to write the condition in operator form.

[9] **Corollary:** (to [1], [7] or [8])

If in condition (2) of [1] or [7] it is required that 
$F(A, x_n) \subset F(A, x_n)$ \quad n \geq n_0$ then (5), (5)' can be replaced by

$\lim_{n \to \infty} f'(x_n)(h) \in S.$

**Proof:** In this case one can choose $k_n = h, \ z = 0.$
Second Order Sufficiency Conditions in Reflexive Normed Spaces

Second order conditions which are sufficient for the existence of a minimum when the first order sufficiency conditions of chapter 5 are not met have been studied by McCormick (1967), Fiacco (1968), Guignard (1969) and Zlobec (1971). The results were all phrased in $\mathbb{R}^n$ or in finite dimensional normed spaces for real valued objective functions. The finite dimensionality was required to employ a compactness argument. By considering weak pseudotangent cones in reflexive normed spaces an infinite dimensional extension can be made. Two definitions are needed first.

[10] Definition: A point $x_0$ is called an isolated intermediate (local) minimum for $f$ over $A$ with respect to a cone $S$ if there is no sequence $\{x_n\} \subset A$ with $x_n \neq x_0$, $x_n \to x_0$ and such that $f(x_n) - f(x_0) \leq -S$. Clearly such a minimum is a weak minimum if $S^\circ \neq \emptyset$.

[11] Definition: A point $x_0$ will be said to have property (F) (with respect to $S$, $f$, $A$) if whenever there is a sequence $\{x_n\}$ in $A/\{x_0\}$ with $x_n \to x_0$, $f(x_n) \leq f(x_0)$ then there is a sequence $x'_n$ such that $x'_n \to x_0$, $x'_n \in A/\{x_0\}$, $f(x'_n) \leq f(x_0)$ and such that $\|x'_n - x_0\|^{-1}(x'_n - x_0)$ has a weakly convergent subsequence with limit $y_0 \neq 0$.

Remark: (1) The substance of property (F) lies in the assertion that $y_0 \neq 0$. This is because any bounded sequence in a reflexive normed space has a weakly convergent subsequence. This subsequence may, of course, have limit zero. This possibility is excluded for $\|x'_n - x_0\|^{-1}(x'_n - x_0)$ by property (F).

(2) If $X$ is finite dimensional every point has property (F) since in
this case \( \| x_n - x_0 \|^{-1} (x_n - x_0) \) has a convergent subsequence \( k_n \) with \( \| k_n \| = 1 \). This means that the limit can not be 0.

(3) Clearly any isolated intermediate local minimum has property (F) since no such sequence \( \{ x_n \} \) can be found initially.

One is now ready to formulate the following extension of Guignard's sufficiency condition.

\[ \text{[12] Theorem: Suppose } f : X \to Y, g : X \to Z \text{ are twice continuously Fréchet differentiable at } x_0 \text{ and that } x_0 \text{ has property (F) with respect to a closed convex pointed cone } S. \text{ Suppose the following conditions hold (where } A = g^{-1}(B) \cap C) : \]

1. \( G \) is a closed convex cone such that if \( x \in A \) and \( \| x - x_0 \| < \varepsilon \) for some \( \varepsilon > 0 \) then \( x - x_0 \in G \).
2. There is some \( s^+ \in S^+ / \{ 0 \} \) and \( u^+ \in \text{wp}^+ (B, g(x_0)) \) with \( s^+ f'(x_0) - u^+ g'(x_0) \in G^+ \).
3. \( \text{wp}(B, g(x_0)) \) is such that if \( \| g(x) - g(x_0) \| \leq \varepsilon \) for some given \( \varepsilon > 0 \) then \( g(x) - g(x_0) \in \text{wp}(B, g(x_0)) \).

Then a sufficient condition for \( x_0 \) to be an isolated local intermediate minimum for \( f \) over \( A \) w.r.t. \( S \) is the following:

For any nonzero element \( h \in G \) such that either

4. \( (i) s^+ (f'(x_0))(h) = 0 \) and \( (g'(x_0))(h) \in \text{wp}(B, g(x_0)) \cap -\text{wp}(B, g(x_0)) \);

or

4. \( (ii) u^+(g'(x_0))(h) = 0 \) and \( (g'(x_0))(h) \in \text{wp}(B, g(x_0)) \cap -\text{wp}(B, g(x_0)) \)

one has \( s^+ f''(x_0)((h, h)) - u^+(g''(x_0))((h, h)) > 0 \).

Proof: Suppose by way of contradiction that there is a sequence \( \{ x_n \} \) with \( x_n \to x_0 \) and \( f(x_n) \leq f(x_0) \).
Let $k_n = \|x_n - x_o\|^{-1}(x_n - x_o)$. Since $x_o$ is assumed to have property (F) there is another sequence with $k_n' = \|x_n' - x_o\|^{-1}(x_n' - x_o)$ such that $k_n' \to k_o \neq 0$. There is no loss of generality in assuming $x_n = x_n'$.

Since $G$ is a closed convex cone satisfying (1) there is some $n_o$ such that for $n \geq n_o$, $x_n - x_o \in G$ which in turn implies that $k_n \in G$. Since $G$ is closed and convex $G$ is weakly closed and $k_o \in G$. Clearly $k_o \in wP(A, x_o) \subset wP(A, x_o)$.

Suppose that $s^+(f'(x_o))(k_o)$ is nonzero. By proposition [32] of chapter two $(g'(x_o))(k_o) \in wP(B, g(x_o))$ since $k_o \in wP(A, x_o)$. Since $k_o \in G$, (2) produces $s^+(f'(x_o))(k_o) > 0$. By proposition [26] of chapter 2

$$s^+(f'(x_o))(k_o) = \lim_{n \to \infty} s^+(f(x_o) + \frac{\|x_n - x_o\|}{\|x_n - x_o\|} k_n) - s^+f(x_o).$$

For $n \gg n_1$ one has $s^+(f(x_n) - f(x_o)) > 0$ which since $s^+ \in S^+$ contradicts $f(x_n) \leq f(x_o)$. Hence $s^+(f'(x_o))(k_o) = 0$.

(i) Suppose now that $(g''(x_o))(k_o) \in wP(B, g(x_o)) \cap -wP(B, g(x_o))$ then

$$s^+(f'(x_o))(k_o) = 0 \quad (4)(i) \text{ is satisfied.}$$

(ii) Suppose next that $(g''(x_o))(k_o) \in wP(B, g(x_o)) / -wP(B, g(x_o))$.

If $u^+(g'(x_o))(k_o) > 0$, (2) produces $s^+(f'(x_o))(k_o) > 0$ which is again a contradiction. Thus $u^+(g'(x_o))(k_o) = 0$.

In either case (4)(i) or (4)(ii) then guarantees that since $k_o \neq 0$

(5) $s^+f'(x_o)(k_o, k_o) - u^+ g''(x_o)((k_o, k_o)) > 0$.

Let $s^+f(x) - u^+ g(x) = L(x)$. Taylor's theorem produces (since $f, g$ are twice continuously differentiable)

(6) $L(x_n) - L(x_o) = L'(x_o)(x_n - x_o) + \frac{1}{2} L''(x_o)(x_n - x_o, x_n - x_o) + o(\|x_n - x_o\|^2)$.

For $n \gg n_o$, $x_n - x_o \in G$ so that $(L'(x_o))(x_n - x_o) \geq 0$. Also, (5) produces $(L''(x_o))(k_o, k_o) > 0$. Since $L''(x_o)$ is continuous, and
hence weakly continuous, in each variable it follows that

\[ (L^*(x_0))(k_n, k_n) > \epsilon_3 \text{ if } n > n_2, \text{ for some } \epsilon_3 > 0. \]

One can derive from (6), therefore, that

\[ \frac{L(x_n) - L(x_0)}{\|x_n - x_0\|^2} > 0 \quad \text{if } n > n_3. \]

This in turn produces

\[ s^+(f(x_n) - f(x_0)) > u^+(g(x_n) - g(x_0)) \quad \text{if } n \geq n_3. \]

From hypothesis (3) it is apparent (since \( g \) is continuous and \( x_n \to x_0 \)) that \( u^+(g(x_n) - g(x_0)) \geq 0 \) if \( n \geq n_4 \).

Thus \( s^+(f(x_n) - f(x_0)) > 0 \) which again contradicts \( s^+ \in S^+ \) and \( f(x_n) - f(x_0) \leq 0 \).

[13] Corollary: (Guignard) Suppose in the statement of [12] that \( f \) is real valued and \( X \) is finite dimensional and that all weak cones are replaced by corresponding strong cones, then one has the Guignard
sufficiency condition for an isolated local minimum.

Proof: Since \( X \) is finite dimensional the unit ball is compact and
\( k_0 \in P(A, x_0) \). Property \((F)\) holds as was remarked in [11] and
\[ (g^*(x_0))(k_0) \in P(B, g(x_0)). \]
Since \( \omega P(B, g(x_0)) \) only enters the theorem through this relationship it can clearly be replaced by \( P(B, g(x_0)) \).

The theorem is then included in [12] since, with \( S = \mathbb{R}^+ \), \( s^+ \) can be taken to be \( 1 \) and since an isolated local minimum is an isolated intermediate minimum in the terminology of [10].

[14] Remarks: (1) It is clear that if \( g^*(x_0) \) is completely continuous, as is the case if \( Z \) is finite dimensional, \( \omega P(B, g(x_0)) \) can be replaced by \( P(B, g(x_0)) \) in [10].

(2) Guignard did not include (4)(ii) in her statement of [13].

The proof given in Guignard (1970) contains the erroneous assertion that if \( (g^*(x_0))(h) \in P(B, g(x_0))/P(B, g(x_0)) \) then \( u^+(g^*(x_0))(h) \) is positive which enables her to exclude the possibility of (ii).
Zlobec's asymptotic generalization (1970) did include (4)(ii) but he still gives Guignard's result as a corollary without it.

The next example shows that McCormick's sufficiency condition is included in [12] with $G = X$.

Example: When one applies [12] to (P_1) one notices that if one replaces $L(x)$ by the reduced Lagrangian,

$$L_1(x) = f(x) + \sum_{i_1>0} \int u_i^+ g_i(x) + \sum_{j=n+1}^p z_j^+ h_j(x),$$

that the proof method can be applied to $L_1(x)$ instead of $L(x)$.

