A symmetric mixed finite element method for nearly incompressible elasticity based on biorthogonal systems

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Abstract

We present a symmetric version of the non-symmetric mixed finite element method presented in [23] for nearly incompressible elasticity. The displacement–pressure formulation of linear elasticity is discretized using a Petrov–Galerkin discretization for the pressure equation in [23] leading to a non-symmetric saddle point problem. A new three-field formulation is introduced to obtain a symmetric saddle point problem which allows us to use a biorthogonal system. Working with a biorthogonal system, we can statically condense out all auxiliary variables from the saddle point problem arriving at a symmetric and positive-definite system based only on the displacement. We also derive a residual based error estimator for the mixed formulation of the problem.

Keywords: Mixed finite elements, symmetric system, Petrov–Galerkin discretization, biorthogonal system

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1 Introduction

It is a well-known fact that linear, bilinear or trilinear finite elements based on the standard displacement–based formulation exhibit a locking effect when applied to nearly incompressible elasticity. A popular approach is to use a mixed formulation introducing pressure as an extra unknown. Then one has to work with two finite element spaces: one for discretizing the displacement field, and the other for discretizing the pressure. These finite element spaces should also be compatible in the sense that a suitable inf-sup condition is satisfied. One is

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also interested in statically condensing out the pressure from the system arriving at a formulation based only on the displacement. The pressure variable can be obtained as a post-processing step.

There are many different methods to alleviate the locking effect in nearly incompressible elasticity using quadrilateral or hexahedral meshes. Some popular methods are enhanced assumed strain [32, 31, 18, 1], mixed strain method [19, 20] and strain gap method [30]. These methods are based on a three-field formulation, popularly known as the Hu-Washizu formulation, which is a mixed formulation based on stress, strain and displacement. Using the stability of the mini-element [2], the mixed enhanced formulation is extended to simplicial meshes in [34], where the pressure variable is assumed to be continuous. The mathematical analysis of the enhanced assumed strain for linear elasticity can be found in [29, 7, 26]. We have shown a uniform convergence of the three-field formulation in the nearly incompressible elasticity using a modified Hu-Washizu formulation. However, the mathematical analysis in [26] is quite restrictive and only covers a class of quadrilateral meshes. Moreover, we could not negate the presence of spurious pressure modes in the pressure or the volumetric part of the stress [7, 26].

One of the simplest mixed formulations for elasticity is a displacement–pressure formulation [11, 6]. This formulation is obtained by introducing pressure as an extra variable, and writing a variational equation for the pressure. Starting with this simplest mixed formulation, a non-symmetric mixed finite element method based on linear finite elements on simplicial meshes is presented in [23] for linear elasticity and in [25] for nonlinear elasticity. The stability of the formulation is shown by recourse to the mini-element formulation [2] of the Stokes problem. As the variational equation for the pressure is obtained by using a Petrov–Galerkin formulation, where the trial and test spaces for the pressure are different, the discrete system is non-symmetric. The trial and test spaces for the pressure form a biorthogonal system, and hence the associated Gram matrix is diagonal. This allows an easy static condensation of the pressure from the system leading to a reduced system. As the non-symmetricity is a main drawback of this approach, in this paper, we present a symmetric version of this approach. The symmetric version is obtained by introducing a Lagrange multiplier variable as in the case of the biharmonic equation [14]. The continuous formulation is given in Section 2, and the discretization is presented in Section 3. The numerical analysis of the discrete formulation is presented in Section 4. The formulation is extended to nonlinear hyperelasticity in Section 5. We have also derived residual based error estimator for our formulation in Section 6. Finally, we have shown some numerical experiments in the last section.
2 The boundary value problem of linear elasticity

This section is devoted to the introduction of the boundary value problem of linear elasticity. We consider a homogeneous isotropic linear elastic material body occupying a bounded domain $\Omega$ in $\mathbb{R}^d$, $d \in \{2, 3\}$ with Lipschitz boundary $\Gamma$. For a prescribed body force $f \in [L^2(\Omega)]^d$, the governing equilibrium equation in $\Omega$ reads

$$-\text{div} \, \sigma = f,$$

where $\sigma$ is the symmetric Cauchy stress tensor. The stress tensor $\sigma$ is defined as a function of the displacement $u$ by the Saint-Venant Kirchhoff constitutive law

$$\sigma = \frac{1}{2} C (\nabla u + [\nabla u]^T),$$

where $C$ is the fourth-order elasticity tensor. The action of the elasticity tensor $C$ on the strain tensor $\varepsilon(u) := \frac{1}{2} (\nabla u + [\nabla u]^T)$ is defined as

$$\sigma = C \varepsilon(u) = \lambda (\text{tr} \, \varepsilon(u)) 1 + 2\mu \varepsilon(u).$$

Here, $\mathbf{1}$ is the identity tensor, and $\lambda$ and $\mu$ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. The displacement is assumed to satisfy the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \Gamma.$$

We are interested in the nearly incompressible case, which corresponds to $\lambda$ being very large.

