Parametric Euler Sum Identities

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1 Introduction

A somewhat unlikely-looking identity is

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m-x} = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}, \tag{1}$$

valid for all complex x not a positive integer. For $x = 0$, (1) becomes $\zeta(2,1) = \zeta(3)$—the most central Euler sum identity—as discussed below.

In this note we begin with an *ab initio* proof of (1), and then explore various consequences and extensions. We conclude by studying other generating function identities of which

$$\sum_{n=1}^{\infty} \frac{\dfrac{n}{n^2+y^2} + \displaystyle\sum_{m=1}^{n-1} \frac{2m}{m^2+y^2}}{n^2+4y^2} = (\coth(\pi y) + \coth(2\pi y)) \sum_{n=1}^{\infty} \frac{2\pi n y}{(n^2+y^2)(n^2+9y^2)} \tag{2}$$

is a pretty example.

Though we do not belabour the point, all our work was assisted by the use of computer algebra systems—and some of it would have been impossible, at least for us, without such tools. This is true of both our discoveries and of our proofs. The joys of such symbolic and numeric computation ... and much more are discussed in detail in [1] and [2].

2 A Parametric Euler sum

We begin by defining various special functions which we shall exploit in the sequel. For $\mathrm{Re}(x) > 0$, the gamma function and its logarithmic derivative are defined by

$$\Gamma(x) := \int_0^{\infty} e^{-t} t^{x-1}\, dt, \qquad \Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

Note that

$$\sum_{n=1}^{\infty} \frac{x}{n(n-x)} = -\Psi(1-x) - \gamma, \quad \text{where} \quad \gamma := \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log n \right)$$

is Euler's constant.

Recall that the classical *Riemann zeta function* is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Correspondingly,

$$\zeta(s,t) := \sum_{n>m>0} \frac{1}{n^s m^t} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m=1}^{n-1} \frac{1}{m^t} \tag{3}$$

defines a double Euler sum. The two-place function (3) was first introduced by Euler, who noted the *reflection formula*

$$\zeta(s,t) + \zeta(t,s) = \zeta(s)\zeta(t) - \zeta(s+t), \qquad \mathrm{Re}(s) > 1, \quad \mathrm{Re}(t) > 1, \tag{4}$$

and the reduction formula (6) below. An obvious extension of (3) is

$$\zeta(s_1, s_2, \ldots, s_N) := \sum_{n_1 > n_2 > \cdots > n_N > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_N^{s_N}},$$

which defines an *Euler sum* of *depth* N and *weight* $\sum_i s_i$. Some authors reverse the order of the variables.

Euler sums may be studied through a profusion of methods: combinatorial, analytic and algebraic. The reader is referred to [2, Ch. 3] for a concise overview of Euler sums and their applications.

We now prove identity (1) directly.

Theorem 1 *If x is any complex number not equal to a positive integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m-x} = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}.$$

Proof. Fix $x \in \mathbf{C} \setminus \mathbf{Z}^+$. Let S denote the left hand side. By partial fractions,

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \left(\frac{1}{n(n-m)(m-x)} - \frac{1}{n(n-m)(n-x)} \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m-x} \sum_{n=m+1}^{\infty} \frac{1}{n(n-m)} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{n-m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=m+1}^{\infty} \left(\frac{1}{n-m} - \frac{1}{n} \right) - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}. \end{aligned}$$

Now for fixed $m \in \mathbf{Z}^+$,

$$\sum_{n=m+1}^{\infty}\left(\frac{1}{n-m}-\frac{1}{n}\right)=\lim_{N\to\infty}\sum_{n=m+1}^{N}\left(\frac{1}{n-m}-\frac{1}{n}\right)=\sum_{n=1}^{m}\frac{1}{n}-\lim_{N\to\infty}\sum_{n=1}^{m}\frac{1}{N-n+1}$$
$$=\sum_{n=1}^{m}\frac{1}{n},$$

since m is fixed. Therefore, we have

$$S=\sum_{m=1}^{\infty}\frac{1}{m(m-x)}\sum_{n=1}^{m}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n(n-x)}\sum_{m=1}^{n-1}\frac{1}{m}=\sum_{n=1}^{\infty}\frac{1}{n(n-x)}\left(\sum_{m=1}^{n}\frac{1}{m}-\sum_{m=1}^{n-1}\frac{1}{m}\right)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^2(n-x)}.$$