In fact in this case $(g^*(x_o))(k_o) \in P(B,g(x_o)) /\ P(B,g(x_o))$

means that

$$(h^*_j(x_o))(k_o) = 0 \quad j = n+1, \ldots, p \quad \text{and}$$

$$(g^*_i(x_o))(k_o) < 0 \quad \text{for some } i_1 \text{ with } g_i^*(x_o) = 0.$$ 

so that $u_i^+ g^*(x_o) = 0$ implies that $u_i^* = 0$.

Now

$$0 = f^*((x_o)(k_o)) = -\int u_i^+ g^*(x_o)(k_o)$$

so that from this point on in the proof it suffices to examine $L_1(x)$ and the sufficiency condition:

1. If $h \neq 0 \in G$ and $(f^*(x_o))(h) = 0$, $(h_j^*(x_o))(h) = 0 \quad j = n+1, \ldots, p$

and $(g^*_i(x_o))(h) = 0$ if $i \in \{i | u_i^* > 0\}$

then

$$f^*(x_o) + \int_{i_1=1}^n u_i^+ (g_i^*(x_o))(h,h) + \int_{j=n+1}^p z_j^+ (h_j^*(x_o))(h,h) > 0$$

is adequate to prove the result.

Note that in the general case there is no way of separating out inactive multipliers.
Example: McCormick gives the following example to show the second order conditions can isolate behavior that first order conditions do not.

Let \( f(x, y) = (x-1)^2 + y^2 \); \( g(x, y) = x - \frac{1}{k} y^2 \), \( k > 0 \).

Then \( f'(0, 0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \), \( g'(0, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and
\[
\begin{align*}
 f'(0, 0) + 2 g'(0, 0) & \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \forall k > 0.
\end{align*}
\]

For \( k = \frac{1}{2} (0, 0)^T \) is not a local minimum of \( (P_1) \) while for \( k = 3 \) it is. Thus the first order condition is inadequate for the purpose of locating the minima. In the second order condition
\[
(1) \quad f''(0, 0) + 2 g''(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 - \frac{4}{k} \end{bmatrix} = M.
\]

It is easy to verify that all the conditions for the second order sufficiency and necessity theorems are met. The only one worth remarking is that since there is only one constraint Remark [6](2) guarantees the constraint qualification.

The sufficiency result then elicits that for those \( y \) for which \( (g'(x_o))(y) = 0 \) one must have \( y^T M y > 0 \).

These \( y \) are all of the form \((0, a)^T \) and (1) implies that \( 2 - \frac{4}{k} > 0 \) is sufficient for \( x_o \) to be a minimum. Thus \((0, 0)^T \) is a minimum for \( (P_1) \) if \( k > 2 \).

The necessity condition shows similarly that if \( k < 2 \) \((0, 0)^T \) is not a local minimum. For \( k=2 \) the necessary condition holds but the sufficiency condition does not.

Finally, the condition which is lacking for the application of the first order sufficiency condition is the pseudoconvexity of the constraint set. Since \( f \) is convex and \( G = X \) the other conditions are met.

\[
A_k = \left\{ (x, y) \mid y^2 / k \geq x \right\}
\]

is clearly not pseudo-convex at \((0, 0)\) since \( (a_1, a_2) \in P(A_k,(0,0)) \) implies that \( a_1 \leq 0 \) and hence
\[
A_k \notin P(A_k,(0,0)).
\]
Asymptotic second order sufficiency:

[17] The next result extends Zebec's asymptotic version of [13].

Theorem: Suppose in the statement of [12] that (1) and (3) hold and that (2) is replaced by (2'),

\[(2') \lim \left( s^+ f'(x_0) - u^+_1 g'(x_0) \right) (g) = g^+(g), \forall g \in G \]

where \( \{u^+_1\} \) is a sequence in \( w^+(B, g(x_0)) \), \( g^+ \in G^+ \).

In addition

\[(5') \lim \left( s^+ f'(x_0) - u^+_1 g''(x_0) \right) (h, h) = L( ) \text{ exists for all } w \in \overline{W} \]

where

\[ W = \left\{ w \mid w = (z - x_0) \right\} \forall z - x_0 \neq 0, 0 < \| z - x_0 \| < \varepsilon \}

and \( v_k \rightarrow h \Rightarrow L(v_k) \rightarrow L(h) \forall w_k \in \overline{W}, \)

and (4) becomes:

If \( 0 \neq h \in G \) is such that either

\[ 4'(1) \lim \left( u^+_1 g''(x_0) \right)(h) = 0 \text{ and } (g''(x_0))(h) \in \wp(B, g(x_0))/\wp(B, g(x_0)) \]

or

\[ 4'(ii) \left( f'(x_0) \right)(h) = 0 \text{ and } (g''(x_0))(h) \in \wp(B, g(x_0)) \cap \wp(B, g(x_0)) \]

it then follows that

\[ \lim \left( s^+ f''(x_0) - u^+_1 g''(x_0) \right)(h, h) > 0; \]

from which it follows that \( x_0 \) is an intermediate local minimum.

Proof: As before one may assume (using (F)) that \( \{x_n\} \subset A \)

\[ \| x_n - x_0 \|^2 \rightarrow k_0 \rightarrow k_0 \neq 0 \text{ and with } x_n \rightarrow x_0, f(x_n) \leq f(x_0). \]

Then by (2') rather than (2) since \( k_0 \in G \)

\[ (6) \lim \left( s^+ f'(x_0) - u^+_1 g'(x_0) \right)(k_0) \geq 0. \]

As before \( (s^+ f'(x_0))(k_0) \leq 0 \text{ and } (g'(x_0))(k_0) \in \wp(B, g(x_0)) \).

Hence if \( (g'(x_0))(k_0) \in \wp(B, g(x_0)) \)

then \( u^+_1 (g'(x_0))(k_0) = 0 \text{ and using (6) } \]

\[ s^+(f'(x_0))(k_0) \geq 0 \]

which means \( s^+(f'(x_0))(k_0) = 0. \)
If \((g'(x_0))(k_o)\in \wp(B,g(x_0)) / -\wp(B,g(x_0))\) and
\[\lim u_1^+(g'(x_0))(k_o) > 0\] then, using (6) again, \(s'(f(x_0))(k_o) > 0\) which is impossible. Again the sufficiency condition (4)'(i), (ii) must hold for \(k_o\).

Setting \(L_i(x) = s^+ f(x) - u_1^+(g(x))\) and using Taylor's Theorem
\[\lim \frac{L_i(x_n) - L_i(x_o)}{\|x_n - x_o\|} = \frac{1}{2} \lim \frac{L_i''(x_o)(k_n,k_n)}{\|x_n - x_o\|^2}\]
where the limit on the right exists because of (5').

This limit is
\[\lim (L_i''(x_o)(k_o,k_o)) = L(k_o) > 0\]
since \(k_n \to k_o\). As in [12] this sufficiency condition then implies that
\[\lim L_i(x_n) - L_i(x_o) > 0\] if \(n > n_0\).

But \(\lim u_1^+ [g(x_n) - g(x_o)] \geq 0\) (since \(u_1^+ \in \wp^+(B,g(x_o))\)) if \(n > n_1\).
This means that for \(n > \max(n_1,n_0)\)
\[s^+ f(x_n) - s^+ f(x_o) > 0\] which contradicts the choice of \(\{x_n\}\).

A stronger condition but one which emphasises the operator nature of the conditions is given by the next result. It is stated for [12], but could have been stated for [17].

[18] Theorem: Suppose in [12] that (1), (3) continue to hold and that (2) is replaced by (2'')
\[(2'') f'(x_o) - T g'(x_o) \in G^S, \ T \in \wp(B,g(x_o))^S.\]

A sufficient condition for a point \(x_o\) with property (f) to be a local intermediate minimum is:

For any non trivial \(h \in G\) such that
\[(4'')(i) (f'(x_o))(h) = 0 \text{ and } (g'(x_o))(h) \in \wp(B,g(x_o)) \cap -\wp(B,g(x_o))\]
or
\[(4'')(ii) (Tg'(x_o))(h) = 0 \text{ and } (g'(x_o))(h) \in \wp(B,g(x_o)) / -\wp(B,g(x_o))\]
one has
\[(f''(x_0))(h,h) - T(g''(x_0))(h,h) \in \text{weak interior } (S) \neq \emptyset.\]

**Proof:** With \(\{x_n\}\) as in [12] suppose that \((f'(x_0))(k_0) \notin -S\).

Then
\[
\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|} \rightarrow (f'(x_0))(k_0) \in X/-S
\]

since \(k_n \rightarrow k_0\). \(X/-S\) is a positively homogenous weakly open set

since \(-S\) is a (weakly) closed convex cone. Hence (5) would imply that

\[f(x_n) - f(x_0) \in X/-S \quad \text{for } n \geq n_0\]

which contradicts \(f(x_n) - f(x_0) \in -S\). So \((f'(x_0))(k_0) \in -S\).

(2") gives \((f'(x_0))(k_0, k_0) \geq T(g'(x_0))(k_0)\)

and \(T(g'(x_0))(k_0) \geq 0\) since \((g'(x_0))(k_0) \in \wp(B, g(x_0))\).