Here we use standard notations $L^2(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ for Sobolev spaces, see [8, 15] for details. Let $V := [H^1_0(\Omega)]^d$ be the vector Sobolev space with inner product $(\cdot, \cdot)_{1, \Omega}$ and norm $\| \cdot \|_{1, \Omega}$ defined in the standard way: $(u, v)_{1, \Omega} := \sum_{i=1}^d (u_i, v_i)_{1, \Omega}$, and the norm being induced by this inner product.

We define the bilinear form $A(\cdot, \cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : V \times V \to \mathbb{R}, \quad A(u, v) := \int_{\Omega} C \varepsilon(u) : \varepsilon(v) \, dx,$$

$$\ell : V \to \mathbb{R}, \quad \ell(v) := \int_{\Omega} f \cdot v \, dx.$$

Then the standard weak form of the linear elasticity problem is as follows: given $\ell \in V'$, find $u \in V$ that satisfies

$$A(u, v) = \ell(v), \quad v \in V,$$

where $V'$ is the space of continuous linear functionals on $V$. The assumptions on $C$ guarantee that $A(\cdot, \cdot)$ is symmetric, continuous, and $V$-elliptic. Hence by using standard arguments it can be shown that (5) has a unique solution $u \in V$. 

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Furthermore, we assume that the domain $\Omega$ is convex and polygonal or polyhedral such that $u \in [H^2(\Omega)]^d \cap V$, and there exists a constant $C$ independent of $\lambda$ such that
\[
\|u\|_{2,\Omega} + \lambda \|\text{div } u\|_{1,\Omega} \leq C \|f\|_{0,\Omega}.
\] (6)

The a priori estimate (6) has been shown in [9] for the two-dimensional linear elasticity posed in a convex domain with polygonal boundary, see [21] for the three-dimensional case with convex domain and polyhedral boundary.

As pointed out in the introduction, the linear elasticity problem can be recast into different mixed formulations. The easiest mixed formulation is given by introducing pressure as an extra variable, which leads to penalized Stokes equations. Defining $p := \lambda \text{div } u$ and $R := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}$, a mixed variational formulation of linear elastic problem (5) is given by: find $(u, p) \in V \times R$ such that
\[
\tilde{a}(u, v) + \tilde{b}(v, p) = \ell(v), \quad v \in V,
\]
\[
\tilde{b}(u, q) - \frac{1}{\lambda} \tilde{c}(p, q) = 0, \quad q \in R,
\] (7)

where
\[
\tilde{a}(u, v) := 2\mu \int_\Omega \varepsilon(u) : \varepsilon(v) \, dx,
\]
\[
\tilde{b}(v, q) := \int_\Omega \text{div } v \, q \, dx,
\]
\[
\tilde{c}(p, q) := \int_\Omega pq \, dx.
\]

We note that $p \in R$ because of the homogeneous Dirichlet boundary condition. The existence, uniqueness and the stability of the mixed formulation (7) can be established by using the standard saddle point theory [11, 5]. Using a pair of finite element bases forming a biorthogonal system to discretize the pressure equation, an efficient numerical scheme arises [23, 25]. Since a Petrov–Galerkin discretization is applied to the pressure equation, where the trial and test spaces are different, the arising linear system is non-symmetric even after eliminating the pressure. Therefore, we consider a new formulation, which allows us to use a biorthogonal system and leads to a symmetric formulation. To this end, we write the standard weak formulation of the linearly elastic problem as a minimization problem:
\[
\min_{u \in V} \frac{1}{2} A(u, u) - \ell(u).
\] (8)

Since
\[
A(u, u) = 2\mu \int_\Omega \varepsilon(u) : \varepsilon(u) \, dx + \lambda \int_\Omega \text{div } u \, \text{div } u \, dx,
\]
we introduce a pressure-like variable
\[
\phi := \sqrt{\lambda} \text{div } u \in R,
\]
and write the minimization problem (8) as a constrained minimization problem:
\[
\min_{(u, \phi) \in V \times R} \phi \quad \min_{\phi = \sqrt{\lambda} \text{div } u} \frac{1}{2} \left( \int_\Omega 2\mu \varepsilon(u) : \varepsilon(u) \, dx + \int_\Omega \phi^2 \, dx \right) - \ell(u). \quad (9)
\]
We write a weak variational equation for the equation $\phi = \sqrt{\lambda} \text{div} \, u$ in terms of the Lagrange multiplier space $R$ to obtain the saddle point problem of the minimization problem (9). The saddle point formulation is to find $(u, \phi, \xi) \in V \times R \times R$ such that

$$
a((u, \phi), (v, \psi)) + b((v, \psi), \xi) = \ell(v), \quad (v, \psi) \in V \times R, \\
b((u, \phi), \eta) = 0, \quad \eta \in R,
$$

