Effective dimension for weighted ANOVA and anchored spaces

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Outline

1 Motivation
   • High Dimensional Integral: Effective dimension
   • Compare ANOVA and Anchored Decompositions and Spaces

2 Main Results
   • Relate Variances to Norms
   • Implications

3 Summary
1 Motivation
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High Dimensional Integral

\[ \int_{[0,1]^d} f(x) \, dx \]

- Applications: e.g., in finance \( d = 360 = 12 \times 30 \).
- Some integration problems are easier than others, e.g.,

\[ f(x) = \sum_{i=1}^{d} f_i(x_i). \]
Multivariate Decomposition

\[ f(x) = \sum_{u \subseteq [1:d]} f_u(x), \]

where \( u \) is a subset of \( \{1, 2, \ldots, d\} := [1 : d] \), and \( f_u(x) \) depends only on \( x_j \) for \( j \in u \).

ANOVA Decomposition

\[ f_{A, \emptyset} = \int_{[0,1]^d} f(x) \, dx \quad \text{and} \quad f_{A, u}(x) = \int_{[0,1]^{|u_c|}} f(x) \, dx_{u_c} - \sum_{v \subseteq u} f_{A, v}, \quad (1) \]

where \( x_u = (x_j)_{j \in u} \).

- Integrate out coordinates not in \( u \) (hard to evaluate)
Example

\[ f(x) = \sum_{i=1}^{d} (x_i - \frac{1}{2}) \]

ANOVA decomposition

\[ f_{A,\emptyset} = 0, \]
\[ f_{A,\{i\}} = x_i - \frac{1}{2}, \]
\[ f_{A,u} = 0 \text{ if } |u| \geq 2. \]
Element II: Variance of Functions

**Definition**

The variance of an integrable function

\[ f : [0, 1]^d \to \mathbb{R} \]

is defined as

\[ \sigma^2(f) := \int_{[0,1]^d} [f(x)]^2 \, dx - \left( \int_{[0,1]^d} f(x) \, dx \right)^2. \]
Definition of Effective Dimension
Caflisch et al. (1997)

Truncation dimension
The smallest integer $k$ such that

$$\sigma^2 \left( \sum_{u \subseteq [1:k]} f_{A,u} \right) \geq (1 - \varepsilon)\sigma^2(f),$$

(2)

where $\varepsilon$ is small, e.g., $\varepsilon = 0.01$.

Superposition dimension
The smallest integer $k$ such that

$$\sigma^2 \left( \sum_{|u| \leq k} f_{A,u} \right) \geq (1 - \varepsilon)\sigma^2(f).$$

(3)

- Previous example
- Why is it important?
- How to calculate it?
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**Anchored Decomposition**

**Definition**

\[
f_{a,\emptyset} = f(0) \quad \text{and} \quad f_{a,u}(x) = f(x_u, 0_{u^c}) - \sum_{v \subset u} f_{a,v},
\]

where \( f(x_v, 0_{v^c}) = f(x) \bigg|_{x_j = 0, j \in v^c} \).

- Fix some coordinates at 0 (the anchor).
- Cf. the ANOVA case which integrates out other components.

**Cf. ANOVA Decomposition**

\[
f_{A,\emptyset} = \int_{[0,1]^d} f(x) \, dx \quad \text{and} \quad f_{A,u}(x) = \int_{[0,1]^{u^c}} f(x) \, dx_{u^c} - \sum_{v \subset u} f_{A,v}.
\]
## Comparison: ANOVA and Anchored Decomposition

