Subsequences of Automatic Sequences

with Polynomial Growth

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In Memory of Alf van der Poorten
DENSITIES

Let $A \subseteq \mathbb{N}$

NATURAL (or ASYMPTOTIC) DENSITY

$$d(A) = \lim_{N \to \infty} \frac{1}{N} \# \left\{ n \leq N \mid n \in A \right\} \quad \text{(if it exists)}$$

LOGARITHMIC DENSITY

$$d_{\log}(A) = \lim_{N \to \infty} \frac{1}{\log(N)} \sum_{n \leq N \mid n \in A} \frac{1}{n} \quad \text{(if it exists)}$$
Let $E$ denote a finite set and let $\nu : \mathbb{N} \rightarrow E$. For $a \in E$, we shall consider the existence and the value of $\delta(\nu = a) = \delta(\{n/\nu(n) = a\})$, for $\delta = d$ or $d_{log}$.

Let $c > 1$. We denote by $\nu_c$ the sequence defined by $\forall n : \nu_c(n) = \nu(\ln^{\log n})$.

Can we compare $\delta(\nu = a)$ and $\delta(\nu_c = a)$?
Theorem (Harman - Rivat, 1995)

If $d(v = a)$ exists, then
\[ \forall c \in [1, 2] : d(v_c = a) \text{ exists and is equal to } d(v = a) \]

Theorem (Mauduit - Rivat, 2005)

Let $1 < c < 7/5$, $q \geq 2$, $m \geq 2$. Then
\[ \forall a : d(\{ n \in \mathbb{N} / s_q(l_{n+1}) = a \ (\text{mod } m) \}) = \frac{1}{m}, \]
where $s_q(m)$ denotes the sum of the digits of $m$ in base $q$. 
Fact. The function $s_q$ is $q$-additive, i.e.

$$\forall r \geq 1, \forall a \geq 0, \forall b \in [0, q^r - 1]:$$

$$s_q(a q^r + b) = s_q(a q^r) + s_q(b),$$

and $s_q \mod m$ is $q$-additive with values in $\mathbb{Z}/m\mathbb{Z}$.

Consequence. The function $f(n) = e_m(h s_q(n))$ is $q$-multiplicative. This permits to "factorize"

$$\sum_{n \leq x} f(n)$$

and show that it is $o(x^{1-\delta})$ for some $\delta > 0$. 
How to treat \( \sum_{m \leq \gamma} f(Lm^{c}) \)?

We have to select those \( n \)'s which have the shape \( \lfloor m^{c} \rfloor \), for some \( m \).

This is equivalent to \( m^{c} \leq n < m^{c+1} \)

or \( m \leq n^{\gamma} < (m^{c+1})^{\gamma} = m + \gamma m^{1-c} + \ldots \), with \( \gamma = 1/c \).

We have (essentially)

\[
\sum_{m \leq \gamma} f(Lm^{c}) = \sum_{n \leq x} f(n)
\]

\[
\lfloor n^{\gamma} \rfloor \leq \gamma n^{\gamma-1}
\]
Build a machine which reads in written in base 2 and produces $s_2(n)$ modulo 3.

Example $91 = \underline{1011011}_2 \rightarrow 2$. 
Finite set $R$ (states)
Initial state $i \in R$
$\Sigma = \{0, 1, \ldots, q-1\}$
$R \times \Sigma \xrightarrow{\delta} R$
$R \xrightarrow{\tau} E$
Reading the most significant digit of $n$ in base 3:

$92 = \underbrace{10102}_3$

from the right

from the left
Theorem (M. Drmota – J. Morgenbesser – 2nd)

Let $c \in (1, 7/5)$, $q \geq 2$ and $u$ a $q$-automatic sequence with value in a (finite) set $E$.

For any $a \in E$, the logarithmic density

$$d_{\log} (u_c = a) \text{ exists and is equal to } d_{\log} (u = a).$$

Moreover, $d(u_c = a)$ exists if and only if $d(u = a)$ exists (and then there are equal).
**Definition** \( F : \mathbb{N} \to M_d(\mathbb{C}) \) is \( q \)-multiplicative if there exist \( G_r^{(1)}, G_r^{(2)} : \mathbb{N} \to M_d(\mathbb{C}) \) : \( F(q^k a + b) = G_r^{(1)}(b) G_r^{(2)}(a) \) (for \( k \geq 1, a \geq 0, 0 \leq b < q^k \)).

**Proposition** Let \( \| \cdot \|_s \) be a Banach norm on \( M_d(\mathbb{C}) \), \( F \) \( q \)-multiplicative with \( \| G_r^{(1)}(b) \|_s \leq 1 \). Let \( c \in (1, 7/5), \delta \in (0, (7-5c)/8) \). Then

\[
\| \sum_{1 \leq n \leq x} F(L_n c) - \sum_{1 \leq m \leq x^c} \gamma m^{-1} F(m) \|_s = O(\alpha^{-5}).
\]
Mahler (1927) proved that for the Thue-Morse sequence 
\( E(n) \equiv S_2(n) \pmod{2} \) , that for \( k > 0 \) and \( (\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2 \):

\[
P_k(\varepsilon_1, \varepsilon_2) = \lim_{x \to \infty} \frac{1}{x} \sum_{1 \leq n \leq x} \chi\left\{ (E(n), E(n+k)) = (\varepsilon_1, \varepsilon_2) \right\}
\]

exists and is \( \neq 1/4 \) for infinitely many \( k \)’s.

**Theorem (M. Drmota, J. Morgenbesser, 1970)**

Let \( k > 0 \), \( (\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2 \).

For \( c \in (1, 7/5) \):
\[
\lim_{x \to \infty} \frac{1}{x} \sum_{1 \leq n \leq x} \chi\left\{ (E(\lfloor cn^2 \rfloor), E(\lfloor cn^2 + k \rfloor)) = (\varepsilon_1, \varepsilon_2) \right\} = \frac{P_k(\varepsilon_1, \varepsilon_2)}{2}
\]

For \( c \in (1, 10/9) \):
\[
\lim_{x \to \infty} \frac{1}{x} \sum_{1 \leq n \leq x} \chi\left\{ (E(\lfloor (n+1) \log n \rfloor), E(\lfloor (n+1) \log (n+1) \rfloor)) = (\varepsilon_1, \varepsilon_2) \right\} = \frac{1}{4}
\]

\[\times 1 \rightarrow\frac{1}{x} \sum_{1 \leq n \leq x} \chi\left\{ (E(\lfloor (n+1) \log (n+1) \rfloor), E(\lfloor (n+1) \log (n+1) \rfloor)) = (\varepsilon_1, \varepsilon_2) \right\}
\]

has no limit.