(10)

where

$$
a((u, \phi), (v, \psi)) = 2\mu \int_{\Omega} \varepsilon(u) : \varepsilon(v) \, dx + \int_{\Omega} \phi \psi \, dx, \quad \text{and}
$$

$$
b((u, \phi), \eta) = \int_{\Omega} \left( \text{div} \, u - \frac{\phi}{\sqrt{\lambda}} \right) \eta \, dx.
$$

Here the Lagrange multiplier plays the role of pressure of the formulation (7), i.e.,

$$
\xi = \sqrt{\lambda} \phi = \lambda \text{div} \, u.
$$

This idea can be easily generalized to nonlinear elasticity if the problem can be written as a minimization problem (9). In order to show that the saddle point problem (10) has a unique solution, we want to apply a standard saddle point theory \cite{11, 6, 8}. To this end, we need to show the following three conditions of well-posedness.

1. The linear form $\ell(\cdot)$, the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous on the spaces on which there are defined.

2. The bilinear form $a(\cdot, \cdot)$ is coercive on the space $K$ defined as

$$
K = \{(v, \psi) \in V \times L^2_0(\Omega) : b((v, \psi), \eta) = 0, \eta \in L^2_0(\Omega)\}.
$$

3. The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition:

$$
\sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1, \Omega} + \|\psi\|_{0, \Omega}} \geq \beta \|\eta\|_{0, \Omega}, \eta \in L^2_0(\Omega) \quad (11)
$$

for a constant $\beta > 0$ independent of $\lambda$.

It is easy to show that the linear form $\ell(\cdot)$, and the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous. The coercivity of the bilinear form $a(\cdot, \cdot)$ follows from Korn’s inequality:

$$
a((u, \phi), (u, \phi)) = 2\mu \|\varepsilon(u)\|_{0, \Omega}^2 + \|\phi\|_{0, \Omega}^2 \geq C(\|u\|_{1, \Omega}^2 + \|\phi\|_{0, \Omega}^2).
$$

(12)

Thus the bilinear form $a(\cdot, \cdot)$ is coercive on the whole space $V \times L^2_0(\Omega)$.

Now we show that the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition uniformly in $\lambda$. 

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Lemma 1 There exists $\beta > 0$ independent of $\lambda$ such that

\[
\sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1,\Omega} + \|\psi\|_{0,\Omega}} \geq \beta \|\eta\|_{0,\Omega}, \quad \eta \in L^2_0(\Omega).
\]

Proof

\[
\sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1,\Omega} + \|\psi\|_{0,\Omega}} = \frac{1}{2} \left( \sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1,\Omega} + \|\psi\|_{0,\Omega}} + \sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1,\Omega} + \|\psi\|_{0,\Omega}} \right).
\]

Now we substitute $\psi = 0$ in the supremum for the first term, and $v = 0$ in the supremum for the second term to get

\[
\sup_{(v, \psi) \in V \times L^2_0(\Omega)} \frac{b((v, \psi), \eta)}{\|v\|_{1,\Omega} + \|\psi\|_{0,\Omega}} \geq 1 \sup_{v \in V} \int_\Omega \text{div} v \eta dx + \frac{1}{2} \sqrt{\lambda} \sup_{\psi \in L^2_0(\Omega)} \int_\Omega \psi \eta dx.
\]

The result follows from the fact that [17]

\[
\sup_{v \in V} \int_\Omega \text{div} v \eta dx \geq \beta_1 \|\eta\|_{0,\Omega}.
\]

Summarizing we have proved the following theorem.

**Theorem 2** The saddle point problem (10) has a unique solution $(u, \phi, \xi) \in V \times L^2_0(\Omega) \times L^2_0(\Omega)$ and

\[
\|u\|_{1,\Omega} + \|\phi\|_{0,\Omega} + \|\xi\|_{0,\Omega} \leq \|f\|_{0,\Omega}.
\]

3 Finite element discretization

We consider a quasi-uniform triangulation $T_h$ of the polygonal or polyhedral domain $\Omega$, where $T_h$ consists of simplices, either triangles or tetrahedra. Making use of the standard linear finite element space $S_h$ defined on the triangulation $T_h$

\[
S_h := \{ v \in H^1(\Omega) : v_T \in \mathcal{P}_1(T), T \in T_h \}
\]

and the space of bubble functions

\[
B_h := \left\{ b_T \in \mathcal{P}_{d+1}(T) : b_T|_{\partial T} = 0, \text{ and } \int_T b_T dx > 0, T \in T_h \right\},
\]

we introduce our finite element space for the displacement as $V_h = (S_h \oplus B_h)^d \cap V$. The bubble function on an element $T$ is most often defined as

