Effective dimension for weighted ANOVA and anchored spaces

Chenxi Fan[1,2]
hellocIety@gmail.com

[1]School of Mathematics and Statistics
University of New South Wales

[2]School of Mathematical Sciences
Zhejiang University

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Outline

1. Motivation
   - High Dimensional Integral: Effective dimension
   - Compare ANOVA and Anchored Decompositions and Spaces
2. Main Results
   - Relate Variances to Norms
   - Implications
3. Summary

Outline

High Dimensional Integral

$$\int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x}$$

- Applications: e.g., in finance $d = 360 = 12 \times 30$.
- Some integration problems are easier than others, e.g.,

$$f(\mathbf{x}) = \sum_{i=1}^{d} f_i(x_i).$$

Element I: ANOVA Decomposition

Multivariate Decomposition

$$f(\mathbf{x}) = \sum_{u \subseteq [1:d]} f_u(\mathbf{x}),$$

where u is a subset of $\{1, 2, \ldots, d\} := [1 : d]$, and $f_u(\mathbf{x})$ depends only on x_j for $j \in u$.

ANOVA Decomposition

$$f_{A,\emptyset} = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{and} \quad f_{A,u}(\mathbf{x}) = \int_{[0,1]^{|u^c|}} f(\mathbf{x})\mathrm{d}\mathbf{x}_{u^c} - \sum_{v \subset u} f_{A,v}, \qquad (1)$$

where $\mathbf{x}_u = (x_j)_{j \in u}$.

- Integrate out coordinates not in u (hard to evaluate)

Element I: ANOVA Decomposition

Example

$$f(\mathbf{x}) = \sum_{i=1}^{d} (x_i - \frac{1}{2})$$

ANOVA decomposition

$$f_{A,\emptyset} = 0,$$
$$f_{A,\{i\}} = x_i - \frac{1}{2},$$
$$f_{A,u} = 0 \text{ if } |u| \geq 2.$$

Element II: Variance of Functions

Definition

The variance of an integrable function

$$f : [0, 1]^d \to \mathbb{R}$$

is defined as

$$\sigma^2(f) := \int_{[0,1]^d} [f(\mathbf{x})]^2 \mathrm{d}\mathbf{x} - \left[\int_{[0,1]^d} f(\mathbf{x}) \mathrm{d}\mathbf{x} \right]^2.$$

Definition of Effective Dimension

Caflisch et al. (1997)

Truncation dimension

The smallest integer k such that

$$\sigma^2\Big(\sum_{u\subseteq[1:k]} f_{A,u}\Big) \geq (1-\varepsilon)\sigma^2(f), \tag{2}$$

where ε is small, e.g., $\varepsilon = 0.01$.

Superposition dimension

The smallest integer k such that

$$\sigma^2\Big(\sum_{|u|\leq k} f_{A,u}\Big) \geq (1-\varepsilon)\sigma^2(f). \tag{3}$$

- Previous example
- Why is it important?
- How to calculate it?

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Anchored Decomposition

Definition

$$f_{a,\emptyset} = f(\mathbf{0}) \quad \text{and} \quad f_{a,u}(\mathbf{x}) = f(\mathbf{x}_u, \mathbf{0}_{u^c}) - \sum_{v \subset u} f_{a,v}, \tag{4}$$

where $f(\mathbf{x}_v, \mathbf{0}_{v^c}) = f(\mathbf{x})\big|_{x_j=0,\ j\in v^c}$.

- Fix some coordinates at 0 (the anchor).
- Cf. the ANOVA case which integrates out other components.

Cf. ANOVA Decomposition

$$f_{A,\emptyset} = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{and} \quad f_{A,u}(\mathbf{x}) = \int_{[0,1]^{|u^c|}} f(\mathbf{x})\mathrm{d}\mathbf{x}_{u^c} - \sum_{v \subset u} f_{A,v}.$$

Comparison: ANOVA and Anchored Decomposition

ANOVA	Anchored
$\sum_{v\subseteq u} f_{A,v} = \int_{[0,1]^{d-\lvert u\rvert}} f(\mathbf{x})\mathrm{d}\mathbf{x}_{u^c}$	$\sum_{v\subseteq u} f_{a,v} = f(\mathbf{x}_u, \mathbf{0}_{u^c})$
$\int_0^1 f_{A,u}(\mathbf{x})\mathrm{d}x_j = 0,\ j\in u$	$f_{a,u}(\mathbf{x})\big\rvert_{x_j=0} = 0,\ j\in u$
L_2 Orthogonality $\int f_{A,u}(\mathbf{x})f_{A,v}(\mathbf{x})\mathrm{d}\mathbf{x} = 0,\ \text{if } u\neq v.$	No L_2 Orthogonality
Variance decomposition $\sigma^2(f) = \sum_{u\neq\emptyset}\sigma^2(f_{A,u})$	Cross terms may appear e.g., $\int f_{a,u}(\mathbf{x})f_{a,v}(\mathbf{x})\mathrm{d}\mathbf{x}$
ANOVA = ANalysis Of VAriance	Anchor = $\mathbf{0}$