Since \(S\) is pointed

\[(f'(x_0))(k_0) \in S\setminus -S = 0.\]

Thus either (4")(i) or (4")(ii) holds and

\[(f'(x_0))(k_0, k_0) - T(g''(x_0))(k_0, k_0) \in \text{weak int } (S).\]

Setting \(L(x) = f(x) - Tg(x)\) and applying the generalised Taylor

Theorem (which can be done because \(L\) is twice continuously differentiable)

one again sees that:

\[L(x_n) - L(x_0) = g'(x_0) \cdot (x_n - x_0) + \frac{1}{2} (L''(x_0) \cdot (x_n - x_0, x_n - x_0))\]

\[+ o(\|x_n - x_0\|^2).\]

(2") gives \((L'(x_0))(x_n - x_0) \in S.\) Since \(L''(x_0)\) is continuous and hence

weakly continuous

\[L''(x_0) \cdot \frac{x_n - x_0}{\|x_n - x_0\|} \rightarrow L''(x_0)((k_0, k_0))\]

thus for \(n \geq n_0\) and some weak neighbourhood \(N\) of 0

\[N + (L''(x_0)) \left( \frac{x_n - x_0}{\|x_n - x_0\|}, \frac{x_n - x_0}{\|x_n - x_0\|} \right) \in \text{weak int } S\]

and

\[N + L(x_n) - L(x_0) \rightarrow o(\|x_n - x_0\|^2) \rightarrow o(\|x_n - x_0\|^2) \in \text{weak int } S.\]
Since any point of \( \frac{\|x_n - x_0\|^2}{\|x_n - x_0\|^2} \) belongs to any given neighbourhood \( N \) of \( 0 \) for \( n \gg n_1 \), one has for \( n \gg n_2 \) that

\[
\frac{L(x_n) - L(x_o)}{\|x_n - x_o\|^2} \in \text{weakint} S.
\]

Proceeding as before

\[
f(x_n) - f(x_o) - T(g(x_n) - g(x_o)) \in \text{weakint} S.
\]

and using (3) again since \( g \) is continuous and \( x_n \rightarrow x_o \)

\[
g(x_n) - g(x_o) \in \omega P(B, g(x_o)) \text{ if } n \gg n_3.
\]

Hence

\[
f(x_n) - f(x_o) \in \text{weakint} S + S \subseteq \text{weakint} S.
\]

Thus once again a contradiction has been reached.  

\[19\] Remarks: (1) Again it is apparent that if \( g'(x_o) \) is completely continuous that \( \omega P(B, g(x_o)) \) can be replaced by \( P(B, g(x_o)) \).

(2) If in addition \( f'(x_o) \) is completely continuous \( S^0 \) can be used instead of \( \text{weakint}(S) \). This follows because when \( g'(x_o) \) is completely continuous \( g''(x_o) \) is and so, therefore, is \( L''(x_o) \). One would then have

\[
(L''(x_o))(k_n, k_n) \rightarrow (L''(x_o))(k_n, k_n) \in S^0
\]

which would suffice to deduce that

\[
\frac{L(x_n) - L(x_o)}{\|x_n - x_o\|^2} \in S^0 \text{ if } n \gg n_o.
\]

(3) Finally it seems worth mentioning that the two part sufficiency conditions in [12], [17], [18] could easily have been written in simpler form. This was not done to facilitate comparison with Guignard's and Zlobec's results.
Chapter Eight

OPTIMIZATION IN HILBERT SPACE AND VARIATIONAL INEQUALITIES FOR OPTIMIZATION
Section one: Optimization in Hilbert Space

In this section a variety of results specific to Hilbert space are discussed. The central advantages of Hilbert Space lie in the existence of a unique closest point to a closed convex set $C$ and in the fact that dual cones can be more fully described since they lie in the space itself.

The first propositions give simpler equivalent constraints for $A(x) \in B$ when $A: X \rightarrow Z$ is a linear map between Hilbert spaces.

[1] Definition: A densely defined map $A$ between two Hilbert spaces $H_1$ and $H_2$ is said to have pseudo-inverse $A^\dagger$ if there is a map satisfying $D(A^\dagger) = H_2$ and

\begin{align*}
(1) & \quad R(A)CD(A^\dagger) = R(A^\dagger)CD(A) \\
(2) & \quad AA^\dagger = F \quad \text{and} \quad A^\dagger A = F \quad \text{in} \quad R(A^\dagger).
\end{align*}

A is often called the Moore-Penrose or generalized inverse. Such inverses have been studied by many mathematicians. The basic properties used here are given in CHARNEK & BEN-ISRAEL (1963) which includes an extensive bibliography. Extensions to non Hilbert spaces have been considered by Hille & Phillips (1957) among others.

[2] Proposition: (CHARNEK & BEN-ISRAEL)

(1) Every closed densely defined linear operator has a unique closed pseudo-inverse.

(2) $N(A^*) = R(A) = N(A^\dagger)$.

(3) $R(A^\dagger) = N(A)$.

(4) $(A^\dagger)^\dagger = A$.

(5) If $A^{-1}$ exists $A^\dagger = A^{-1}$.

Moreover, if $A$ is bounded with $R(A)$ closed, $A$ is bounded and

(6) $A^\dagger AA^\dagger = A^\dagger$  \quad $AA^\dagger A = A$. 
Proposition: Suppose \( A \in B [X, Z] \) with \( R(A) \) closed then
\[ Ax \in D \iff x \in \hat{D} \]
where \( \hat{D} = A^\top (D \cap R(A)) \ominus N(A) \).

Proof: \( \Rightarrow \) If \( Ax \in D \) then \( Ax \in D \cap R(A) \) and \( A^\top Ax \in A^\top (D \cap R(A)) \).

Using (2) of [1]
\[ P \xrightarrow{R(A^\top)} x \in A^\top (D \cap R(A)). \]

Using (3) of [2]
\[ P_{N(A)} \perp x \in A^\top (D \cap R(A)) \]
and since \( P_{N(A)} \perp = \frac{I - P_{N(A)}}{N(A)} \)
\[ x \in A^\top (D \cap R(A)) \ominus N(A). \]

Since \( A \) is continuous \( \frac{N(A)}{N(A)} = N(A) \) and \( x \in \hat{D} \).

Conversely if \( x \in \hat{D} \)
\[ P \xrightarrow{R(A^\top)} x \in A^\top (D \cap R(A)) \]
so that
\[ AA^\top Ax \in AA^\top (D \cap R(A)) \]

Using (6) of [2] at (2) of [1]
\[ Ax \in P \xrightarrow{R(A)} (D \cap R(A)) = D \cap R(A). \]

If \( R(A) \) is not closed one still has that \( Ax \in D \) implies
\[ x \in A^\top (D \cap R(A)) \ominus N(A) \]

[4] Definition: \( E_A(x) \) will be used to denote the closed point to \( x \) in a closed convex set \( A \) and will be called the projection of \( x \) on \( A \).

The fact that \( E_A \) is well defined is a standard result of Hilbert space convexity theory and is proved in Luenberger (1969). The use of the term projection is suggestive of linear projections on closed subspaces to which the notion in [4] reduces if \( A \) is a closed subspace.
Proposition: Suppose \( f: X \to Y \) is compactly differentiable at \( x_0 \) and that \( A: X \to X \) is a continuous linear map with \( R(A) \) closed. A necessary condition for \( x_0 \) to be a strong minimum for \( f \) with respect to \( S \) subject to \( Ax \in D \) is
\[
(f'(x_0))(h) \in S \quad \forall h \in P(A^\top (D \cap R(A)) \cap N(A), x_0).
\]
If \( f \) is pseudoconvex with respect to \( S \) at \( x_0 \), this is also sufficient.

Proof: Since \( Ax \in D \) if and only if \( x \in \hat{D} \) this is just [2.24].

If \( f \) is supposed Fréchet differentiable then the cone \( P \) can be replaced by \( wP \) when \( f'(x_0) \) is completely continuous.

Proposition: Suppose that \( D \) is a convex set then if \( f: X \to \mathbb{R} \) a necessary condition for \( x_0 \) to be a minimum for \( f \) subject to \( Ax \in D \) is
\[
P_{T(D, x_0)}(-f'(x_0)) = 0.
\]
Again, if \( f \) is pseudoconvex at \( x_0 \), the condition is sufficient.

Proof: It is well known that the nearest point can be characterised by
\[
(1) \quad (P_A a_0 - a_0, P_A a_0 - a) \leq 0 \quad \forall a \in A.
\]
Since \( D \) is supposed convex, \( \hat{D} \) is convex and
\[
P(\hat{D}, x_0) = T(\hat{D}, x_0).
\]
The condition of [5] gives
\[
(2) \quad (0, -f'(x_0), 0, -t) \leq 0 \quad \forall t \in T(\hat{D}, x_0)
\]
which on inspection of (1) is equivalent to
\[
P_{T(\hat{D}, x_0)}(-f'(x_0)) = 0.
\]
This result says in geometric terms that the closest point to the tangent cone for \(-f'(x_0)\) is the vertex. In the last result it is apparent that convexity is only used to replace \( P(D, x_0) \) or \( wP(D, x_0) \) by \( T(D, x_0) \).

This gives rise to the following generalisation.
Proposition: Suppose \( f : X \to \mathbb{R} \) is (Fréchet) compactly differentiable. A necessary condition for \( x_0 \) to be a minimum of \( \min f(x) \) subject to \( Ax \in D \) with \( A \) pseudo-invertible is

\[
\mathbb{P}_F(D, x_0) (-f'(x_0)) = 0
\]

\[
(<\mathbb{P}_w F(D, x_0), -f'(x_0)) = 0 \quad .
\]

Projections have been studied by McCormick & Tapia (1972) to provide gradient descent methods for solving \( \min f(x) \) subject to \( x \in B \), a closed convex set. The most extensive investigation of projections has been made by Zarantonello (1971).

The result of [7] can be rewritten as follows.

Proposition: If \( y \) is any point of \( X \)

\[
\mathbb{P}_F(f, x_0)(y) = \mathbb{P}_F(A^+(D \cap R(A)), A^+ A x_0)(A^+ A y) \oplus \mathbb{P}_N(A)(y)
\]

Proof: \( x_0 \in A^+ A x_0 \oplus \mathbb{P}_N(A)(x_0) \) since \( R(A^+ A) \) is closed by definition.

(1) \( F(f, x_0) = F(A^+(D \cap R(A)), A^+ A(x_0)) \oplus N(A) \quad .
\]

It is clear that the right hand side contains the left. Conversely, since \( N(A) \) is a closed subspace,

\[
T(A^+(D \cap R(A)), A^+ A x_0) \oplus N(A) \subset F(f, x_0)
\]

which easily gives (1).

The definition of \( \mathbb{P} \) implies that

\[
\mathbb{P}_F(f, x_0)(y) = \mathbb{P}_F(A^+(D \cap R(A)), A^+ A x_0)(A^+ A y) \oplus \mathbb{P}_N(A)(y) .
\]

Corollary: The condition of [7] is equivalent to

(1) \( \mathbb{P}_F(A^+(D \cap R(A)), A^+ A x_0)(-A^+ A f'(x_0)) = 0 \)

(2) \( \mathbb{P}_N(A)(f'(x_0)) = 0 \quad .
\]
If $A$ is invertible (1) just becomes $P_{(A^{-1})}(x_0)(f'(x_0)) = 0$ while (2) disappears. And if $A$ is the zero mapping (1) collapses while (2) reduces to $f'(x_0) = 0$ which is the standard result for unconstrained minimization.