\[
b_T(x) = c_n \Pi_{i=1}^{d+1} \lambda_T(x),
\]
where \( \lambda_T(x) \) are the barycentric coordinates of the element \( T \) associated with vertices \( x_T \) of \( T \), \( i = 1, \ldots, d + 1 \). The constant \( c_b \) is computed in such a way that the value of the bubble function at the barycenter of \( T \) is one. Let \( N \) be the number of nodes in the finite element mesh, and \( \{ \varphi_1, \ldots, \varphi_N \} \) be the finite element basis of \( S_h \). Starting with the basis of \( S_h \), we construct a dual space \( M_h \) spanned by the basis \( \{ \xi_1, \ldots, \xi_N \} \) so that the basis functions of \( S_h \) and \( M_h \) satisfy a condition of biorthogonality relation

\[
\int_{\Omega} \xi_i \varphi_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \ 1 \leq i, j \leq N,
\]

where \( \delta_{ij} \) is the Kronecker symbol, and \( c_j \) a scaling factor, which is chosen so that \( \int_T \xi_i \, dx = \int_T \varphi_i \, dx \). Hence, the sets of basis functions of \( S_h \) and \( M_h \) form a biorthogonal system. The basis functions of \( M_h \) are constructed locally on a reference element \( \hat{T} \) so that the basis functions of \( S_h \) and \( M_h \) have the same support \[23\], and in each element the sum of all the basis functions of \( M_h \) is one. In the following, we give the local basis functions of \( M_h \) on the reference triangle \( \hat{T} := \{(x,y) : 0 \leq x, 0 \leq y, x + y \leq 1\} \) and on the reference tetrahedron \( \hat{T} := \{(x,y,z) : 0 \leq x, 0 \leq y, 0 \leq z, x + y + z \leq 1\} \). For the the reference triangle, we have

\[
\hat{\xi}_1 := 3 - 4x - 4y, \quad \hat{\xi}_2 := 4x - 1, \quad \text{and} \quad \hat{\xi}_3 := 4y - 1,
\]

associated with its three vertices \((0,0)\), \((1,0)\) and \((0,1)\), respectively, and for the reference tetrahedron \( \hat{T} := \{(x,y,z) : 0 \leq x, 0 \leq y, 0 \leq z, x + y + z \leq 1\} \), we have

\[
\hat{\xi}_1 := 4 - 5x - 5y - 5z, \quad \hat{\xi}_2 := 5x - 1, \quad \text{and} \quad \hat{\xi}_3 := 5y - 1, \quad \hat{\xi}_4 := 5z - 1,
\]

associated with its four vertices \((0,0,0)\), \((1,0,0)\), \((0,1,0)\) and \((0,0,1)\), respectively. Moreover, \( \sum_{i=1}^{d+1} \hat{\xi}_i = 1 \). The global basis functions for the space \( M_h \) are constructed by glueing the local basis functions together. This procedure of constructing global basis functions for \( M_h \) from the local ones is the same as of constructing global basis functions for the standard finite element space \( S_h \) from the local ones. These global basis functions then satisfy the condition of biorthogonality \((14)\) with global finite element basis functions, and \( \text{supp} \varphi_i = \text{supp} \xi_i \), \( 1 \leq i \leq N \). We can see that a local basis function of \( M_h \) does not assume value one at one vertex and zero at other vertices. Since the global basis functions are extended by zero beyond their support, they are not continuous.

**Remark 3** Such a biorthogonal system has been very popular in the context of mortar finite elements \([35, 22]\). Construction of basis functions of \( M_h \) satisfying the biorthogonality and an optimal approximation property for a higher order finite element space is considered in \([22]\). However, construction of such a basis for higher order finite element in a simplicial mesh is not so easy \([27]\).
We also need a subspace of $S_h$ and a subspace of $M_h$ having zero average on $\Omega$ defined as

$$S_h^0 := \left\{ v_h \in S_h : \int_{\Omega} v_h \, dx = 0 \right\}, \quad M_h^0 := \left\{ \xi_h \in M_h : \int_{\Omega} \xi_h \, dx = 0 \right\}. $$

In [23, 25], the first equation of (7) is discretized by using a Galerkin formulation, and the second equation of (7) is discretized by using a Petrov–Galerkin formulation. The Petrov–Galerkin formulation is chosen so that the pressure solution is taken from $S_h$, whereas the test functions are taken from $M_h$. Hence the discrete formulation of variational equation (7) is a non-symmetric saddle point problem. Although the existence, uniqueness and stability of the solution are established in [23] using the theory of non-symmetric saddle point problems [28, 4], many well-known linear solvers perform better for symmetric linear systems. Therefore, we consider the discretization of (10) using our biorthogonal system. The discrete formulation is to find $(u_h, \phi_h, \xi_h) \in V_h \times S_h^0 \times M_h^0$ such that

$$a((u_h, \phi_h), (v_h, \psi_h)) + b((v_h, \psi_h), \xi_h) = \ell(v_h), \quad (v_h, \psi_h) \in V_h \times S_h^0, \quad b(u_h, \phi_h), \eta_h) = 0, \quad \eta_h \in M_h^0.$$  