Weighted ANOVA and anchored Spaces

Assume $\{\gamma_u\}_{u\subseteq[1:d]}$ is a sequence of assigned weights. The weighted ANOVA (and anchored) space is the space of functions with the norm

$$\|f\|_A = \left(\sum_{u\subseteq[1:d]} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left| \int_{[0,1]^{|u^c|}} f^{(u)}(\mathbf{x}_u, \mathbf{x}_{u^c}) \mathrm{d}\mathbf{x}_{u^c} \right|^2 \mathrm{d}\mathbf{x}_u \right)^{1/2}$$

$$\|f\|_a = \left(\sum_{u\subseteq[1:d]} \gamma_u^{-1} \int_{[0,1]^{|u|}} \left| f^{(u)}(\mathbf{x}_u, \mathbf{0}_{u^c}) \right|^2 \mathrm{d}\mathbf{x}_u \right)^{1/2}$$

where $f^{(u)}(\mathbf{x}) = \frac{\partial^{|u|} f}{\partial \mathbf{x}_u}(\mathbf{x})$.

Embedding theorems between the two spaces
E.g., Hefter and Ritter (2015).

Choices of Weights $\{\gamma_u\}_{u \subseteq [1:d]}$

- Product Weights:

$$\gamma_u = \prod_{j \in u} \gamma_j$$

  - where γ_j is a decreasing sequence of non-negative numbers.
  - E.g., Sloan and Woźniakowski (1998).

- Order-Dependent weights:

$$\gamma_u = \Gamma_{|u|}$$

  - where $\Gamma_1, \Gamma_2, \ldots$ are some non-negative numbers.
  - E.g., Dick et al. (2006).

- Product Order-Dependent (POD) weights:

$$\gamma_u = \Gamma_{|u|} \prod_{j \in u} \gamma_j$$

  - e.g., Kuo et al. (2012).

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Relate Variances to Norms I

Individual Component

Proposition 1

Assume that $f(\mathbf{x})$ is a d-dimensional function in the weighted ANOVA (or anchored) space with weights $\{\gamma_u\}_{u\subseteq[1:d]}$. f has the decomposition $f(\mathbf{x}) = \sum_{u\subseteq[1:d]} f_{*,u}$, where $* \in \{A, a\}$. Then

$$\sigma^2(f_{*,u}) \le C^{(1)}_{*,\gamma,u} \|f_{*,u}\|^2_*,$$

where

$$C^{(1)}_{A,\gamma,u} = \gamma_u \left(\frac{1}{3\sqrt{10}}\right)^{|u|} \xrightarrow[\gamma_u = \prod_{j\in u}\gamma_j]{\text{Product weights}} \prod_{j\in u} \frac{\gamma_j}{3\sqrt{10}}, \tag{5}$$

$$C^{(1)}_{a,\gamma,u} = \gamma_u \left[\left(\frac{1}{\sqrt{6}}\right)^{|u|} - \left(\frac{1}{3}\right)^{|u|}\right]. \tag{6}$$

- $d \to \infty$: $\sigma^2(f_{*,u})$ tends to 0 when $|u| \to \infty$.

Sketch of Proof

- Starting point: Lemmas 1 and 6, Hefter et al. (2015)

$$f(\mathbf{x}) = \sum_{u\subseteq[1:d]} \int \int f^{(u)}(\mathbf{t}_u, \mathbf{t}_{u^c}) d\mathbf{t}_{u^c} \prod_{j\in u} \left(1_{[0,x_j)}(t_j) - (1 - t_j)\right) d\mathbf{t}_u,$$

$$f(\mathbf{x}) = \sum_{u\subseteq[1:d]} \int f^{(u)}(\mathbf{t}_u, \mathbf{0}_{u^c}) \prod_{j\in u} 1_{[0,x_j)}(t_j) d\mathbf{t}_u.$$

- f is represented in terms of the key elements of the corresponding norm. So are the components of the decompositions.
- Hölder's inequality