It is apparent that when $(P_1)$ is the minimization with respect to $S$ given by $(P_1) = \text{min } f(x)$ subject to $g(x) \in D$ with $D$ closed and convex and $g$ nonlinear there is not the same possibility of complete solution. It is, however, immediate on setting $G(x) = (I - P_D)g(x)$ that $(P_1)$ is equivalent to $(P_1')$
\[
\text{min } g(x) \text{ subject to } G(x) = 0.
\]

Unfortunately, $G$ need not be differentiable even when $g$ is. The following differential result does hold though.

10 Proposition: Suppose $g: X \to Z$ is compactly differentiable at $x_0$, $d^+G(x_0;h)$ exists and equals
\[
(g'(x_0))(h) = P_T(D,g(x_0))((g'(x))(h)) = P_T(D,g(x_0))((g'(x_0))(h)).
\]
Proof: $d^+G(x_0;h) = \lim_{t \to 0^+} g(x_0 + th) - g(x_0) - \lim_{t \to 0^+} P_D g(x_0 + th) - C_D g(x_0).
\]
Now $\lim_{t \to 0^+} P_D g(x_0 + th) - P_D g(x_0) = lim_{t \to 0^+} t^{-1}(P_D g(x)(g(x_0 + th) - g(x_0)).
\]
This last equality follows from
\[
P_D g(x_0)(y-g(x_0)) = P_D y - g(x_0) \text{ and } P_D g(x_0) = g(x_0).
\]
From the characterisation of $P_A x$ one has
\[
(P_A x - P_A y, x - y) = (P_A x - P_A y, x - P_A x) + (P_A y - P_A x, y - P_A y) +
\]
\[
\|P_A x - P_A y\|^2 \geq \|P_A x - P_A y\|^2
\]
since the first two terms on the right are non negative by virtue of (1) $(P_A - t, z - P_A) \geq 0 \forall t \in A$. This now yields
\[
\|x - y\| \geq \|P_A x - P_A y\|
\]
In particular

\[ (2) \quad \left\| \mathcal{P}^{-1}_t(D-g(x_0))^{-1} \left[ g(x_0 + h) - g(x_0) \right] - \mathcal{P}^{-1}_t(D-g(x_0)) (g'(x_0))(h) \right\| \]

\[ \leq \left\| \frac{g(x_0 + th) - g(x_0)}{t} - (g'(x_0))(h) \right\|. \]

It is a consequence of (1) or the definition of \( \mathcal{P}_A \) that

\[ \mathcal{P}^{-1}_t(D-g(x_0))^{-1} ((g(x_0 + th) - g(x_0)) = t^{-1} \mathcal{P}_t(D-g(x_0))(g(x_0 + th) - g(x_0)). \]

This was attributed to Rockafellar by McCormick & Tapia.

Rockafellar also noted that for \( x \in Z \)

\[ (3) \quad \mathcal{P}^{-1}_t(D-g(x_0))^x \rightarrow \mathcal{P}(D, g(x_0))^x \text{ as } t \rightarrow 0^+. \]

Combining (3) with \( x = (g'(x_0))(h) \) and (2) one sees that

\[ (4) \quad \mathcal{P}_t(D, g(x_0))(g'(x_0))(h) = \lim_{t \rightarrow 0^+} \mathcal{P}^{-1}_t(D-g(x_0))^{-1} \left[ g(x + th) - g(x_0) \right]. \]

Thus

\[ d^+g(x_0;h) = (g'(x_0))(h) - \mathcal{P}(D, g(x_0))(g'(x_0))(h). \]

The final equality is a consequence of Zarantonello's result that for convex closed cones \( I - \mathcal{P}_c = \mathcal{P}_c^- \), which is proved later.

[11] \textbf{Remark:} It is apparent from the proof of [10] that if only \( d^+g(x_0; \cdot) \) exists, \( d^+G(x_0;h) = \mathcal{P}^{-1}(D, g(x_0)) d^+g(x_0;h). \)


\textbf{Proposition:} Suppose \( D \) is closed and convex in [6] then a necessary condition for \( x_0 \) to be a minimum for \( f(x) \) subject to \( Ax \leq D \) is

\[ \frac{d}{dt} \mathcal{P}_D(x_0 - tf'(x_0)) \big|_{t=0^+} = 0. \]

\textbf{Proof:} \( \hat{D} \) is clearly convex since \( D \) is. Suppose now that \( d'_n \in \hat{D} \) and \( d'_n \rightarrow d'_o \). Then \( d'_n \in A^+(D \cap R(A)) \oplus N(A) \), \( Ad'_n \in (D \cap R(A)). \)
The continuity of $\mathcal{A}$ implies that $\mathcal{A}^+ \subset (D \cap \mathcal{R}(A))$ and $\mathcal{A}^+ \subset (D \cap \mathcal{R}(A))$ which in turn implies that $d_0 \in \mathcal{A}^+(D \cap \mathcal{R}(A)) \cap \mathcal{N}(A) = \hat{\mathcal{D}}$. Thus $\hat{\mathcal{D}}$ is closed. 

[10] with $g(x) = x$ provides that
\[
\lim_{t \to 0^+} \frac{\mathcal{P}_D(x_0 - t f'(x_0)) - \mathcal{P}_D(x_0)}{t} = \mathcal{P}_T(\hat{\mathcal{D}}, x_0)(-f'(x_0)).
\]
The last quantity is 0 from [6] and this is the desired result.

The differential condition of [10], while interesting, is not of much use in optimization because $d^+ g(x, \cdot)$ need not in general be convex, let alone linear. The only case in which $d^+ g$ will certainly be convex occurs when $\mathcal{D}$ is a closed subspace and then $d^+ g$ is in fact $(\mathcal{P}_D g)'$ and the optimization results can be applied directly. A necessary condition can be developed, though, using the notion of a projection (constraint) qualification.

[13] Definition: The projection qualification will be said to be satisfied at $x_0$ if there is a closed convex cone $\mathcal{G}$ such that
\[
\mathcal{P}_G \mathcal{P}_K = \mathcal{P}_G \mathcal{K} = \mathcal{P}(A, x_0),
\]
where $K$ and $A$ retain their usual meanings. Zareantonello has shown that this implies that $\mathcal{G} \cap K = P(A, x_0)$.

The following propositions on projections are necessary to the first order conditions.

[14] Proposition: (Zareantonello). If $C$ is a closed convex cone in a Hilbert space then any point $x$ can be expressed uniquely as
\[
x = x_1 + x_2, \quad x_1 \in C,
\]
\[
x_2 \in C^\perp \quad \text{and} \quad x_1 = \mathcal{P}_C x, \quad x_2 = \mathcal{P}_C x.
\]
Proof: From the projection inequality
\[(1) \quad (x - P_C x, x - P_C x) \leq 0 \quad \forall c \in C.\]
Since C is closed convex cone, 0 and $2 P_C x_0$ belong to C. Thus
\[ (x - P_C x, P_C x) = 0. \]
Moreover, C + $P_C x \subset C$ and hence (1) implies $x - P_C x \in C^{-}$.
Now let $x = x_1 + x_2$, $x_1 \in C$, $x_2 \in C^{-}$, $(x_1, x_2) = 0$. Then
\[ (x - x_1, e - x_1) = (x_2, e - x_1) = (x_2, e) \leq 0 \quad \forall c \in C \]
and $x_1 = P_C x$, similarly $x_2 = P_{C^{-}} x$.
Since $x = P_C x + (x - P_C x)$ and $x - P_C x \in C^{-}$ and $(P_{C'} x, x - P_C x) = 0$
this means that $x - P_C x = P_{C^{-}}(x)$. Note that this includes:
$P_C x = 0$ if and only if $x \in C^{-}$.

[15] Proposition: (Zarantonello) Let $C_1, C_2$ be closed convex cones. Suppose $P_{C_1} P_{C_2} = P_{C_2} P_{C_1}$ then
\[(1) \quad P_{C_1} P_{C_2} x = P_{C_1} P_{C_2} x \quad \text{if and only if} \]
\[(2) \quad (x - P_{C_1} P_{C_2} x, P_{C_1} P_{C_2} x) = 0. \]
It is immediate that [15] holds for any $x$ if one of $C_1, C_2$ is the whole space or if both $C_1, C_2$ are closed subspaces. Zarantonello has proved in addition that if $C_1, C_2$ are finite dimensional closed convex cones then $P_{C_1} P_{C_2} = P_{C_1} P_{C_2}$ whenever $C_1, C_2$ commute.

[16] Proposition: Let $H = \mathbb{C}^n | h = u^+ g^i(x), u^+ P^+(E, g(x_0)) \}$,
\[ K = \left\{ x \left| (g^i(x_0))(k) \in P(E, g(x_0)) \right. \right\}. \]
Then $K^+ = \overline{H}$.

Proof: That $K^+ \subset \overline{H}$ is a consequence of [31] of chapter five.
Suppose, conversely, that $h \in \overline{H}$. Then $h = u^+ g^i(x_0)$.
\[ u^+ \in \mathcal{P}^+(B, g(x_o)) \].

For any \( k \in K \), \((g'(x_o))(k) \in \mathcal{P}(B, g(x_o))\) by definition.

Thus \((h,k) = u^+(g'(x_o))(k) \geq 0\) and since \( h \) was arbitrary \( h \in K^+ \).

Since \( K \) is a closed cone and \( K^+ \subset \overline{H} \) one has \( K^+ = \overline{H} \).

[17] The cone \( G = \mathcal{P}(A, x_o) \) will always satisfy the constraint qualification since

\[ \mathcal{P}_{\mathcal{P}(A, x_o)}(\mathcal{P}_K) = \mathcal{P}(A, x_o) \cap K = \mathcal{P}_K \mathcal{P}(A, x_o) = \mathcal{P}(A, x_o) \]

as can be easily verified. For the purpose of proving a necessary condition it would suffice to require that

\[ \mathcal{P}(A, x_o)(-f'(x_o)) = \mathcal{P}_G \mathcal{P}_K(-f'(x_o)). \]

This would be satisfied by the cone \( G = \mathcal{P}_K(-f'(x_o)) \) but,

since for this cone \( G \) one has \( \mathcal{P}_G \mathcal{P}_K(-f'(x_o)) = 0 \) whether

\[ \mathcal{P}_P(A, x_o)(-f'(x_o)) = 0 \] or not, it would fail to discriminate and would have no chance of giving any useful necessary condition.