□

For $x=\sqrt{-1}$, equation (1) becomes the pair of tangent sum evaluations

$$\sum_{n=1}^{\infty}\frac{1}{n(n^2+1)}\sum_{m=1}^{n-1}\frac{mn-1}{m^2+1}=\sum_{n=1}^{\infty}\frac{1}{n(n^2+1)}=\gamma+\operatorname{Re}\Psi(1+\sqrt{-1})$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n(n^2+1)}\sum_{m=1}^{n-1}\frac{n+m}{m^2+1}=\sum_{n=1}^{\infty}\frac{1}{n^2(n^2+1)}=\frac{\pi^2}{6}-\operatorname{Im}\Psi(1+\sqrt{-1})$$
$$=\frac{\pi^2}{6}-\frac{\pi\coth\pi-1}{2}.$$

Setting $x=0$ in equation (1) gives $\zeta(2,1)=\zeta(3)$. More generally, differentiating (1) k times with respect to x produces a corresponding formula for $\zeta(k+3)$. For example, differentiating once and setting $x=0$ (equivalently, comparing coefficients of x on both sides) produces

$$\zeta(4)=\zeta(3,1)+\zeta(2,2).$$

Since, by the *reflection* formula (4), $\zeta(2,2)=[\zeta^2(2)-\zeta(4)]/2$, we also evaluate

$$\zeta(3,1)=\frac{1}{4}\zeta(4)=\frac{\pi^4}{360}.$$

This is the first case of a remarkable identity discovered by Zagier, and first proved in [4]. See (11) below.

Integrating equation (1) produces the following corollary.

Corollary 2 *For all complex x not equal to a positive integer,*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log\left(1 - \frac{x}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{n-m} \log\left(\frac{1 - x/m}{1 - x/n}\right). \tag{5}$$

For $x = \sqrt{-1}$, the imaginary part of (5) leads to

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{m} \arctan\left(\frac{m}{n^2 - mn + 1}\right) = \sum_{n=1}^{\infty} \frac{\arctan(1/n)}{n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \zeta(2n+3),$$

where the last evaluation comes from writing $\arctan(1/n) = \int_0^1 n(x^2 + n^2)^{-1}\,dx$ and exchanging the order of summation and integration.

3 Euler's Reduction Formula

Euler's *reduction formula* is

$$\zeta(s, 1) = \frac{1}{2} s\zeta(s+1) - \frac{1}{2} \sum_{k=1}^{s-2} \zeta(k+1)\zeta(s-k), \qquad 1 < s \in \mathbf{Z}, \tag{6}$$

which *reduces* the double Euler sum $\zeta(s,1)$ to a sum of products of classical Riemann ζ-values. Another marvellous fact is the *sum formula*

$$\sum_{\substack{\Sigma a_i = s \\ a_i \geq 0}} \zeta\left(a_1 + 2, a_2 + 1, \cdots, a_r + 1\right) = \zeta(r + s + 1), \tag{7}$$

valid for all integers $s \geq 0$, $r \geq 1$. It is easy to show that (7) is equivalent to

$$\sum_{k_1 > k_2 > \cdots > k_r > 0} \frac{1}{k_1} \prod_{j=1}^{r} \frac{1}{k_j - x} = \sum_{n=1}^{\infty} \frac{1}{n^r (n-x)}, \qquad r \in \mathbf{Z}^+. \tag{8}$$

The first three non-trivial cases of (7) are $\zeta(3) = \zeta(2,1)$, $\zeta(4) = \zeta(3,1) + \zeta(2,2)$ and $\zeta(2,1,1) = \zeta(4)$.