Recall $* \in \{A, a\}$ and define

$$S_{*,k} = \sum_{v \subseteq [1:k]} f_{*,v}.$$

Proposition 2

$$\sigma^2(f - S_{*,k}) \le C^{(2)}_{*,\gamma,k} \Big\| \sum_{v \bigcap [1:k]^c \neq \emptyset} f_{*,v} \Big\|_*^2$$

where

$$C^{(2)}_{A,\gamma,k} = \max_{v \bigcap [1:k]^c \neq \emptyset} \gamma_v \left(\frac{1}{3\sqrt{10}} \right)^{|v|}, \tag{7}$$

$$C^{(2)}_{a,\gamma,k} = \sum_{v \bigcap [1:k]^c \neq \emptyset} \gamma_v \left(\frac{1}{3} \right)^{|v|}. \tag{8}$$

Relate Variances to Norms II

Partial Sum of Tails

Consider

- $d \to \infty$
- Choose the product weights $\gamma_u = \prod_{j \in u} \gamma_j$ with
  - $3\sqrt{10} \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq \cdots \geq 0$
  - $\sum_{j=1}^{\infty} \gamma_j < \infty$
- When k is increasing,

$$C^{(2)}_{A,\gamma,k} = \max_{v \bigcap [1:k]^c \neq \emptyset} \prod_{j \in v} \frac{\gamma_j}{3\sqrt{10}} = \frac{\gamma_{k+1}}{3\sqrt{10}} \to 0$$

$$C^{(2)}_{a,\gamma,k} = \sum_{v \bigcap [1:k]^c \neq \emptyset} \prod_{j \in v} \frac{\gamma_j}{3} = \Big(\sum_{|v| < \infty} - \sum_{v \subseteq [1:k]} \Big) \Big(\prod_{j \in v} \frac{\gamma_j}{3} \Big)$$

$$= \prod_{j=1}^{d} \Big(1 + \frac{\gamma_j}{3} \Big) - \prod_{j=1}^{k} \Big(1 + \frac{\gamma_j}{3} \Big) \to 0$$

Denote

$$\Delta_k = S_{A,k} - S_{a,k} = \sum_{v \subseteq [1:k]} f_{A,v} - \sum_{v \subseteq [1:k]} f_{a,v}.$$

Proposition 3

$$\sigma^2(\Delta_k) \le C^{(3)}_{\gamma,k} \|f\|^2_A,$$

where

$$C^{(3)}_{\gamma,k} = \sum_{\substack{v \bigcap [1:k]^c \neq \emptyset \\ v \bigcap [1:k] \neq \emptyset}} \gamma_v \left(\frac{1}{3}\right)^{|v|}. \tag{9}$$

Relate Variances to Norms III

Difference between Two Decompositions

- Denote $L = \prod_{j=1}^{d}\left(1+\frac{\gamma_j}{3}\right)$ and $\alpha_k = \prod_{j=1}^{k}\left(1+\frac{\gamma_j}{3}\right)$.

$$
\begin{aligned}
C_{\gamma,k}^{(3)} &= \sum_{\substack{u\bigcap[1:k]^c\neq\emptyset \\ u\bigcap[1:k]\neq\emptyset}} \prod_{j\in u} \frac{\gamma_j}{3} \\
&= \Bigg(\sum_{\emptyset\neq u\subseteq[1:d]} - \sum_{\emptyset\neq u\subseteq[1:k]} - \sum_{\emptyset\neq u\subseteq[1:k]^c} \Bigg) \prod_{j\in u} \frac{\gamma_j}{3} \\
&= \prod_{j=1}^{d}\left(1+\frac{\gamma_j}{3}\right) - \prod_{j=1}^{k}\left(1+\frac{\gamma_j}{3}\right) - \prod_{j=k+1}^{d}\left(1+\frac{\gamma_j}{3}\right) + 1 \\
&= L + 1 - \left(\alpha_k + \frac{L}{\alpha_k}\right).
\end{aligned}
$$

- $C_{\gamma,k}^{(3)}$ will first increase then decrease to 0 when k is increasing.

Outline

Summary

- Conclusions
  - The variance of a particular component $f_{*,u}$ is decreasing with the rate $\gamma_u \left(\frac{1}{3\sqrt{10}}\right)^{|u|}$ or $\gamma_u \left(\frac{1}{6}\right)^{|u|}$.
  - The variance of the difference between the ANOVA and anchored decompositions is decreasing with the rate $\sum_{\substack{v \bigcap [1:k]^c \neq \emptyset \\ v \bigcap [1:k] \neq \emptyset}} \gamma_v \left(\frac{1}{3}\right)^{|v|}$.
- Connection to effective dimension?
  - Ongoing work.

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