(15)

The goal of choosing the different finite element bases to discretize $\phi$ and $\xi$ is to be able to statically condense out these variables from the system leading to a displacement-based formulation. In order to obtain an algebraic formulation of the discrete saddle point problem (15), we use the same notation for the vector representation of the solution and the solution as elements in $V_h, S_h$ and $M_h$. Using $\psi_h = 0$ and $v_h = 0$ subsequently in (15), we have

$$2\mu \int_{\Omega} \varepsilon(u_h) : \varepsilon(v_h) \, dx + \int_{\Omega} \nabla v_h \xi_h \, dx = \ell(v_h), \quad (v_h) \in V_h,$$

$$\int_{\Omega} \phi_h \psi_h - \int_{\Omega} \sqrt{\lambda} \varepsilon \xi_h \, dx = 0, \quad \psi_h \in S_h^0.$$  

(16)

Let $A, M, B$ and $D$ be the matrices associated with the bilinear forms $2\mu \int_{\Omega} \varepsilon(u_h) : \varepsilon(v_h) \, dx, \int_{\Omega} \phi_h \psi_h \, dx, \int_{\Omega} \nabla u_h \, dx$, and $\int_{\Omega} \phi_h \eta_h \, dx$, respectively. Then the algebraic formulation of the saddle point problem (15) can be written as

$$\begin{bmatrix}
A & 0 & B^T \\
0 & M & -\frac{1}{\sqrt{\lambda}}D^T \\
B & -\frac{1}{\sqrt{\lambda}}D & 0
\end{bmatrix}
\begin{bmatrix}
u_h \\
\phi_h \\
\xi_h
\end{bmatrix}
= 
\begin{bmatrix}
f_h \\
0 \\
0
\end{bmatrix}.$$  

(17)

where $f_h$ is the vector of discretization of the linear form $\ell(\cdot)$. Note that the first two equations in (17) correspond to the two equations in (16), and the last equation in (17) corresponds to the last equation in (15). If we look closely at the linear system (17), we find that if the matrix $D$ is diagonal, we can easily eliminate the degrees of freedom corresponding to $\phi_h$ and $\xi_h$ arriving at the formulation involving only one unknown $u_h$. After statically condensing out variables $\phi_h$ and $\xi_h$, we arrive at the reduced system

$$(A + \lambda B^T D^{-1} M D^{-1} B) u_h = f_h.$$  

As the matrix $D^{-1}$ is also diagonal, the system matrix is sparse. The variational formulation of this reduced system is given in (24).


4 A priori error estimate

Now we turn our attention to showing the well-posedness and error estimate for our discrete formulation. In order to show the existence, uniqueness and stability of the solution of (15), we have to prove three conditions of well-posedness stated in Section 2 before Lemma 1. As the discretization is conforming the linear form \( \ell(\cdot) \), and the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are continuous on the discrete spaces as well. An application of Korn’s inequality again guarantees the coercivity of the bilinear form \( a(\cdot, \cdot) \):

\[
a((u_h, \phi_h), (u_h, \phi_h)) = 2\mu \|\varepsilon(u_h)\|^2_{\Omega} + \|\phi_h\|^2_{0, \Omega} \geq C(\|u_h\|^2_{1, \Omega} + \|\phi_h\|^2_{0, \Omega}). \tag{18}
\]

Hence the bilinear form \( a(\cdot, \cdot) \) is coercive on the whole space \( V_h \times S^0_h \). It remains to show the third condition. That is, the bilinear form \( b(\cdot, \cdot) \) satisfies the discrete inf-sup condition:

\[
\sup_{(v_h, \psi_h) \in V_h \times S^0_h} \frac{b((v_h, \psi_h), \eta_h)}{\|v_h\|_{1, \Omega} + \|\psi_h\|_{0, \Omega}} \geq \beta \|\eta_h\|_{0, \Omega}, \quad \eta_h \in M^0_h \tag{19}
\]

for a constant \( \beta > 0 \) independent of mesh-size \( h \) and \( \lambda \). To show that the bilinear form \( b(\cdot, \cdot) \) satisfies the inf-sup condition (19), we need the following lemma. A proof of this lemma can be found in [23].

Lemma 4 There exists a constant \( \tilde{\beta} \) independent of the mesh-size \( h \) such that

\[
\sup_{v_h \in V_h} \frac{\int_\Omega \text{div } v_h \eta_h \, dx}{\|v_h\|_{1, \Omega}} \geq \tilde{\beta} \|\eta_h\|_{0, \Omega}, \quad \eta_h \in M^0_h. \tag{20}
\]

With the help of this lemma, we can show that the bilinear form \( b(\cdot, \cdot) \) satisfies the inf-sup condition (19). The proof of this lemma is similar to the one in the continuous case but we give the proof for completeness.