For this reason the constraint qualification of [13] is used since in many cases it is also sufficient.

[18] Theorem: Suppose \( X \) is a Hilbert Space and \( f:X \to R, g:X \to Z \) are compactly differentiable at a point \( x_o \). Suppose that \( G \) is a closed convex cone satisfying the projection constraint qualification at \( x_o \). A necessary condition for \( x_o \) to minimize

\[ (P) = \min f(x) \text{ subject to } g(x) \in B, x \in C \] is given by

\[ \lim_{i} f'(x_o) - u^+_i g'(x_o) \in G^+, u^+_i \in \mathcal{P}^+(B, g(x_o)), \text{ where} \]

\[ \lim_{i} u^+_i g'(x_o) = \mathcal{P}_{\mathcal{H}} e'(x_o). \]

**Proof:** The result of proposition [7] gives \( \mathcal{P}_{\mathcal{P}(A, x_o)}(-f'(x_o)) = 0 \). The projection qualification gives \( \mathcal{P}_G \mathcal{P}_K(-f'(x_o)) = 0 \) or

\[ \mathcal{P}_K(-f'(x_o)) \in - G^+. \]
This is equivalent to \( f'(x_o) + (I-P_K)(-f'(x_o)) \in G^+ \).

Using [14] and [16] \( I-P_K = P_{-H} \).

Moreover, \( P_{-H}f'(x_o) = -P_Hf'(x_o) \)

so that \( f'(x_o) - P_Hf'(x_o) \in G^+ \).

This last equation is equivalent to the claimed result.

As a special case of [18] one has

[19] \textbf{Proposition:} Suppose that \( H \) is closed and that

\[ g'(x_o)^{-1} \left[P(P,B,g(x_o))\right] = P(g^{-1}(B),x_o), \]

then a necessary condition for \( \min f(x) \) subject to

\( g(x) \in B \) is (1) \( f'(x_o) - u^+ g'(x_o) = 0 \) or (2) \( u^+ g'(x_o) = P_H f'(x_o) \)

and \( f'(x_o) \notin H \).

\textbf{Proof:} Since \( H = H \cap P_{-H} x \). Since \( K = P(A,x_o) = P(A,x_o) \),

\( G \) can be taken to be \( x \) and the result follows.

[20] \textbf{Theorem:} (Sufficiency) Let \( f:X \to \mathbb{R} \), \( g:X \to Z \) be differentiable at \( x_o \). Suppose (1) \( P_G P_K(-f'(x_o)) = P_P(A,x_o) (-f'(x_o)) \)

and (2) \( A \) is pseudoconvex at \( x_o \) and \( f \) is pseudoconvex at \( x_o \)

over \( A \) then \( f'(x_o) - P_H f'(x_o) \in G^+ \) is sufficient for \( x_o \) to be

a minimum for (P).

\textbf{Proof:} Working back through the proof of [38]

\[ P_G P_K(-f'(x_o)) = 0. \]

Using (1) \( P_P(A,x_o) (-f'(x_o)) = 0 \) Equivalently \( f'(x_o) \in f^+((A,x_o)) \).

The result now follows from the standard sufficiency argument.

[21] \textbf{Remarks:}

(1) [16] and [19] can equally well be framed for bounded
differentiation and weak pseudotangent cones.

(2) By using $s^+ f'(x_0), s^+ S^+ / f(0)$ instead of $f'$ the previous results can be adjusted to deal with \((P)\) when $f: X \to Y$ and $x_0$ is a weak minimum with respect to $S$.

(3) If $P(A, x_0) = 0$ then $C$ can be chosen to be $S^+$. This covers the case, for example, when $C$ is a discrete set and provides an example in which $C$ can be considerably larger than $P(A, x_0)$.

An alternative formulation of a first order necessary condition but one which appears to avoid restraint qualification is given by the next theorem.

[22] **Proposition:** Suppose $x_0$ is a minimum for \((P)\) then a necessary condition is given by:

$$ f'(x_0) - P^H f'(x_0) \in P^+_K P(A, x_0). $$

**Proof:**

$$ f'(x_0) - P^H f'(x_0) = (I - P^H f'(x_0)) $$

$$ = P(-H^+) f'(x_0) $$

$$ = P^+_K f'(x_0). $$

Then, since $x_0$ is a minimum, $f'(x_0) \in P^+(A, x_0)$ and

$$ f'(x) - P^H f'(x_0) \in P^+_K P^+(A, x_0). $$

[23] **Theorem:** Let $H = \overline{H}$ and $G = [P^+_K (P^+(A, x_0))]^+$. Then a necessary condition is

$$ f'(x_0) - u^+ g'(x_0) \in G^+; u^+ g'(x_0) = P^+_H f'(x_0). $$

If

(1) $P^+_K P^+(A, x_0) \subset P^+(A, x_0)$,

(2) $A$ is pseudoconvex at $x_0$,

(3) $f$ is pseudoconvex over $A$ at $x_0$; then the condition is sufficient as well.

**Proof:** Necessity follows from $H = \overline{H}$ and

$$ \left[ P^+_K P^+(A, x_0) \right]^+ = (P^+_K P^+(A, x_0))^{++} = G^+. $$
whence \( f'(x_0) - u^+ g'(x_0) \in \mathbb{F}_{-K} \mathbb{P}^+(A, x_0) \subseteq G^+ \).

Sufficiency is proved by noting that with \( G = \mathbb{F}_{-K} \mathbb{P}^+(A, x) \)
\[
f'(x_0) - u^+ g'(x_0) \in G^+
\]
so that by (1) \( f'(x_0) - u^+ g'(x_0) \in \mathbb{P}^+(A, x_0) \)
and hence \( (f'(x_0))(h) \triangleright u^+ ((g'(x_0))(h)) \quad \forall h \in \mathbb{P}(A, x_0). \)

Since \( A \) is pseudoconvex \( (f'(x_0))(x-x_0) \triangleright u^+ (g'(x_0))(x-x_0) \geq 0, \quad \forall x \in A. \)

Which, since \( f \) is pseudoconvex implies that \( x_0 \) is a minimum for \( f \) over \( A. \)

**Remarks:**

(1) Condition (1) \( \mathbb{F}_{-K} \mathbb{P}^+(A, x_0) \subseteq \mathbb{P}^+(A, x_0) \) is essentially Guignard's sufficiency condition that \( A-x_0 \subseteq G \) because \( A \) is pseudoconvex by (2).

(2) It is only in cases in which \( G \) does satisfy some sort of constraint condition that [22] is at all useful. This constraint condition might actually be \( \mathbb{F}_{-K} \mathbb{P}^+(A, x_0) \subseteq \mathbb{P}^+(A, x_0) \) which is certainly met if \( K = \mathbb{P}(A, x_0). \)

(3) McCormick and Tapia give an explicit characterisation of \( \mathbb{F}_C \) when \( C \) is what they call a positive cone with respect to an orthogonal set in \( X. \) That is \( C = \{ x \in X \mid (x, \delta_\alpha) \triangleright d_\alpha \quad \forall \alpha \in A \} \) and \( \{ \delta_\alpha \} \)
is orthogonal in \( X. \) In the case that \( K \) is positive this allows an explicit statement of [22] to be made.

The final Hilbert space result uses a series of results by Zarantonello (1971) on spectral mappings with respect to non linear projections.

**Definition:** \( J: X \rightarrow X \) is said to be a spectral mapping with respect
to a spectral resolution $\{E_\lambda^j\}$ if $J$ is a mapping which can be written as

$$J = \int_{-\infty}^{\infty} d\lambda D_\lambda$$

where (roughly) the meaning of integration is analogous to that in the standard (linear) projection theory. Zarantonello has developed a complete theory of such spectral integrals with respect to projections on convex cones.

[26] Theorem: Suppose $f: X \rightarrow \mathbb{R}$ is compactly differentiable and $J: X \rightarrow X$ is spectral with respect to a spectral resolution $\{E_\lambda^j\}$ if $E_\lambda^j = E_D^\lambda$. Suppose that $D$ is a closed convex cone such that $D_\lambda = D$ for some $D_\lambda$ in the resolution. Then a necessary condition for $x_0$ to minimize $f(x)$ subject to $Jx \in D$ is given by

$$\mathcal{P}_D(\partial T^*(x_0), x_0)(f'(x_0)) = 0$$

where $T(x) = \frac{1}{2} (E_D^j(x), J(x))$ and $\partial T^*$ is the subgradient of the convex conjugate of $T$.

Proof: $J(x) \in D$ if and only if $(I - E_D^j)(J(x)) = 0$

Using [14] $J(x) \in D$ if and only if $P_D^j - (J(x)) = 0$.

By Lemma 9.1 of Zarantonello (1971) $P_D^j J(x)$ is a spectral map since $J$ is, and by Theorem 9.8 of Zarantonello

(1) $P_D^j (J(x)) = \frac{d}{dx} (E_D^j, J(x)) = \frac{d}{dx} T(x)$.

It is reasonably simple to verify that so defined $T$ is convex from $X$ to $\mathbb{R}$. In fact any spectral map is the gradient of a convex map.

Now $0 = P_D^j J(x)$ if and only if $0 = \frac{d}{dx} T(x) = \partial T(x)$

and $0 \in \partial T(x) \iff x \in \partial T^*(0)$ by a theorem of Rockafellar (1966).

Hence, using (1),

$$0 \in P_D^j J(x) \iff x \in \partial T^*(0).$$
Since $\partial T^*(0)$ is a convex set and since the problem is equivalent to
\[
\min f(x) \text{ subject to } x \in \partial T^*(0),
\]

[27] Remarks:

(1) Spectral resolutions containing any given $P_0$ can be simply constructed, as Zarantonello indicates.

(2) Again sufficiency is guaranteed by the pseudoconvexity of $f$ at $x_0$. The constraint set is necessarily pseudoconvex at $x_0$ since $\partial T^*(0)$ is convex.

Section two: Variational Inequalities

Suppose that $X$ is a convex space and $f:X \rightarrow \mathbb{R}$ is lower semicontinuous and convex. Let $T:X \rightarrow X'$ be a given mapping. Let $(\cdot, \cdot)$ be the associated bilinear form.