The ordinary generating function of the sequence $\{\zeta(s) : 1 < s \in \mathbf{Z}\}$ is

$$Z(x) := \sum_{s=2}^{\infty} \zeta(s)\, x^{s-1} = \sum_{n=1}^{\infty} \frac{x}{n(n-x)} = -\Psi(1-x) - \gamma.$$

One can check that Euler's reduction (6) is equivalent to the ordinary generating function identity

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^{\infty} \frac{1}{n(n-x)^2} - \frac{x}{2} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)^2} + \left(\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \right)^2 \right\}. \tag{9}$$

This relies on observing that the right hand side of (6) involves the square and the derivative of $Z(x)$. In turn, (9) is equivalent to (1). This equivalence may be demonstrated as follows: Let

$$S_1 := \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{k=1}^{n-1} \frac{1}{k}, \qquad S_2 := \sum_{n=1}^{\infty} \frac{1}{n(n-x)^2}, \quad S_3 := \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)^2},$$
$$S_4 := \sum_{n=1}^{\infty} \frac{1}{n(n-x)}, \qquad S_5 := \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}, \quad S_6 := \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{k=1}^{n-1} \frac{1}{k-x}.$$

It suffices to show that $S_1 - S_2 + (S_3 + S_4^2)x/2 \equiv S_6 - S_5$. Observe that

$$S_1 - S_6 = -x \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{k=1}^{n-1} \frac{1}{k(k-x)}, \qquad S_5 - S_2 = -xS_3,$$

and

$$S_4^2 - S_3 = 2 \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{k=1}^{n-1} \frac{1}{k(k-x)}.$$

Combining these yields the desired equivalence.

Correspondingly, equation (1) is equivalent to

$$\zeta(s+3) = \sum_{\substack{a+b=s \\ a,b\geq 0}} \zeta(2+a, 1+b), \qquad 0 \leq s \in \mathbf{Z},$$

which is an inversion of Euler's reduction formula (6), and simultaneously recaptures the case $r = 2$ of equation (7).

Thus, Theorem 1 has established all of these results directly, without using analytic methods, or sophisticated partial fraction decompositions, as is usual. The first case, once more, is $\zeta(2,1) = \zeta(3)$.

4 More Generating Functions

Euler sums arise very naturally and it is perhaps a trick of time that $\zeta(2n)$ is viewed as more natural than

$$\zeta(\{2\}_n) := \zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \sum_{k_1 > k_2 > \cdots > k_n > 0} \prod_{j=1}^{n} \frac{1}{k_j^2}.$$

Indeed, Euler's infinite product for the sine function is precisely equivalent to

$$\frac{\sin(\pi x)}{\pi x} = \sum_{n=0}^{\infty} (-1)^n \, \zeta(\{2\}_n) \, x^{2n},$$

from which we may deduce that

$$\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n+1)!}, \qquad 0 \le n \in \mathbf{Z}.$$

Similarly, with ω a primitive cube root of unity,

$$\begin{aligned}\sum_{n=0}^{\infty} \zeta(\{3\}_n)\, x^{3n} &= \prod_{n=1}^{\infty}\left(1 + \frac{x^3}{k^3}\right) = \frac{1}{\Gamma(1+x)\Gamma(1+\omega x)\Gamma(1+\omega^2 x)} \\ &= 1 + \zeta(3)x^3 + \left(\frac{1}{2}\zeta^2(3) - \frac{8}{21}\frac{\pi^6}{6!}\right)x^6 + O\big(x^9\big),\end{aligned}$$

and allows one to show that $\zeta(\{3\}_n)$ is always in the ring generated by the numbers $\zeta(3k)$ $(k = 1, 2, \ldots, n)$.

By various methods, one can show that

$$\zeta(\{3\}_n) = \zeta(\{2,1\}_n)$$

for all $0 \le n \in \mathbf{Z}$, while a proof of

$$\zeta(\{2,1\}_n) \stackrel{?}{=} 2^{3n}\, \zeta(\{\overline{2},1\}_n), \qquad 0 \le n \in \mathbf{Z} \tag{10}$$

remains elusive. The bar over the 2 on the right hand side of (10) signifies that in the summation, terms of the form $1/k^2$ are to be multiplied by $(-1)^k$. Such sums are called alternating Euler sums. Only the first case of (10), namely

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{k-1} \frac{1}{m} = 8 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sum_{m=1}^{k-1} \frac{1}{m} \qquad (= \zeta(3))$$

has a self-contained proof [2, 4]. Indeed, the only other proven case[1] is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{k-1} \frac{1}{m} \sum_{p=1}^{m-1} \frac{1}{p^2} \sum_{q=1}^{p-1} \frac{1}{q} = 64 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sum_{m=1}^{k-1} \frac{1}{m} \sum_{p=1}^{m-1} \frac{(-1)^p}{p^2} \sum_{q=1}^{p-1} \frac{1}{q} \qquad (= \zeta(3,3)).$$