Lemma 5 There exists \( \tilde{\beta} > 0 \) independent of mesh-size \( h \) and \( \lambda \) such that

\[
\sup_{(v_h, \psi_h) \in V_h \times S^0_h} \frac{b((v_h, \psi_h), \eta_h)}{\|v_h\|_{1, \Omega} + \|\psi_h\|_{0, \Omega}} \geq \tilde{\beta} \|\eta_h\|_{0, \Omega}, \quad \eta_h \in M^0_h. \tag{21}
\]

Proof Let \( \eta_h \in M^0_h \).

\[
\sup_{(v_h, \psi_h) \in V_h \times S^0_h} \frac{b((v_h, \psi_h), \eta_h)}{\|v_h\|_{1, \Omega} + \|\psi_h\|_{0, \Omega}}
= \frac{1}{2} \left( \sup_{(v_h, \psi_h) \in V_h \times S^0_h} \frac{b((v_h, \psi_h), \eta_h)}{\|v_h\|_{1, \Omega} + \|\psi_h\|_{0, \Omega}} + \sup_{(v_h, \psi_h) \in V_h \times S^0_h} \frac{b((v_h, \psi_h), \eta_h)}{\|v_h\|_{1, \Omega} + \|\psi_h\|_{0, \Omega}} \right)
\geq \frac{1}{2} \sup_{v_h \in V_h} \frac{\int_\Omega \text{div } v_h \eta_h \, dx}{\|v_h\|_{1, \Omega}} + \frac{1}{2\sqrt{\lambda}} \sup_{\psi_h \in S^0_h} \frac{\int_\Omega \psi_h \eta_h \, dx}{\|\psi_h\|_{0, \Omega}},
\]
where the last step is obtained by using \( \psi_h = 0 \) in the supremum for the first term, and \( v_h = 0 \) in the supremum for the second term from the previous step. The result follows (with \( \tilde{\beta} = \frac{2}{3} \)) by using Lemma 4.

As all conditions of well-posedness are satisfied, the standard theory of saddle point problem yields the following two theorems [11]:

**Theorem 6** The discrete problem (15) has exactly one solution \((u_h, \phi_h, \xi_h) \in V_h \times S^0_h \times M^0_h\), which is uniformly stable with respect to the data \( f \), and there exists a constant \( C \) independent of mesh-size and Lamé parameter \( \lambda \) such that

\[
\|u_h\|_{1,\Omega} + \|\phi_h\|_{0,\Omega} + \|\xi_h\|_{0,\Omega} \leq C\|f\|_{0,\Omega}.
\]

**Theorem 7** Assume that \((u, \phi, \xi) \in V \times R \times R\) and \((u_h, \phi_h, \xi_h) \in V_h \times S^0_h \times M^0_h\) are the solutions of problems (10) and (15), respectively. Then, the following error estimate holds uniformly in \( \lambda \):

\[
\|u - u_h\|_{1,\Omega} + \|\phi - \phi_h\|_{0,\Omega} + \|\xi - \xi_h\|_{0,\Omega} \\
\leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} + \inf_{\psi_h \in S^0_h} \|\phi - \psi_h\|_{0,\Omega} + \inf_{\eta_h \in M^0_h} \|\xi - \eta_h\|_{0,\Omega} \right).
\]

It is well-known that the space \( V_h \) and \( S^0_h \) have optimal approximation property. We now establish an approximation property of the space \( M_h \). For that purpose, we define a quasi-projection operator \( Q_h : L^2(\Omega) \to M_h \) by

\[
\int_\Omega Q_h v \varphi_h \, dx = \int_\Omega v \varphi_h \, dx, \quad v \in L^2(\Omega), \ \varphi_h \in S_h.
\]

Since \( Q_h v \in M_h \), we have \( Q_h v = \sum_{j=1}^{N} d_j \xi_j \). We multiply both sides of \( Q_h v = \sum_{j=1}^{N} d_j \xi_j \) by \( \varphi_i \), integrate in \( \Omega \) and use the biorthogonality relation between the basis functions of \( S_h \) and \( M_h \) to get

\[
d_i = \int_\Omega \frac{\varphi_i v \, dx}{c_i}.
\]

This allows us to write the action of the operator \( Q_h \) on a function \( v \in L^2(\Omega) \) as

\[
Q_h v = \sum_{i=1}^{N} \int_\Omega \frac{\varphi_i v \, dx}{c_i} \xi_i,
\]

which tells that the operator \( Q_h \) is local in the sense that for any \( v \in L^2(\Omega) \), the value of \( Q_h v \) at any point in \( T \in T_h \) only depends on the value of \( v \) in \( S(T) \), where \( S(T) \) is the patch of an element \( T \in T_h \). Precisely, \( S(T) \) is defined as the interior of the closed set

\[
\tilde{S}(T) = \bigcup \{ \hat{T}' \in T_h : \partial T' \cap \partial T \neq \emptyset \}.
\]
The definition of $Q_h$ allows us to eliminate the auxiliary variables $\phi_h$ and $\xi_h$ from the discrete saddle point problem (15) arriving at a symmetric and positive-definite problem of finding $u_h \in V_h$ such that

$$2\mu \int_{\Omega} \varepsilon(u_h) : \varepsilon(v_h) \, dx + \lambda \int_{\Omega} Q_h \div v_h \, dx = \ell(v_h), \quad v_h \in V_h. \quad (24)$$

Moreover, $Q_h$ is stable in the $L^2$-norm [22].