<table>
<thead>
<tr>
<th>ANOVA</th>
<th>Anchored</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sum_{v \subseteq u} f_{A,v} = \int_{[0,1]^{d-</td>
<td>u</td>
</tr>
<tr>
<td>[ \int_0^1 f_{A,u}(x) dx_j = 0, \ j \in u ]</td>
<td>[ f_{a,u}(x)</td>
</tr>
<tr>
<td>\text{L}_2 \text{ Orthogonality}</td>
<td>No \text{L}_2 \text{ Orthogonality}</td>
</tr>
<tr>
<td>[ \int f_{A,u}(x)f_{A,v}(x) dx = 0, \ \text{if} \ u \neq v. ]</td>
<td>Cross terms may appear</td>
</tr>
<tr>
<td>Variance decomposition [ \sigma^2(f) = \sum_{u \neq \emptyset} \sigma^2(f_{A,u}) ]</td>
<td>e.g., [ \int f_{a,u}(x)f_{a,v}(x) dx ]</td>
</tr>
<tr>
<td>ANOVA = ANalysis Of VAriance</td>
<td>Anchor = 0</td>
</tr>
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</table>
Assume \( \{ \gamma_u \}_{u \subseteq [1:d]} \) is a sequence of assigned weights. The weighted ANOVA (and anchored) space is the space of functions with the norm

\[
\| f \|_A = \left( \sum_{u \subseteq [1:d]} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left| \int_{[0,1]^{u^C}} f(u)(x_u, x_u^c) \, dx_u^c \right|^2 \, dx_u \right)^{1/2}
\]

\[
\| f \|_a = \left( \sum_{u \subseteq [1:d]} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left| f(u)(x_u, 0_{u^c}) \right|^2 \, dx_u \right)^{1/2}
\]

where \( f(u)(x) = \frac{\partial^{|u|} f}{\partial x_u}(x) \).

Embedding theorems between the two spaces

E.g., Hefter and Ritter (2015).
Choices of Weights \( \{ \gamma_u \}_{u \subseteq [1:d]} \)

- **Product Weights:**
  \[
  \gamma_u = \prod_{j \in u} \gamma_j
  \]
  - where \( \gamma_j \) is a decreasing sequence of non-negative numbers.

- **Order-Dependent weights:**
  \[
  \gamma_u = \Gamma_{|u|}
  \]
  - where \( \Gamma_1, \Gamma_2, \ldots \) are some non-negative numbers.
  - E.g., Dick et al. (2006).

- **Product Order-Dependent (POD) weights:**
  \[
  \gamma_u = \Gamma_{|u|} \prod_{j \in u} \gamma_j
  \]
  - e.g., Kuo et al. (2012).
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Proposition 1

Assume that $f(x)$ is a $d$-dimensional function in the weighted ANOVA (or anchored) space with weights $\{\gamma_u\}_{u \subseteq [1:d]}$. $f$ has the decomposition $f(x) = \sum_{u \subseteq [1:d]} f_*,u$, where $* \in \{A, a\}$. Then

$$\sigma^2(f_*,u) \leq C^{(1)}_{*,\gamma,u} \|f_*,u\|^2_*,$$

where

$$C^{(1)}_{A,\gamma,u} = \gamma_u \left( \frac{1}{3\sqrt{10}} \right)^{|u|} \frac{\text{Product weights}}{\gamma_u = \prod_{j \in u} \gamma_j} \prod_{j \in u} \frac{\gamma_j}{3\sqrt{10}}, \tag{5}$$

$$C^{(1)}_{a,\gamma,u} = \gamma_u \left[ \left( \frac{1}{\sqrt{6}} \right)^{|u|} - \left( \frac{1}{3} \right)^{|u|} \right]. \tag{6}$$

$\bullet$ $d \to \infty$: $\sigma^2(f_*,u)$ tends to 0 when $|u| \to \infty$. 
Sketch of Proof

- Starting point: Lemmas 1 and 6, Hefter et al. (2015)

\[
f(x) = \sum_{u \subseteq [1:d]} \int \int f(u)(t_u, t_{uc}) \, dt_{uc} \prod_{j \in u} (1_{[0,x_j]}(t_j) - (1 - t_j)) \, dt_u,
\]

\[
f(x) = \sum_{u \subseteq [1:d]} \int f(u)(t_u, 0_{uc}) \prod_{j \in u} 1_{[0,x_j]}(t_j) \, dt_u.
\]

- \( f \) is represented in terms of the key elements of the corresponding norm. So are the components of the decompositions.
- Hölder’s inequality
Recall \( * \in \{ A, a \} \) and define

\[
S_{*,k} = \sum_{v \subseteq [1:k]} f_{*,v}.
\]

