Stabilized Mixed Finite Element Method for Poisson Problem Based on a Three-Field Formulation

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Outline

Poisson Problem

In mathematics, Poisson's equation is a partial differential equation of elliptic type with broad utility in electrostatics, mechanical engineering and theoretical physics.
Consider a bounded polyhedral domain Ω in $\mathbb{R}^d$, $d \in \{2, 3\}$. In its most general form, Poisson's equation is written as

$$-\Delta u = f \quad \text{in } \Omega$$

where $u(\mathbf{x})$ is some scalar function to be determined and $f(\mathbf{x})$ is a known "source function" for all $\mathbf{x} \in \Omega$. Usually it comes with known boundary condition.

Minimization Problem

- We define $L^2(\Omega)$ be the space of square-integrable function defined as
$$L^2(\Omega) = \left\{ u \mid \|u\|_{L^2,\Omega} < \infty \right\}$$
where
$$\|u\|_{L^2,\Omega} = \|u\|_{0,\Omega} = \sqrt{\int_\Omega |u(x)|^2 \, dx}$$
- We also define Sobolev space $H_0^1(\Omega)$ as
$$H_0^1 = \left\{ u \mid \|u\|_{H^1,\Omega} < \infty, u|_{\partial\Omega} = 0 \right\}$$
where
$$\|u\|_{H^1,\Omega} = \|u\|_{1,\Omega} = \left(\left\|\partial^1 u\right\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2 \right)^{\frac{1}{2}}$$
and $\partial^1 u$ is defined as first-order weak derivative of u.

Minimization Problem (2)

- Let $V = H_0^1(\Omega)$ and $L = \left[L^2(\Omega)\right]^d$, $d \in \{2,3\}$. We start with the following minimization problem for the Poisson problem:

$$\arg\min_{v \in V} \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx - \ell(v),$$

where $\ell(v) = \int_\Omega fv \, dx$ and $f \in L^2(\Omega)$.

- Introduce $\sigma = \nabla u$, we could write the following constrained minimization problem

$$\arg\min_{\substack{(u,\sigma)\in V\times L \\ \sigma = \nabla u}} \frac{1}{2}\left\|\sigma^2\right\|_{0,\Omega} - \ell(u).$$

- The norm for the product space $V \times L$ is defined by

$$\|(u,\sigma)\|_{V\times L} = \sqrt{\|\sigma\|_{0,\Omega}^2 + \|u\|_{1,\Omega}^2}$$

for $(u,\sigma) \in V \times L$.

Saddle Point Formulation

The saddle-point formulation is to find $(u, \sigma, \varphi) \in V \times L \times M$ such that

$$\widetilde{a}\left((u,\sigma),(v,\tau)\right) + b\left((v,\tau),\varphi\right) = \ell\left(v\right), \quad (v,\tau) \in V \times L,$$
$$b\left((u,\sigma),\psi\right) = 0, \quad \psi \in M,$$

where

$$\widetilde{a}\left((u,\sigma),(v,\tau)\right) = \int_\Omega \sigma \cdot \tau \, dx,$$
$$b\left((u,\sigma),\psi\right) = \int_\Omega (\sigma - \nabla u) \cdot \psi \, dx,$$
$$\ell\left(v\right) = \int_\Omega fv \, dx.$$

and here $V = H_0^1(\Omega)$, $L = M = \left[L^2(\Omega)\right]^2$.

Well-Posedness Conditions (Lax-Milgram Theorem)

In order to show that saddle-point problem has a unique solution, we need to show that three conditions of well-posedness are satisfied.

1. The linear form $\ell(\cdot)$, the bilinear form $\widetilde{a}(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on the spaces in which they are defined.
2. The bilinear form $\widetilde{a}(\cdot,\cdot)$ is coercive on the kernel space which is defined by $K = \{(u,\sigma) \in V \times L |\, b((u,\sigma),\psi) = 0, \forall \psi \in M\}$.
3. The bilinear form $b(\cdot,\cdot)$ satisfies the following condition

$$\inf_{\psi \in M} \sup_{(v,\tau) \in V \times L} \frac{b((v,\tau),\psi)}{\|(v,\tau)\|_{V \times L} \|\psi\|_M} \geq \gamma, \quad \gamma > 0.$$

Problem in Finite Element Methods

- Here the bilinear form $\widetilde{a}(\cdot,\cdot)$ is not coercive on the whole space $V \times L$. It is only coercive on the kernel subspace $K \subset V \times L$,

$$\widetilde{a}((u,\sigma),(u,\sigma)) = \|\sigma\|_{0,\Omega}^2 \geq \alpha \|(u,\sigma)\|_{V\times L}^2 .$$

- On the discrete formulation, it is not easy to select appropriate kernel space to satisfy the coercivity condition.
- It is easier to develop a finite element method if the bilinear form $\widetilde{a}(\cdot,\cdot)$ is coercive on the whole space $V \times L$.
- One way to make the bilinear form $\widetilde{a}(\cdot,\cdot)$ coercive on the whole space $V \times L$ is to stabilize it.