**Lemma 8** For each element $T \in \mathcal{T}$, if $v \in L^2(S(T))$,

$$\|Q_h v\|_{0,T} \leq C \|v\|_{0,S(T)}. \quad (25)$$

**Lemma 9** Let $v \in H^1(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|v - Q_h v\|_{0,\Omega} \leq C h |v|_{1,\Omega}. \quad (26)$$

**Proof** We first show that if $w \in L^2(\Omega)$ and $w = 1$ on $S(T)$, then

$$Q_h w|_T = w \quad \text{on} \quad T. \quad (27)$$

Let $l_1, \ldots, l_{n_{S(T)}}$ be the indices of vertices in $S(T)$ ordered in such a way that $l_1, \ldots, l_{n_T}$ are vertices of element $T$, and $l_{n_T+1}, \ldots, l_{n_{S(T)}}$ are indices of remaining vertices in $S(T)$. Denoting the support of $\varphi_i$ by $S_i$ and using the expression of $Q_h w$ from (22), we have

$$Q_h w|_T = \sum_{i=1}^{n_T} \int_{S_i} \varphi_i \, w \, dx \xi_i. \quad (28)$$

Since $w = 1$ on $S(T)$, and the sum of all basis functions of $M_h$ at each element is one, $w = \sum_{i=1}^{n_{S(T)}} \xi_i$ on $S(T)$. We substitute this expression of $w$ in (28) and obtain

$$Q_h w = \sum_{i=1}^{n_T} \xi_i,$$

which concludes that $Q_h w = w$ on $T$. Let $|S(T)|$ be the area or the volume of the set $S(T)$, $v \in H^1(\Omega)$, and $c_v := \frac{1}{|S(T)|} \int_{S(T)} v \, dx$. Then using a triangle inequality and Lemma 8, we obtain

$$\|v - Q_h v\|_{0,T} \leq \|v - c_v\|_{0,T} + \|Q_h v - c_v\|_{0,T} = \|v - c_v\|_{0,T} + \|Q_h(v - c_v)\|_{0,T} \leq C \|v - c_v\|_{0,S(T)}.$$

An application of the Bramble–Hilbert lemma [15] then yields a constant $C > 0$ such that

$$\|v - Q_h v\|_{0,T} \leq C h |v|_{1,S(T)}. \quad (29)$$

The estimate (26) is then obtained by summing (29) over all elements of $\mathcal{T}_h$ and noting that each element is contained in only a fixed number of $S(T')$ for $T' \in \mathcal{T}$.

Then Theorem 7 yields the linear convergence of the energy norm of the error to zero with respect to the mesh-size if $u \in [H^2(\Omega)]^d$, $\phi \in H^1(\Omega)$ and $\xi \in H^1(\Omega)$.
5 Extension to finite elasticity

In this section, we briefly outline the extension of this symmetric formulation to the nonlinear elasticity. The non-symmetric formulation proposed in [23] is extended to nonlinear hyperelastic material law in [24]. Here we propose a symmetric mixed formulation for finite elasticity, where we can apply a biorthogonal system for the discretization so that other auxiliary variables can be statically condensed out from the system as in the linear case.

The variational approach to the nonlinear elasticity is based on the minimization of the energy functional of the form

\[ \int_{\Omega} \left( W(\mathbf{F}(\mathbf{u})) - \ell(\mathbf{u}) \right) dx \]  \hspace{1cm} (30)

over a suitable class \( \mathcal{W} \) of the displacements [16, 13, 33], where \( \mathbf{F}(\mathbf{u}) \) is the deformation gradient defined as \( \mathbf{F}(\mathbf{u}) = \nabla \mathbf{u} + 1 \). Let \( J = \det(\mathbf{F}) \). We assume that the polyconvex energy function \( W(\mathbf{F}) \) can be written as

\[ W(\mathbf{F}) = H(\lambda, J)^2 + G_2(\mathbf{F}), \]

where \( G_2 \) is independent of \( \lambda \), and \( H(\lambda, J) \) is a function of \( \lambda \) and \( J \).