There has been abundant evidence amassed to support (10) since it was first conjectured [3] in 1996. For example, very recently Petr Lisoněk checked the first $n \le 85$ cases to 1000 decimal places in about 41 hours with only the *expected roundoff error*. And he checked $n = 163$ in ten hours. If true, (10) would be the *only* known identification thus far of an infinite parametrized class of alternating Euler sums with a corresponding class of non-alternating Euler sums. An intriguing reformulation of (10) is stated below.

[1] This is an outcome of a complete set of equations for MZV's of depth four.

Conjecture 1 *Define a sequence of polynomials* $a_n = a_n(t)$ *for positive integers* n *by* $a_1 = a_2 = t^3$ *and*

$$n(n+1)^2 a_{n+2} = n(2n+1)a_{n+1} + (n^3 + (-1)^{n+1}t^3)a_n, \qquad n \geq 1.$$

Then

$$\lim_{n\to\infty} a_n = t^3 \prod_{n=1}^{\infty} \left(1 + \frac{t^3}{8n^3}\right).$$

A more recondite generating function identity is

$$\sum_{n=0}^{\infty} \zeta(\{3,1\}_n)\, x^{4n} = \frac{\cosh(\pi x) - \cos(\pi x)}{\pi^2\, x^2},$$

which is equivalent to Zagier's conjecture (subsequently proved):

$$\zeta(\{3,1\}_n) = \frac{2\pi^{4n}}{(4n+2)!}, \qquad 0 \leq n \in \mathbf{Z}. \tag{11}$$

The proof of (11) (see [2, p. 160], [4], [5]) devolves from a remarkable factorization of the generating function in terms of Gaussian hypergeometric functions:

$$\sum_{n=0}^{\infty} \zeta(\{3,1\}_n)\, x^{4n} = {}_2F_1(t, -t; 1; 1)\; {}_2F_1(it, -it; 1; 1),$$

where $t = (1+i)x/2$.

5 Further extensions

Relatedly and more centrally, Euler's reduction formula (6) has many extensions. For instance, with even $a > 0$ and odd $b > 1$ with $a + b = 2N + 1$, one has [3]

$$\begin{aligned} \zeta(a,b) &= \zeta(a)\zeta(b) + \frac{1}{2}\left\{\binom{a+b}{a} - 1\right\}\zeta(a+b) \\ &\quad - \sum_{r=1}^{N-1}\left\{\binom{2r}{a-1} + \binom{2r}{b-1}\right\}\zeta(2r+1)\zeta(a+b-1-2r), \end{aligned} \tag{12}$$

and hence by the reflection formula (4),

$$\begin{aligned} \zeta(b,a) &= -\frac{1}{2}\left\{1 + \binom{a+b}{a}\right\}\zeta(a+b) \\ &\quad + \sum_{r=1}^{N-1}\left\{\binom{2r}{a-1} + \binom{2r}{b-1}\right\}\zeta(2r+1)\zeta(a+b-1-2r). \end{aligned} \tag{13}$$

Although $\zeta(1)$ is undefined, Euler's reduction (6) implies that (12) also holds in the case $b=1$ if we interpret $\zeta(1)=0$. Thus, (12) is indeed an extension of (6).

We next recast (12) using generating functions, and then develop some further consequences and interesting special cases. We initially keep $b=2t+1$ fixed and want to express

$$\sum_{N=t+1}^{\infty} \zeta(a,b)x^{2(N-t-1)} = \sum_{N=t+1}^{\infty}\sum_{n=1}^{\infty} \frac{x^{2(N-t-1)}}{n^{2(N-t)}} \sum_{m=1}^{n-1} \frac{1}{m^b} = \sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \sum_{m=1}^{n-1} \frac{1}{m^b}.$$