First Approach

We modify the bilinear form $\widetilde{a}(\cdot,\cdot)$ so that we have the new bilinear form

$$a_1\left((u,\sigma),(v,\tau)\right) = r\int_\Omega \sigma\cdot\tau\,dx + (1-r)\int_\Omega \nabla u\cdot\nabla v\,dx,$$

where $0 < r < 1$.

Second Approach

We modify the bilinear form $\widetilde{a}(\cdot,\cdot)$ so that we have the new bilinear form

$$a_2\left((u,\sigma),(v,\tau)\right) = \int_\Omega \sigma \cdot \tau \, dx + r \int_\Omega (\sigma - \nabla u) \cdot (\tau - \nabla v) \, dx,$$

where $r > 0$.
Non parameterized form is used in:

- Mindlin-Reissner plate problem (Arnold and Brezzi, 1993)
- Poisson problem (Lamichhane, 2013)

We use the standard linear finite element space $V_h \subset H_0^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$V_h := \{v \in C^0(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}.$$

We define the local basis functions of V_h and M_h on the reference triangle $\hat{T} := \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$ associated with vertices $(0, 0), (1, 0)$ and $(0, 1)$.

- Let local basis functions $\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\}$ of V_h be defined as

$$\hat{\rho}_1 = 1 - x - y, \quad \hat{\rho}_2 = x, \quad \hat{\rho}_3 = y$$

- We construct local basis functions $\{\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3\}$ of M_h as

$$\hat{\mu}_1 = 3 - 4x - 4y, \quad \hat{\mu}_2 = 4x - 1, \quad \hat{\mu}_3 = 4y - 1$$

so that the basis functions of V_h and M_h satisfy a condition of biorthogonality relation

$$\int_\Omega \rho_i \mu_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j, \leq N,$$

where δ_{ij} is the Kronecker symbol, and c_j a scaling factor.

Poisson Problem

Our problem is to find $(u_h, \sigma_h, \varphi_h) \in V_h \times [V_h]^2 \times [M_h]^2$ such that

$$\begin{aligned} \hat{a}\left((u_h, \sigma_h), (v_h, \tau_h)\right) + b\left((v_h, \tau_h), \varphi_h\right) &= \ell(v_h) \\ b\left((u_h, \sigma_h), \psi_h\right) &= 0 \end{aligned} \tag{1}$$

where for the first approach,

$$\hat{a}\left((u_h, \sigma_h), (v_h, \tau_h)\right) = a_1\left((u_h, \sigma_h), (v_h, \tau_h)\right)$$

and for second approach,

$$\hat{a}\left((u_h, \sigma_h), (v_h, \tau_h)\right) = a_2\left((u_h, \sigma_h), (v_h, \tau_h)\right)$$

and

$$b\left((u_h, \sigma_h), \psi_h\right) = \int_\Omega (\sigma_h - \nabla u_h) \cdot \psi_h \, dx$$

$$\ell(v_h) = \int_\Omega f v \, dx.$$

Extended Céa's Lemma

Theorem

Suppose that well-posedness conditions are satisfied. In addition, suppose (u, σ) and (u_h, σ_h) are the solutions of the continuous and discrete variational problem, respectively. Then $\|(u, \sigma) - (u_h, \sigma_h)\|_{V \times L}$ is bounded from above by

$$\left(1 + \frac{C_a}{\alpha}\right) \inf_{(v_h, \tau_h) \in V_h \times L_h} \|(u, \sigma) - (v_h, \tau_h)\|_{V \times L} + \frac{C_b}{\alpha} \inf_{\psi_h \in M_h} \|\varphi - \psi_h\|_M \,,$$

where C_a and C_b are the continuity constants for bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, respectively, and α is the coercivity constant of $a(\cdot, \cdot)$.

First approach

Approximation give us the following constants as a function of parameter.

- $C_a = \sqrt{2}\max\{r, 1-r\}$ depends only on r.
- $\alpha_1 = \frac{1}{\max\left\{\frac{1}{r}, \frac{c^2+1}{1-r}\right\}}$ depends on r and c, Poincaré inequality constant.