[28] Definition: An inequality of the form
\[
(1) \quad (Tu, u-v) \leq f(v)-f(u) \quad u, v \in \text{dom} f
\]
is called a variational inequality and is said to have solution $u_0$ if
\[
(Tu_0, u_0-v) \leq f(v)-f(u_0) \quad \forall v \in \text{dom} f.
\]

The study of such abstract variational inequalities is well developed in the work of Browder, Stampacchia, Lions and others. The primary motivation for the study has come from partial differential equation theory, but as Browder has remarked (1966,a) there are very close correspondences with optimization theory. Kosca (1969) has shown that any variational inequality can be considered as one in which $f$ is just the indicator of a closed convex set.
It will be seen from (1) that a solution to the variational inequality is equivalent to $0 \in R(T + \partial f)$. Thus any theory which guarantees solutions to operator equations $T_1(u) = 0$ also can be invoked in the variational context. Such theorems are usually somewhat less constructive than the corresponding direct proof of solutions to (1) but are generally much more immediate. In this section a brief survey of results from monotone operator theory is made and these results are then applied to two optimization problems. For the remainder of this discussion $X$ is a real Banach space with dual $X'$.

[29] **Definition:** (Browder & Hess' (1971)) A mapping (multivalued) $T$ from $X$ into $X'$ is said to be **generalised pseudo-monotone** if the following holds:

For any sequences $\{u_j\}$ in $X$ and $\{w_j\}$ in $X'$ with $w_j \in Tu_j, u_j \rightharpoonup u_0, w_j \rightharpoonup w_0$ such that $\limsup (w_j, u_j - u_0) \leq 0$; $w_0 \in Tu_0$ and $(w_j, u_j) \rightharpoonup (w_0, u_0)$.

[30] **Definition:** $T: X \to X'$ is said to be **monotone** if

$$(w-v, x-y) \geq 0 \quad \forall x, y \in D(T), \forall w \in Tx, \forall v \in Ty.$$  

[31] **Definition:** $T_1$ is said to be **maximal monotone** if $G(T_1) \subset (X, X')$ is maximal among the graphs of monotone maps $G(T)$; or equivalently if $(z-w, u-v) \geq 0$ $\forall u \in D(T)$ and $\forall z \in Tu$ implies $v \in D(T)$ and $w \in Tv$.

Browder and Hess have proved that any maximal monotone map is generalised pseudomonotone so in particular any subgradient map is generalised pseudomonotone. Maximal monotonicity of such maps was proved in Rockafellar (1970).
Definition: Let $T: X \to X$ be a multivalued map. $T$ is said to be **quasibounded** if $\forall M > 0$ there exist $K(M) > 0$ such that if $w \in Tu$ and $(w, u) \leq M \Rightarrow u \leq M$ then $w \leq K(M)$.

Definition: $T$ is **strongly quasibounded** if for each $M > 0$ there exists $K(M) > 0$ such that if $w \in Tu$ and $(w, u) \leq M$, $u \leq M$ then $w \leq K(M)$.

In addition to bounded maps strongly quasibounded maps include those maximal monotone maps which have $0 \in D(T)^\circ$ (Rockafellar 1969).

Definition: $T$ is **coercive** if there is a map $c: \mathbb{R} \to \mathbb{R}^+$ with $\lim_{r \to \infty} c(r) = \infty$ such that $(w, u) \geq c(\|u\|) \|w\| \\forall (u, w) \in G(T)$.

With these definitions one can state the following theorem of Browder and Hess.

Theorem: Let $X$ be a reflexive Banach space and $T$ be a maximal monotone map from $X$ into $2^{X'}$ with $0 \in D(T)$. Let $T_0$ be a generalised pseudomonotone and coercive with the property (2) that $T_0$ is regular, that is $R(T_0 + T_2) = X'$ for any bounded, everywhere defined, single-valued maximal monotone mapping $T_2$. Suppose further that $T_0$ is quasibounded or $T$ is strongly quasibounded; then $R(T_0 + T) = X'$.

From the previous discussion it is clear that this includes the result $0 \in R(T_0 + \overline{w}_0 + \partial f)$, $\overline{w}_0 \in X'$ when the conditions on $T$ hold with $T = \partial f$. For these maps, therefore, one has a solution $w_0 \in Tu_0$ to

\[(3) \quad (w_0 - \overline{w}_0, u_0 - v) \leq f(v) - f(u_0) \quad \forall v \in \text{dom}f.\]
If in fact one only wishes to solve (3) with \( w_0 = 0 \) the coercivity conditions can be weakened in many situations. For example, in the case that \( T_0 \) is quasibounded and \( 0 \in T(0) \) the condition: \( \exists M_1 \geq (T_0 u, u) > 0 \) if \( \|u\| > M_1 \), suffices as can be seen by inspecting the proofs in Browder and Hess.

Many alterations are possible in the type of theorem that can be proved. Browder and Hess (1971) and Mosca (1969) provide more than enough variations to indicate the depth of the subject. Mosca's paper which deals with approximations of inequalities provides conditions under which monotone mappings, which are not necessarily maximal, can be used.

With this brief discussion behind one can turn to the use of variational inequalities in optimization. The most immediate example is provided by the notion of the subgradient itself. As has already been pointed out any variational inequality can be considered as a statement that the subgradient of \( f \) contains vectors of certain forms. Within this framework it seems worth noting the following theorem which relies on a result of Rockafellar (1970, c) that:

**Proposition:** If \( K(u, v) \) is a convex-concave semicontinuous saddle function then \( \partial K \) is maximal monotone where \( \partial K \) denotes \( \partial K(u, v) = (\partial_u K, -\partial_v K) \). This last notation is used to denote the standard subgradients of \( K(\cdot, v) \) and \( -K(u, \cdot) \) both of which are convex functions.

**Theorem:** Let \( X = (Y, Z) \) where \( Y \) and \( Z \) are reflexive Banach Spaces. Let \( K(y, z) \) be a convex-concave semicontinuous saddle function on \( X \) into \( R \). Let \( T_0 \) be a coercive generalised pseudomonotone map
satisfying (2) of [35]. Suppose that either

1) $T_0$ is quasibounded and $(0,0) \in D(\partial K)$, or

2) $\partial K$ is strongly quasibounded (i.e. $(0,0) \in D^0(\partial K)$).

Then given any $(y'_0,z'_0) \in X'$ there exist $(y_o,z_o) \in D(\partial K)$ simultaneously solving

\[ (3a) \quad K(y_o,z_o) - K(y'_0,z'_0) \succ (y'_0 - \pi_1 T_0(y_o,z_o), y - y_o) \]
\[ (3b) \quad K(y_o,z_o) - K(y'_0,z'_0) \prec (z'_0 - \pi_2 T_0(y_o,z_o), z - z_o) \]

for all $(y,z) \in (Y,Z)$ where $\pi_1$, $\pi_2$, are the projections of $X'$ on $Y',Z'$ respectively.

**Proof:** (3a) can be rewritten as $y'_0 - \pi_1 T_0 \in \partial_{y_o} K(y_o,z_o)$

and (3b) as $-z'_0 - \pi_2 T_0 \in \partial_{z_o} K(y_o,z_o)$,

or using [37] $(y'_0,z'_0) - (\pi_1 T_0(x_o,y_o), \pi_2 T_0(x_o,y_o)) \in \partial K(y_o,z_o)$.

This in turn is equivalent to requiring that $R(T_0 + \partial K) = X'$.

By [37] $K$ is maximal monotone. An application of the Theorem of [35] gives the desired results.

Note that $(0,0) \in D(\partial K)$ if and only if $K$ has a saddlepoint at some point $(y_o,z_o)$ since the existence of such a saddle point is equivalent to

\[ K(y,z) - K(y_o,z_o) \succ (y - y_o,0) \]
\[ -K(y_o,z) - (-K(y_o,z_o)) \prec (z-z_o,0) \]

which in turn says

\[ 0 \in \partial K(y,z) | y_o \quad 0 \in \partial K(y_o,z) | z_o \]

or

\[ (0,0) \in \partial K(y_o,z_o). \]

It is simple to verify that if $T_0(y,z) = (T_1 y, T_2 z)$ and $T_1, T_2$ are both coercive, strongly quasibounded generalised pseudomonotone then $T_0$ has these properties. It is clear that [38] includes the standard variational inequality as the case $Z = 0$ in which case (3b) is vacuously satisfied.
Theorem [38] might be said to give solutions to a pair of coupled variational inequalities with the coupling taking place \( T_0 \).

The next results return to simple variational inequalities and to a discussion of the complementarity problem. Suppose that \( C \) is a closed convex cone in a Banach Space and that \( T: X \rightarrow X' \) is a single valued mapping which will subsequently be required to be of various monotone types. The complementarity problem is defined to be: \( (P) \) minimize \((Tu, u)\) subject to \( Tu \in C^+, \ u \in C \).

This problem finds its origins in linear programming when given an \( n \) vector \( b \) and an \( n \times n \) matrix \( M \) one wants to find \( y, x \in \mathbb{R}^n \) such that \( y = Mx + b \) and \((y_i, x_i) = 0, \ i = 1, \ldots, n \).

Karamardian (1969) showed that when \( C \in \mathbb{R}^n \) and \( T = f \) was continuous and satisfied \((f(u) - f(v), u - v) \geq k \| u - v \|^2 \) on \( C \) that the minimum in \((P)\) is 0.

Bazaraa (et al.) (1972) have showed that 0 is the minimum when \( T \) is bounded hemicontinuous and satisfies \( C \in D(T) \) and

\[
(Tu - Tv, u - v) \geq \alpha(\|u - v\|) \|u - v\| \quad \text{for some strictly increasing} \quad \alpha \quad \text{with} \quad \lim_{r \to \infty} \alpha(r) = \infty . \quad \text{This property is called } \alpha \text{-monotonicity.}
\]

Their proof relies on variational inequalities proved in Mosco (1969) concerning perturbations of mappings satisfying the various conditions listed above. In the case that \( T \) is everywhere defined, such a \( T \) is maximal monotone and their result is included in the following results.