**Proposition 2**

\[
\sigma^2 (f - S_{*,k}) \leq C^{(2)}_{*,\gamma,k} \left\| \sum_{v \cap [1:k]^c \neq \emptyset} f_{*,v} \right\|_*^2
\]

where

\[
C_{A,\gamma,k}^{(2)} = \max_{v \cap [1:k]^c \neq \emptyset} \gamma v \left( \frac{1}{3 \sqrt{10}} \right)^{|v|},
\]

\[
C_{a,\gamma,k}^{(2)} = \sum_{v \cap [1:k]^c \neq \emptyset} \gamma v \left( \frac{1}{3} \right)^{|v|}.
\]
Consider

- \( d \to \infty \)
- Choose the product weights \( \gamma_u = \prod_{j \in u} \gamma_j \) with
  - \( 3\sqrt{10} \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq \cdots \geq 0 \)
  - \( \sum_{j=1}^{\infty} \gamma_j < \infty \)
- When \( k \) is increasing,

\[
C_{A,\gamma,k}^{(2)} = \max_{\nu \cap [1:k]^c \neq \emptyset} \prod_{j \in \nu} \frac{\gamma_j}{3\sqrt{10}} = \frac{\gamma_{k+1}}{3\sqrt{10}} \to 0
\]

\[
C_{a,\gamma,k}^{(2)} = \sum_{\nu \cap [1:k]^c \neq \emptyset} \prod_{j \in \nu} \frac{\gamma_j}{3} = \left( \sum_{|\nu| < \infty} - \sum_{\nu \subseteq [1:k]} \right) \left( \prod_{j \in \nu} \frac{\gamma_j}{3} \right)
\]

\[
= \prod_{j=1}^{d} \left( 1 + \frac{\gamma_j}{3} \right) - \prod_{j=1}^{k} \left( 1 + \frac{\gamma_j}{3} \right) \to 0
\]
Denote
\[ \Delta_k = S_{A,k} - S_{a,k} = \sum_{v \subseteq [1:k]} f_{A,v} - \sum_{v \subseteq [1:k]} f_{a,v}. \]

**Proposition 3**

\[ \sigma^2(\Delta_k) \leq C_{\gamma,k}^{(3)} \| f \|_{A}^2, \]

where
\[ C_{\gamma,k}^{(3)} = \sum_{v \cap [1:k]^c \neq \emptyset \atop v \cap [1:k] \neq \emptyset} \gamma_v \left( \frac{1}{3} \right) |v|. \]
Denote $L = \prod_{j=1}^{d} \left(1 + \frac{\gamma_j}{3}\right)$ and $\alpha_k = \prod_{j=1}^{k} \left(1 + \frac{\gamma_j}{3}\right)$.

$$C_{\gamma,k}^{(3)} = \sum_{u \subseteq [1:k]} \prod_{j \in u} \frac{\gamma_j}{3}$$

$$= \left( \sum_{\emptyset \neq u \subseteq [1:d]} - \sum_{\emptyset \neq u \subseteq [1:k]} - \sum_{\emptyset \neq u \subseteq [1:k]^c} \right) \prod_{j \in u} \frac{\gamma_j}{3}$$

$$= \prod_{j=1}^{d} \left(1 + \frac{\gamma_j}{3}\right) - \prod_{j=1}^{k} \left(1 + \frac{\gamma_j}{3}\right) - \prod_{j=k+1}^{d} \left(1 + \frac{\gamma_j}{3}\right) + 1$$

$$= L + 1 - \left(\alpha_k + \frac{L}{\alpha_k}\right).$$

$C_{\gamma,k}^{(3)}$ will first increase then decrease to 0 when $k$ is increasing.
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Conclusions

- The variance of a particular component $f_{*,u}$ is decreasing with the rate $\gamma_u \left( \frac{1}{3\sqrt{10}} \right) |u|$ or $\gamma_u \left( \frac{1}{6} \right) |u|$.
- The variance of the difference between the ANOVA and anchored decompositions is decreasing with the rate $\sum_{v \cap [1:k] \neq \emptyset} \gamma_v \left( \frac{1}{3} \right) |v|$.

Connection to effective dimension?

- Ongoing work.

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