The smallest possible value of constant in extended Céa's Lemma is obtained when we choose r such that

$$\arg\min_r \frac{C}{\alpha_1} = \arg\min_r \left\{\sqrt{2}\max\{r, 1-r\} \cdot \max\left\{\frac{1}{r}, \frac{c^2+1}{1-r}\right\}\right\}.$$

If we choose $\Omega = [0,1] \times [0,1]$ as our domain, then Poincaré inequality constant is one and $\frac{1}{3} \le r \le \frac{1}{2}$ will approximate optimal value of r.

Approximation give us the following constants as a function of parameter.

- $C_a = \max\{r+1, r\} = r+1.$
- $\alpha_2 = \frac{1}{\max\left\{2c^2+3, \frac{2c^2+2}{r}\right\}}$ depends on r and c, Poincaré inequality constant.

The smallest possible value of constant in Céa's Lemma is obtained when we choose r such that

$$\arg\min_r \frac{C}{\alpha_2} = \arg\min_r \left\{(r+1)\cdot\max\left\{2c^2+3, \frac{2c^2+2}{r}\right\}\right\}.$$

If we choose $\Omega = [0,1]\times[0,1]$ as our domain, then Poincaré inequality constant is one and $r = \frac{4}{5}$ will approximate optimal value of r.

Analytic solution

- Well behaved problem with a smooth solution that has no trouble spots.
- Dirichlet boundary condition.
- Domain $\Omega = [0, 1] \times [0, 1]$.
- Exact solution $u = 2^{4a} x^a (1 - x)^a y^a (1 - y)^a$.

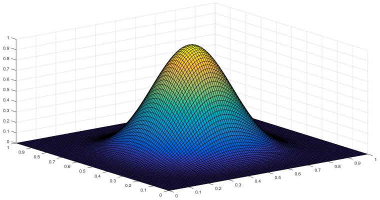

Figure: Analytic solution for $a = 5$

First Approach

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	0.6338		0.9295		1.1758		3.4340	
32	0.1231	5.1498	0.0964	9.6468	0.0824	14.2706	1.1751	2.9223
128	0.0446	2.7582	0.0234	4.1196	0.0132	6.2610	0.1244	9.4447
512	0.0131	3.3954	0.0046	5.0908	0.0022	5.9225	0.0197	6.3079
2048	0.0034	3.8242	0.0010	4.5977	0.0006	3.8025	0.0046	4.3345
8192	0.0009	3.9544	0.0002	4.1719	0.0002	3.8761	0.0011	4.0745

Table: $\|u - u_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	H^1-error	rate	H^1-error	rate	H^1-error	rate	H^1-error	rate
8	4.1224		5.7319		7.1045		19.9649	
32	1.0831	3.8060	1.0872	5.2720	1.2507	5.6806	18.2038	1.0967
128	0.6947	1.5591	0.6738	1.6136	0.7207	1.7352	2.5808	7.0536
512	0.3527	1.9697	0.3499	1.9258	0.3618	1.9923	0.4838	5.3345
2048	0.1762	2.0012	0.1760	1.9884	0.1777	2.0360	0.1911	2.5319
8192	0.0881	2.0011	0.0880	1.9986	0.0883	2.0131	0.0899	2.1256

Table: $\|u - u_h\|_{1,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	2.6314		3.2245		3.7788		9.5791	
32	1.4079	1.8691	1.3227	2.4378	1.2549	3.0112	2.6403	3.6280
128	0.7520	1.8722	0.6005	2.2027	0.0132	2.5306	0.3625	7.2828
512	0.2570	2.9264	0.1729	3.4723	0.1233	4.0205	0.0712	5.0904
2048	0.0707	3.6332	0.0438	3.9469	0.0294	4.1975	0.0167	4.2592
8192	0.0181	3.8991	0.0110	3.9974	0.0072	4.0642	0.0041	4.0599

Table: $\|\sigma - \sigma_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	0.0228		0.8679		1.4653		6.7169	
32	0.0153	1.4910	0.4930	1.7606	0.7198	2.0358	1.0105	6.6474
128	0.0094	1.6285	0.2741	1.7987	0.3708	1.9413	0.2383	4.2394
512	0.0035	2.6497	0.0929	2.9511	0.1170	3.1690	0.0682	3.4951
2048	0.0010	3.4845	0.0252	3.6860	0.0308	3.8016	0.0178	3.8417
8192	0.0003	3.8511	0.0064	3.9205	0.0078	3.9545	0.0045	3.9595