Generally, one has:

\[ [39] \text{Proposition: The solution to } (P) \text{ is 0 if and only if there is } \]
a solution to the variational inequality \((Tv, u - v) \geq 0 \quad \forall u \in C \).

\[ \text{Proof: Note first that setting } f_c(u) = \sum_{u \in C} 1_{u \not\in C} \text{ this has the } \]

form of [28] (1). Suppose $u_o$ is a variational solution, then
$$(Tu_o, u_o) \leq (Tu_o, u) \quad \forall u \in C.$$ Since $0 \in C$, $(Tu_o, u_o) \leq 0$.
For any $\lambda > 0$ \(\lambda u_o \in C\) since $C$ is a cone.
Thus $\forall \lambda > 0$
$$(Tu_o, u_o) \leq \lambda (Tu_o, u_o)$$
which is impossible unless $(Tu_o, u_o) = 0$. This in turn implies
that $(Tu_o, u) > 0 \quad \forall u \in C$ or equivalently that $Tu_o \in C^+$.
Conversely if $(Tu_o, u_o) = 0$, $Tu_o \in C^+$, $u_o \in C$ then
$$(Tu_o, u-u_o) = (Tu_o, u) > 0 \quad \forall u \in C$$ since $Tu_o \in C^+$.

Finally since $u_o \in C$, $u_o$ is actually a solution to the variational inequality.

A solution to $(Tu_o, u_o) = 0$ must be a minimum for $(P)$ because
$Tu \in C^+$, $u \in C$ implies $(Tu, u) > 0$. The solution will be unique if
one has $(Tu-Tu, u-v) > 0$, $u \neq v$, since with $u_o, u_1 \in C, Tu_o, Tu_1 \in C^+$
$$(Tu_o, u_o) = (Tu_1, u_1) = 0$$ implies
$$(Tu_1-Tu_o, u_1-u_o) = -(Tu_1, u_o) - (Tu_o, u_o) \leq 0.$$ Thus one has, using the remark of [36]:

[40] Theorem: $(P)$ has solution $u_o$ with $(Tu_o, u_o) = 0$ whenever $T$ is a
quasibounded generalised pseudomonotone mapping which is regular
and satisfies $(Tu, u) \leq 0$ if $\|u\| \leq M$.

Browder and Hess (1972) remark that for a monotone map,
regularity is equivalent to maximality when $0 \in D(T)$ so that [40]
includes all maximal monotone maps with $0 \in D(T)$.

If one requires a stronger property than generalised pseudomonotonicity then regularity is not needed in [40]. The definition is initially due to Brezis (1968).
Definition: A multivalued mapping \( T: X \to X' \) is said to be pseudomonotone on \( C \subseteq D(T) \) when

(a) \( Tu \) is a nonempty closed convex subset of \( X \) if \( u \in C \).

(b) \( T \) is upper semicontinuous as a multivalued mapping from \( C \cap F \) into \( X \) with the weak topology for any finite dimensional subspace \( F \).

(c) Whenever \( \{ u_j \} \subseteq C \) \( u_j \to u \), \( w_j \in Tu_j \) then
\[
limsup(w_j, u_j - u) \leq 0 \implies \text{for each } v \in C \exists w(v) \in Tu \text{ with } \liminf(w_j, u_j - v) \geq (w(v), u - v).
\]

Definition: A single valued function \( T: X \to X' \) is said to be demicontinuous if \( x_n \to x \) implies \( Tx_n \to Tx \)

while \( T \) is said to be hemicontinuous, if
\[
T(tx + (1 - t)x_0) \to Tx \text{ when } t \to 1.
\]

Proposition: A single valued hemicontinuous monotone mapping with \( C \subseteq D(T) \) is pseudomonotone on \( C \) if (1) \( D(T) \) is open or (2) if \( D(T) \) is a dense subspace and \( T \) is locally bounded.

Proof: (a) is immediate since \( T \) is single valued.

(b) Kato (1967)(1964) has shown that hemicontinuity implies demicontinuity for a class of sets \( D(T) \) which include open sets and includes dense subspaces when \( T \) is locally bounded. Demicontinuity is clearly stronger than the continuity property of [41] (b).

(c) Let \( \{ u_j \} \subseteq C \) with \( u_j \to u \) and suppose
\[
\lim(Tu_j, u_j - u) \leq 0. \quad \text{By monotonicity and } C \subseteq D(T), (Tu_j, u_j - u) \leq (Tu_j, u_j - u)
\]

which means that
\[
(Tu_j, u_j - u) \to 0.
\]
Let \( x \) be any point in \( C \). Then
\[
(Tu_j, u_j - x) = (Tu_j, u_j - u) + (Tu_j, u - x)
\]
so that
\[ \liminf (T_{u_j}, u_j - x) = \liminf (T_{u_j}, u - x) \]
Also \( (T_x, u_j - x) \leq (T_{u_j}, u_j - x) \) so that
\( (T_x, u - x) \leq \liminf (T_{u_j}, u - x) \).

Let \( v \in C \) and let \( x_t = tv + (1-t)u \) which since \( C \) is convex is in \( C \subset D(T) \). Setting \( x_t = x \)
\( (T_{x_t}, u - x_t) \leq \liminf (T_{u_j}, u - x_t) \).

Since \( T \) is hemicontinuous
\[ \lim_{t \to 0} T_{x_t} = Tu \]
and
\[ (Tu, u - v) \leq \liminf (T_{u_j}, u - v) = \lim (T_{u_j}, u_j - v) \]
and \( T \) is pseudoconvex on \( C \).

For pseudomonotone maps one has the following theorem of Browder and Hess, which they prove directly.

**Theorem:** Let \( X \) be a reflexive Banach space, and \( C \) a closed convex subset of \( X \) with \( T \) pseudomonotone on \( C \). Then if
1. \( w \in Tu \), and \( (w, u) \leq 0 \) implies \( \| u \| \leq M \) for some \( M > 0 \),

there is a solution \( u_0 \in C \), \( w_0 \in Tu \) to
\[ (w_0, u - u_0) > 0 \quad \forall u \in C. \]

**Proof:** Browder and Hess prove this theorem for \( T \) coercive and assert that \( (w_0, u - u_0) > 0 \) \( \forall u \in C \) has solution for any \( w \in X'. \)
Inspection shows that their proof holds for the present theorem.

Note, conversely, that coerciveness of \( T \) would provide that
\( T - w \) satisfied the hypothesis (1) for all \( w \in X' \).

Using [42], [39] one has as a corollary

**Corollary:** There is a solution to
\[ (Tu_0, u_0) = 0, \quad u_0 \in C, \quad Tu_0 \in C'. \]
whenever \( T \) is hemi-continuous single valued monotone satisfying

\((1)\) \( (Tu,u) > 0 \) if \( \|u\| > M \)

and either \((2)(a)\) \( C \subset D(T) \) and \( D(T) \) is open or \((2)(b)\) \( T \) is locally bounded with \( D(T) \) a dense subspace.

**Remarks:**

(1) Kato in fact shows (1964) that \( D(T) \) need only be "quasi dense" in \((2)(b)\) by which he means for each \( u \in D(T) \) there is a dense subset \( M_u \) in \( X \) such that for each \( v \in M_u \) \( u + tv \in D \) if \( 0 < t < \in \( v \). In this circumstance hemi-continuity and local boundedness still imply demicontinuity.

(2) Any \( \alpha \)-monotone \( T \) satisfies the condition in \((1)\) so that for \( D(T) \) quasidense \([44]\) includes the result of Bazaraa et al.

**The variational inequalities theorems discussed above are all valid for \( T \) multivalued so that they all provide solutions to**

\((P')0 \in (Tu_0, u_0) \quad Tu_0 \cap C^+ \neq \emptyset , \quad u_0 \in C\)

which might be considered as the multivalued nonlinear complementary problem.

Bazaraa et al. note that their method, which as was already noted relies on Mosca's approximation theory and which doesn't appear to have direct extension to non monotone mappings, also gives information on approximation, perturbation and continuous dependence of solutions. The perturbation theory for the variational inequalities discussed above is buried in the operator analysis. The penultimate results of this section use a perturbation argument directly to establish the existence of a solution to the complementarity problem for generalised pseudomonotone mappings which are demicontinuous and satisfy another weak
monotone type condition.

[47] Definition: (Hess(1972)). A single valued map \( TX \rightarrow X' \) is said to be of type (P) if whenever \( u_n \rightarrow u_0 \) then
\[
\text{limsup}(u_n, u_n - u_0) \not\rightarrow 0.
\]

[48] Definition: \( J: X \rightarrow X' \) is called the duality map and is defined by
\[
Ju = \left\{ v \in X' \mid (u, v) = \| u \| \| v \|, \quad \| v \| = \| u \| \right\}.
\]
It is known (Hess(1972)) that every reflexive Banach space \( X \) has an equivalent norm in which both \( X \) and \( X' \) are locally uniformly convex and that with these equivalent norms \( J \) is single valued and demicontinuous.

[49] Theorem: Let \( X \) be a reflexive Banach space and let \( T: X \rightarrow X' \) be a demicontinuous type (P) mapping then
\[
T_\lambda = T + \lambda J
\]
is pseudomonotone on any closed set contained in \( D(T) \) when \( X, X' \) have the locally uniformly convex norms mentioned above.

Proof: \( J \) is single valued demicontinuous so that \( T_\lambda \) is also \( \forall \lambda > 0. \)

\( J \) is known to have the property that whenever \( u_j \rightarrow u_0 \)
and \( \text{liminf}(Ju_j, u_j - u_0) \leq 0 \) then \( u_j \rightarrow u_0. \) As Hess remarks it is straightforward to verify that \( T_\lambda \) shares this property. This in turn allows one to verify that \( T_\lambda \) is pseudomonotone on any closed \( C \subset D(T). \)

[50] Theorem: Suppose \( X \) is a reflexive Banach Space and that
\( T : X \to X' \) satisfies the following properties for a closed convex cone \( C \).

1. \( T \) is generalized pseudomonotone on \( C \).
2. \( T \) is type (P) on \( C \).
3. \( T \) is demicontinuous on \( C \).
4. \( T \) is quasibounded on \( C \).
5. If \( (Tu,u) < 0 \) and \( u \in C \) then \( \|u\| \leq M \). Then there is a solution \( u_o \) to the complementarity problem with \( (Tu_o,u_o) = 0 \).