We suppose that x^2 is not a positive integer and set $D := \dfrac{d}{dx}$. The requisite component generating functions are

$$\sum_{N=t+1}^{\infty} \zeta(a)x^{2(N-t-1)} = \sum_{N=t+1}^{\infty}\sum_{n=1}^{\infty} \frac{x^{2(N-t-1)}}{n^{2(N-t)}} = \sum_{n=1}^{\infty} \frac{1}{n^2-x^2},$$

$$\sum_{N=t+1}^{\infty} \zeta(a+b)x^{2(N-t-1)} = \sum_{N=t+1}^{\infty}\sum_{n=1}^{\infty} \frac{x^{2(N-t-1)}}{n^{2(N-t)+b}} = \sum_{n=1}^{\infty} \frac{1}{n^b(n^2-x^2)},$$

$$\begin{aligned}\sum_{N=t+1}^{\infty} \zeta(a+b)\binom{a+b}{b} x^{2(N-t-1)} &= x^{-2} \sum_{N=t+1}^{\infty} \zeta(2N+1)\frac{D^b}{b!}x^{2N+1} \\ &= \frac{1}{2x^2}\sum_{n=1}^{\infty} n\left(\frac{1}{(n-x)^{b+1}} + \frac{1}{(n+x)^{b+1}}\right) - \frac{\zeta(b)}{x^2},\end{aligned}$$

$$\begin{aligned}\sum_{N=t+1}^{\infty} x^{2(N-t-1)} &\sum_{r=1}^{N-1} \binom{2r}{b-1} \zeta(2r+1)\zeta(2N-2r) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^2-x^2}\right) \sum_{n=1}^{\infty} \frac{D^{b-1}}{(b-1)!}\left(\frac{x^2}{n(n^2-x^2)}\right) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^2-x^2}\right) \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{(n-x)^b} + \frac{1}{(n+x)^b}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{N=t+1}^{\infty} x^{2(N-t-1)} &\sum_{r=1}^{N-1} \binom{2r}{a-1} \zeta(2r+1)\zeta(2N-2r) \\ &= \sum_{m=1}^{t} \zeta(2m)x^{-1} \sum_{n=1}^{\infty} \frac{D^{b-2m}}{(b-2m)!}\left(\frac{x^2}{n(n^2-x^2)}\right) \\ &= \sum_{m=1}^{t} \zeta(2m)x^{-1} \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{(n-x)^{b+1-2m}} - \frac{1}{(n+x)^{b+1-2m}}\right)\end{aligned}$$

$$= \sum_{m=1}^{t} \zeta(b+1-2m) x^{-1} \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{(n-x)^{2m}} - \frac{1}{(n+x)^{2m}} \right).$$

Combining these, with a fair amount of care, yields a generating function for fixed b:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \sum_{m=1}^{n-1} \frac{1}{m^b} &= \sum_{s=1}^{\infty} \zeta(2s,b) x^{2s-2} \\
&= \zeta(2t+1) \sum_{k=1}^{\infty} \frac{1}{k^2-x^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2t+1}(n^2-x^2)} \\
&\quad + \frac{1}{2} \sum_{n=1}^{\infty} \frac{n}{2x^2} \left\{ \frac{1}{(n-x)^{2t+2}} + \frac{1}{(n+x)^{2t+2}} - \frac{2}{n^{2t+2}} \right\} \\
&\quad - \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \right) \cdot \sum_{n=1}^{\infty} \left\{ \frac{1}{(n-x)^{2t+1}} + \frac{1}{(n+x)^{2t+1}} \right\} \\
&\quad - \sum_{m=1}^{t} \zeta(2t+2-2m) \sum_{n=1}^{\infty} \frac{1}{2x} \left\{ \frac{1}{(n-x)^{2m}} - \frac{1}{(n+x)^{2m}} \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \sum_{m=1}^{n-1} \frac{1}{m^b} &= \frac{\pi \cot(\pi x)}{4x} \sum_{n=1}^{\infty} \left\{ \frac{1}{(n-x)^{2t+1}} + \frac{1}{(n+x)^{2t+1}} - \frac{2}{n^{2t+1}} \right\} \\
&\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^{-2t-1}}{n^2-x^2} - \frac{1}{2x} \sum_{m=1}^{t+1} \zeta(2t+2-2m) \sum_{n=1}^{\infty} \left\{ \frac{1}{(n-x)^{2m}} - \frac{1}{(n+x)^{2m}} \right\},
\end{aligned}$$

using

$$\frac{\pi \cot(\pi x)}{2x} = \frac{1}{2x^2} - \sum_{k=1}^{\infty} \frac{1}{k^2-x^2} \quad \text{and} \quad \zeta(0) = -\frac{1}{2}.$$