Table: $\|\varphi - \varphi_h\|_{0,\Omega}$

Second Approach

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	3.4745		1.1240		0.1016	
32	1.2084	2.8753	0.0873	12.8680	0.1830	0.5551
128	0.1273	9.4934	0.0254	3.4434	0.0850	2.1523
512	0.0200	6.3568	0.0098	2.5944	0.0262	3.2481
2048	0.0046	4.3401	0.0028	3.4777	0.0063	4.1311
8192	0.0011	4.0756	0.0007	3.8841	0.0015	4.1491

Table: $\|u - u_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	H^1-error	rate	H^1-error	rate	H^1-error	rate
8	20.1968		6.8141		1.7076	
32	18.6895	1.0806	1.0993	6.1988	1.4440	1.1825
128	2.6344	7.0945	0.7997	1.3746	0.9252	1.5607
512	0.4865	5.4148	0.3969	2.0148	0.4095	2.2595
2048	0.1914	2.5423	0.1836	2.1612	0.1828	2.2400
8192	0.0899	2.1280	0.0891	2.0621	0.0888	2.0592

Table: $\|u - u_h\|_{1,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	9.6870		3.6589		1.9829	
32	2.7251	3.5547	1.2935	2.8287	1.5807	1.2545
128	0.3793	7.1848	0.4365	2.9632	1.0235	1.5444
512	0.0742	5.1092	0.0589	7.4074	0.3824	2.6761
2048	0.0174	4.2650	0.0072	8.2059	0.1026	3.7288
8192	0.0043	4.0612	0.0014	5.0489	0.0254	4.0307

Table: $\|\sigma - \sigma_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	6.8342		2.5435		2.6502	
32	1.0246	6.6699	1.5920	1.5977	1.9105	1.3872
128	0.2365	4.3322	0.8824	1.8041	1.7696	1.0796
512	0.0676	3.4995	0.2609	3.3823	0.8807	2.0093
2048	0.0176	3.8402	0.0638	4.0907	0.2801	3.1443
8192	0.0044	3.9591	0.0157	4.0503	0.0743	3.7714

Table: $\|\varphi - \varphi_h\|_{0,\Omega}$

Peak problem

- Has an exponential peak in the interior of the domain.
- Dirichlet boundary condition.
- Domain $\Omega = [0,1] \times [0,1]$.
- Exact solution $u = e^{-\alpha\left((x-x_c)^2+(y-y_c)^2\right)}$.

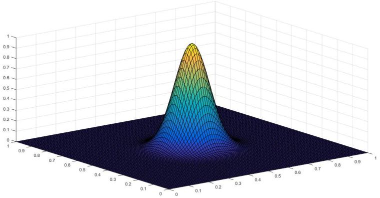

Figure: Peak problem for $\alpha = 100$ and $(x_c, y_c) = (0.5, 0.5)$

First Approach

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	4.4214		5.9092		7.1451		18.4461	
32	1.8856	2.3449	2.1470	2.7523	2.3244	3.0740	6.7012	2.7527
128	0.0706	26.6979	0.0746	28.7921	0.0830	28.0130	1.1121	6.0259
512	0.0219	3.2276	0.0119	6.2680	0.0071	11.6398	0.0790	14.0683
2048	0.0066	3.3216	0.0024	5.0273	0.0012	6.1449	0.0102	7.7180
8192	0.0017	3.7998	0.0005	4.6677	0.0003	3.8382	0.0023	4.4045

Table: $\|u - u_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	H^1-error	rate	H^1-error	rate	H^1-error	rate	H^1-error	rate
8	24.8781		33.4178		40.5141		105.4228	
32	9.6336	2.5824	11.3622	2.9411	12.8159	3.1612	86.5195	1.2185
128	0.9829	9.8012	1.1818	9.6147	1.5169	8.4488	34.7271	2.4914
512	0.7001	1.4040	0.6771	1.7453	0.7240	2.0952	3.7760	9.1968
2048	0.3579	1.9559	0.3546	1.9096	0.3682	1.9664	0.5179	7.2906
8192	0.1789	2.0005	0.1786	1.9855	0.1806	2.0383	0.1967	2.6337

Table: $\|u - u_h\|_{1,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	11.8934		15.9231		19.2768		50.0050	
32	7.5847	1.5681	8.5164	1.8697	9.1203	2.1136	19.0944	2.6188
128	1.2755	5.9467	1.2030	7.0790	1.1593	7.8673	3.6217	5.2722
512	0.7736	1.6488	0.6339	1.8979	0.5357	2.1640	0.4429	8.1780
2048	0.2763	2.8001	0.1891	3.3514	0.1367	3.9190	0.0802	5.5216
8192	0.0773	3.5745	0.0481	3.9297	0.0324	4.2211	0.0186	4.3194