**Proof:** There is no loss in assuming that \( X \) and \( X' \) are locally uniformly convex since all the hypotheses are invariant under equivalent norms as is the conclusion. [49] then implies that \( T \) is pseudomonotone on \( C \) and coercive. Using [43] there is a solution \( u_\lambda \in C \) to

\[
(Tu_\lambda, u_\lambda) = 0 \quad u_\lambda \in C, \quad T u_\lambda \in C^+.
\]

Now

\[
(Tu_\lambda, u_\lambda) = (Tu_\lambda, u_\lambda') + \lambda(J u_\lambda, u_\lambda') = (Tu_\lambda, u_\lambda) + \|u_\lambda\|^2
\]

so that

\[
\forall \lambda > 0, \quad (Tu_\lambda, u_\lambda) < 0.
\]

Property (5) implies that \( \{u_\lambda\} \) is bounded set in \( C \).

Since \( X \) is reflexive there is some sequence \( \lambda_n \to 0 \)

with \( u_n = u_\lambda_n \rightharpoonup u_o \). \( u_o \) belongs to \( C \) because \( C \) is convex and closed and hence weakly closed. Using \( (Tu_n, u_n) \leq 0 \) and \( \|u_n\| \leq M \) one has, by (4), \( \|Tu_n\| \leq K(M) \) so that for some subsequence which will not be relabeled \( Tu_n \rightharpoonup w_o \).

Since

\[
T\lambda_n u_n = Tu_n + \lambda_n Ju_n \in C^+
\]

and

\[
\|T\lambda_n u_n - Tu_n\| = \|\lambda_n Ju_n\| = \lambda_n \|u_n\| \leq \lambda_n M
\]

\[
\|Tu_n - w_o\| \to 0 \quad \text{and since} \quad Tu_n \rightharpoonup w_o.
\]
\( \lambda_n u_n \to w_0 \in C^+ \) as again \( C^+ \) is weakly closed.

Now
\[
(T_n u_n, u_n - u_0) < 0
\]
because \( u_0 \in C \) and \( (T_n u_n, u_n) = 0 \). Thus one has
\[
(T_n u_n - u_0) + \lambda_n (J_n u_n - u_0) \leq 0.
\]
This in turn implies that
\[
\liminf(T_n u_n - u_0) \leq 0.
\]
because \( \lambda_n \to 0 \) and \( \| (J_n u_n - u_0) \| \leq \| u_n \| \| u_n - u_0 \| \leq M_1 \).

Since \( T \) is generalised pseudomonotone and \( u_n \to u_0, T_n \to w_0 \)
one may conclude that \( w_0 \in T u_0 \) and that
\[
(T_n u_n) \to (T u_0, u_0).
\]
Finally \( (T u_0, u_0) \geq 0 \) because \( u_0 \in C \) and \( T u_0 \in C^+ \), but
because \( (T_n u_n) \leq 0 \) one has actually that
\[
(T u_0, u_0) = 0 \quad u_0 \in C, \quad T u_0 \in C^+.
\]

[51] **Remarks:**

(1) This result with \( D(T) = X \) is contained in [35] because one can show that a demicontinuous everywhere defined type \( (P) \)
mapping satisfying the condition \( (Tu, u) \geq -k \| u \| \) is regular.

(2) For [50] to add any new result it must be ascertained that mappings can satisfy [50] without being pseudomonotone. This would appear to be possible since one can envisage examples in which \( u \to u_0 \) but \( \{ T_n u \} \) is unbounded. Note that it is only in this case that the requirement that \( T \) be type \( (P) \) on \( C \) is not implied by generalised pseudomonotonicity on \( C \).

(3) Conditions (4) and (5) could be combined into the weaker condition that when \( u \in C \) and \( (T u, u) \leq 0 \) one has \( \| u \| \leq K \)
and \( \| Tu \| \leq K(N) \). Trivially any strictly monotone map with
\( T(0) = 0 \) satisfies this condition.

It seems worth emphasising that any continuous finite dimensional mapping is pseudomonotone so that [43] holds for all these mappings. Karamardian (1972) has shown that the complementarity problem is solved in \( \mathbb{R}^n \) for any continuous \( f \) satisfying \((x, f(x)) = 0 \) when \( x \in \mathbb{R}^n \cap \sum x \forall x \neq M \) while [39], [43] include any continuous map satisfying \((x, f(x)) > 0 \) when \( x \in \mathbb{R}^n \) and \( \|x\| > M \).

Since lower semicontinuous convex functions have maximal monotone subgradients one might ask if weaker convex type functions have associated with them any monotone type operators. In any case where, for instance, the derivative satisfies some monotonicity requirement one immediately has the whole of monotone operator theory as an adjunct for proving optimization results. A tentative start in this direction is provided by the following two results.

[52] **Proposition:** Let \( X \) be a reflexive Banach space and let \( f: X \to \mathbb{R} \) be a compactly differentiable quasiconvex mapping. Suppose that \( f'(x) \) is bounded as a function of \( x \) and is completely continuous at any local minimum. Then \( f'(x) \) is type (F).

**Proof:** Suppose \( x_n \to x_0 \). In the case that \( x_0 \) is a local minimum
\[
 f'(x_n) \to f'(x_0) \text{ by hypothesis so that}
\]
\[
 \lim (f'(x_n), x_n - x_0) = 0.
\]
Suppose now that \( x_0 \) is not a local minimum. Since \( f \) is continuous and quasiconvex \( f \) is lower semicontinuous in the weak topology and \( \liminf \ f(x_n) \geq f(x_0) \). Equivalently,
\[
 \text{if } n \geq n_0, \quad f(x_0) - \varepsilon_n \leq f(x_n)
\]
where $\epsilon_n$ is a sequence of positive numbers converging to zero. Because $x_0$ is not a local minimum one can find, for $\epsilon_n < \epsilon_{n_1}$ a sequence $\{y_n\}$ with $y_n \to x_0$ and such that for $n \geq n_1$

$$f(x_n) \leq f(x_0) - \epsilon_n \geq f(y_n).$$

Since $f$ is quasiconvex and differentiable one has for $n \geq n_1$

$$f'(x_n)(x_n - y_n) \geq 0.$$

This in turn leads to

$$(f'(x_n), x_n - x_0) = (f'(x_n), x_n - y_n) + (f'(x_n), y_n - x_0) \geq (f'(x_n), y_n - x_0).$$

This last term converges to zero because $y_n \to x_0$ and since $f'(x_n)$ being the image of a bounded set is bounded by hypothesis. Thus $\liminf (f'(x_n), x_n - x_0) \geq 0$ and $f$ is type $(P)$. 

As an application of this result one can prove that solutions exist for the following kind of variational inequality.

$$(Tu - u) \geq 0 \quad \text{if} \quad f(u) \leq f(u_0).$$

[53] Theorem: Let $X$ be a reflexive Banach Space and suppose $f: X \to \mathbb{R}$ is quasiconvex with a demicontinuous type $(P)$ derivative. Let $T: X \to X'$ be an everywhere defined hemiconnected monotone mapping satisfying $(T u - T v, u - v) \geq B(\|u - v\|) \|u - v\|$ for some strictly increasing $B$ with $B(0) = 0$, $B(\infty) = \infty$. Suppose that

$$(f'(x), x) \geq -k \|x\|$$

for some $k > 0$ and $\|x\| > M$, then there is a solution to

$$(Tu - w, u - u_0) \geq 0 \quad \forall u \in \mathcal{F}(u_0) = \{u \mid f(u) \leq f(u_0)\}.$$  

Proof: Let $T_1 = T + f'$. Since $T$ is hemiconnected monotone with $D(T) = X$, $T$ is demicontinuous and $T_1$ is demicontinuous. Suppose $x_0 \to x_0$ and $\limsup(T_1, x_n - x_0) < 0$. Since $f'$ is assumed type $(P)$ one must have $\limsup(T_1, x_n - x_0) < 0$. 


But \((T_{n}, x_{n} - x_{0}) \geq (T_{0}, x_{n} - x_{0}) + B(\|x_{n} - x_{0}\|)(\|x_{n} - x_{0}\|)\) and hence limsup \(B(\|x_{n} - x_{0}\|) \|x_{n} - x_{0}\| = 0\). Suppose that \(x_{n} \not\to x_{0}\) then for some subsequence \(\{x'_{n}\} \|x'_{n} - x_{0}\| \geq \epsilon > 0\) which in turn means that limsup \(B(\|x_{n} - x_{0}\|) \|x_{n} - x_{0}\| \geq B(\epsilon)\epsilon > 0\).

Thus \(x_{n} \to x_{0}\) and as in previous results \(T_{1}\) must be pseudomonotone.

Since \(T\) is \(B\)-monotone and \((f'(x), x) \geq -k \|x\|\), one has

\[(T_{1}, u, u) \geq (T_{0}, u) + (f'(u), u) + B(\|u\|) \|u\| \geq (B(\|u\|) - k \|u\|) \|u\|, \text{ if } \|u\| \geq M \text{ and } T_{1}\]

is coercive.

Applying [43] with \(C = X\) one sees that there is a solution to

\[T_{1}u_{0} = \frac{w}{\omega} \quad \forall \frac{w}{\omega} \in X'.\]

Let \(u \in L(u_{0})\) then \((Tu_{0} - w, u - u_{0}) = -f'(u_{0}), (u - u_{0})) \geq 0\) since \(f\) is quasiconvex.

The condition \((f'(u), u) \geq -k \|u\|\) is always guaranteed if there is a global minimum for \(f\) at \(0\) as one then has \((f'(u), u - 0) > 0\) \(\forall u \in \{u \mid f(u) \geq f(0)\} = X\). It is clear that the condition can be satisfied without any such minimum existing.

In more general terms the theorem opens up the question of when one can find a solution \(u_{0} \in C(u_{0})\)

\[(Tu_{0}, u - u_{0}) \geq 0 \quad \forall u \in C(u_{0}). \text{ (That is, when } C(u) \text{ is a convex set which varies with } u.)\]

The last remarks of this chapter concern the solution of generalised variational inequalities. Specifically, suppose \(T : X \to B[X, Y]\) and that \(S\) is a closed convex cone with interior in \(X\). One can ask for solutions to

1. \((Tu_{0})(u - u_{0}) \in S^{0} \quad \forall u \in C\) or to
2. \((Tu_{0})(u - u_{0}) \in S \quad \forall u \in C\). Conditions for solution of (1) are easily obtained but (2), which appears more interesting, also appears much less tractable.
A particular example of conditions for (1) to be solvable is given by requiring that for some nonzero \( u \in S^+ \), \( u^T \) satisfies the conditions of [43]. Stronger results can be proved by using the natural generalisations of the concepts of this section.
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