Now we sum over the odd parameter, b, and obtain a two-variable ordinary generating function expressible as:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \sum_{m=1}^{n-1} \frac{my}{m^2-y^2} &= \sum_{s>0, t\geq 0} \zeta(2s, 2t+1)\, x^{2s-2} y^{2t-1} \\
&= \frac{\pi}{2} x \cot(\pi x) \sum_{n=1}^{\infty} \frac{ny(4y^2-x^2)/(n^2-y^2)}{((n+x)^2-y^2)\,((n-x)^2-y^2)} \\
&\quad + \frac{\pi}{2} x \cot(\pi x) \sum_{n=1}^{\infty} \frac{ny}{((n+x)^2-y^2)\,((n-x)^2-y^2)} \\
&\quad + \frac{\pi}{2} y \cot(\pi y) \sum_{n=1}^{\infty} \frac{2ny}{((n+x)^2-y^2)\,((n-x)^2-y^2)}
\end{aligned}$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{ny}{(n^2-y^2)\,(n^2-x^2)}. \tag{14}$$

Attractive specializations come with $x = \pm y$, $x = \pm 2i, y$, $x = 0$ and, after dividing by y, for $y = 0$. For example, at $(1/2, 1/2)$ where the right hand side has a removable discontinuity, we deduce that

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\sum_{m=1}^{n-1}\frac{m}{4m^2-1}=\frac{\pi^2}{64}-\frac{1}{16}.$$

Note that the second, third and fourth terms on the right hand side of (14) are expressible as

$$\Sigma_2 := -\frac{\pi}{4}x\cot(\pi x)\,\frac{\Psi(1+x+y)-\Psi(1+x-y)-\Psi(1-x+y)+\Psi(1-x-y)}{4xy},$$

$$\Sigma_3 := \frac{\pi}{2}y\cot(\pi y)\,\frac{\Psi(1+x+y)-\Psi(1+x-y)-\Psi(1-x+y)+\Psi(1-x-y)}{4xy},$$

$$\Sigma_4 := \frac{1-\{\Psi(1+x)-\Psi(1-y)+\Psi(x)-\Psi(y)+\pi\cot(\pi x)\}\,y}{4(y^2-x^2)},$$

respectively while the first term has a corresponding evaluation. It is

$$\begin{aligned}\Sigma_1 :=\; & y^2\pi\cot(\pi)\;\frac{\Psi(1-x-y)+\Psi(1+x-y)+\Psi(1+x+y)+\Psi(1-x+y)}{8x(x^2-4\,y^2)}\\ & -y\pi\cot(\pi x)\,\frac{\Psi(1+x+y)-\Psi(1+x-y)-\Psi(1-x+y)+\Psi(1-x-y)}{16(x^2-4\,y^2)}\\ & -y^2\pi\cot(\pi x)\,\frac{\Psi(y+1)+\Psi(1-y)}{4x(x^2-4\,y^2)}.\end{aligned}$$

We finish by expressing a form of this generating function concisely as:

Theorem 3 *For all x and y with squares not equal to negative integers,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{n}{n^2+y^2}+\sum_{m=1}^{n-1}\frac{2m}{m^2+y^2}\right)\frac{1}{n^2+x^2}\\ &\quad=(x^2-4y^2)\pi x\coth(\pi x)\sum_{n=1}^{\infty}\frac{n}{(n^2+y^2)\,\{(n^2-x^2+y^2)^2+(2nx)^2\}}\\ &\quad+(2\pi y\coth(\pi y)+\pi x\coth(\pi x))\sum_{n=1}^{\infty}\frac{n}{(n^2-x^2+y^2)^2+(2nx)^2}.\end{aligned}$$

Proof. Replace x by ix and y by iy in (14), and then factor denominators and regroup the terms as needed. **QED**

Letting x and y approach zero yields $\zeta(2,1) = \zeta(3)$ again. Setting $x = 2y$ produces (2)—the identity with which we began.

To conclude, we challenge the reader to explicitly obtain the corresponding two variable generating function for $\zeta(2t+1, 2s)$.

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