Table: $\|\sigma - \sigma_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.33$		$r = 0.5$		$r = 0.99$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	0.0838		3.7355		6.7827		34.8716	
32	0.0673	1.2451	2.5031	1.4923	3.9986	1.6963	11.3885	3.0620
128	0.0140	4.7899	0.4489	5.5764	0.6519	6.1338	1.3265	8.5856
512	0.0095	1.4821	0.2807	1.5990	0.3844	1.6960	0.2578	5.1457
2048	0.0038	2.5092	0.1002	2.8020	0.1271	3.0231	0.0745	3.4590
8192	0.0011	3.4040	0.0276	3.6281	0.0338	3.7602	0.0196	3.8090

Table: $\|\varphi - \varphi_h\|_{0,\Omega}$

Second Approach

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	18.6486		6.8851		1.5590	
32	6.8404	2.7262	2.4356	2.8268	1.3139	1.1866
128	1.1359	6.0221	0.0945	25.7796	0.0875	15.0198
512	0.0809	14.0464	0.0132	7.1758	0.0407	2.1499
2048	0.0104	7.7749	0.0049	2.7017	0.0132	3.0930
8192	0.0024	4.4113	0.0014	3.4016	0.0032	4.1022

Table: $\|u - u_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	H^1-error	rate	H^1-error	rate	H^1-error	rate
8	106.5861		39.0209		8.4878	
32	88.6791	1.2019	12.7585	3.0584	6.5998	1.2861
128	35.4242	2.5033	1.4178	8.9986	1.2839	5.1406
512	3.8533	9.1931	0.7872	1.8012	0.9300	1.3805
2048	0.5213	7.3917	0.4071	1.9334	0.4233	2.1970
8192	0.1970	2.6462	0.1877	2.1693	0.1866	2.2689

Table: $\|u - u_h\|_{1,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	50.5559		18.5709		4.2466	
32	19.4404	2.6006	9.6945	1.9156	5.5462	0.7657
128	3.7089	5.2416	1.2114	8.0030	1.4617	3.7944
512	0.4623	8.0223	0.4950	2.4474	1.0231	1.4286
2048	0.0836	5.5286	0.0713	6.9440	0.4100	2.4955
8192	0.0193	4.3262	0.0082	8.6873	0.1117	3.6703

Table: $\|\sigma - \sigma_h\|_{0,\Omega}$

elements	$r = 0.01$		$r = 0.8$		$r = 5$	
	L^2-error	rate	L^2-error	rate	L^2-error	rate
8	35.4758		9.6611		5.9221	
32	11.5484	3.0719	7.7103	1.2530	3.7223	1.5910
128	1.3443	8.5908	1.4764	5.2223	1.9513	1.9076
512	0.2561	5.2486	0.9120	1.6188	1.7018	1.1466
2048	0.0739	3.4681	0.2872	3.1759	0.9157	1.8584
8192	0.0194	3.8072	0.0703	4.0821	0.3036	3.0163

Table: $\|\varphi - \varphi_h\|_{0,\Omega}$

Numerical Example Summary

First Approach

1. For $\|u - u_h\|_{0,\Omega}$, chosen parameter give the least error.
2. For $\|u - u_h\|_{1,\Omega}$, chosen parameter doesn't give significant improvement.
3. For $\|\sigma - \sigma_h\|_{0,\Omega}$, least error obtained when $r \to 1$.
4. But for $\|\varphi - \varphi_h\|_{0,\Omega}$, chosen parameter give worst error.

Second Approach

1. For $\|u - u_h\|_{0,\Omega}$, chosen parameter give the least error.
2. For $\|u - u_h\|_{1,\Omega}$, chosen parameter doesn't give significant improvement.
3. For $\|\sigma - \sigma_h\|_{0,\Omega}$, chosen parameter give the least error.
4. But for $\|\varphi - \varphi_h\|_{0,\Omega}$, least error obtained when $r \to 0$.

Summary

- Two parameterized approaches are formulated to stabilize bilinear form of Poisson problem.
- Approximation of optimal parameters based on extended Céa's Lemma.
- Numerical examples are given to observe optimal parameter approximation.

- Outlook
  - Construct adaptive mesh refinement
  - Extend this result for elasticity problem.
  - Observe another stabilization form.