For the isotropic material the energy function \( W \) depends only on the three principal invariants \( I_C, II_C, \) and \( III_C \) of \( C = \mathbf{F}^T \mathbf{F} \), where \( I_C = \text{tr}(C), II_C = \frac{1}{2}(\text{tr}^2(C) - \text{tr}(C^2)) \), and \( III_C = \det(C) = J^2 \). If the material law satisfies the two-term Mooney–Rivlin law [33], we have

\[ W(\mathbf{F}) = \lambda U(J) + \frac{\mu}{2} \left[ (1 - c_m)(I_C - 3 - 2 \ln(J)) + c_m (II_C - 3 - 2 \ln(J)) \right] \]

where \( \lambda \) and \( \mu \) are Lamé parameters and \( 0 \leq c_m \leq 1 \) is a material constant.

The real-valued function \( U \) is given by \( U(J) = \frac{1}{2}(J - 1)^2 \) or \( U(J) = \frac{1}{2}(\ln J)^2 \) or \( U(J) = \frac{1}{4}(J^2 - 1 - 2 \ln J) \). For this case

\[ H(\lambda, J) = \sqrt{\lambda U(J)}, \quad \text{and} \quad G_2(\mathbf{F}) = \frac{\mu}{2} \left[ (1 - c_m)(I_C - 3 - 2 \ln(J)) + c_m (II_C - 3 - 2 \ln(J)) \right]. \]

We recall that the standard neo-Hookean law is recovered when \( c_m = 0 \). Introducing a pressure-like variable \( \phi = H(\lambda, J) \), the mixed formulation for the finite elasticity can be obtained by minimizing the functional

\[ \int_{\Omega} \left( \phi^2 + G_2(\mathbf{F}) - \ell(\mathbf{u}) \right) dx \]  \hspace{1cm} (31)

over the function space \( L^2(\Omega) \times \mathcal{W} \) under the weak constraint

\[ \int_{\Omega} (\phi - H(\lambda, J)) \eta \, dx = 0, \quad \eta \in L^2(\Omega). \]
Then the symmetric non-linear saddle point problem is given by the Euler-Lagrange equations of the constrained minimization problem:

\[
\min_{(u, \phi) \in W \times L^2(\Omega)} \int_{\Omega} \left( \phi^2 + G_2(F) - \ell(u) \right) dx
\]

subject to \( \int_{\Omega} (\phi - H(\lambda, J)) \eta dx = 0, \ \eta \in L^2(\Omega). \)

6 A posteriori error estimator

In this section, using a similar approach as in [3], we derive an a posteriori error estimator of residual type. Let \( h_T \) be the diameter of element \( T \), and \( h_E \) the size of edge \( E \). We assume shape regularity of the mesh, which, in particular, means \( h_T \lesssim h_T \), for all \( T, T' \in \mathcal{T}_h \) with \( T \cap T' = \emptyset \). Here and in the following \( R \lesssim S \) means that the quantity \( R \) is bounded from above by \( C \cdot S \), where \( C \) is a constant not depending on the mesh size.

**Theorem 10** Let \( (u, \phi, \xi) \in V \times R \times R \) and \( (u_h, \phi_h, \xi_h) \in V_h \times S_h^0 \times M_h^0 \) denote the solutions of problems (10) and (15), respectively. Then there holds the a posteriori estimate

\[
\| u - u_h \|_{1, \Omega} + \| \phi - \phi_h \|_{0, \Omega} + \| \xi - \xi_h \|_{0, \Omega} \lesssim \sum_{T \in \mathcal{T}_h} \left[ h_T \| f + \text{div } 2\mu \varepsilon(u_h) \|_{0,T} + \sum_{E \subset \partial T} h_E^{1/2} \| [2\mu \varepsilon(u_h) n_T] \|_{0,E} \right. \\
\left. + \left( \phi_h - \frac{\xi_h}{\sqrt{\lambda}} \right) \right\|_{0,T} + \| \xi_h \|_{0,T} + \left\| \text{div } u_h - \frac{\phi_h}{\sqrt{\lambda}} \right\|_{0,T}.
\]

**Proof** As in [12], we apply Brezzi’s theory [10] to obtain the following inf-sup condition with a constant \( C > 0 \):

\[
\| u \|_{1, \Omega} + \| \phi \|_{0, \Omega} + \| \xi \|_{0, \Omega} \leq C \sup_{(v, \psi, \eta) \in V \times R \times R} \left\{ a((u, \phi, (v, \psi)) + b((u, \phi), \eta) + b((v, \psi), \xi) \mid \|v\|_{1, \Omega} + \|\phi\|_{0, \Omega} + \|\xi\|_{0, \Omega} = 1 \right\}
\]

Let

\[
A := a((u - u_h, \phi - \phi_h), (v, \psi)) + b((v, \psi), \xi - \xi_h), \quad I := \| u - u_h \|_{1, \Omega} + \| \phi - \phi_h \|_{0, \Omega} + \| \xi - \xi_h \|_{0, \Omega} \quad \text{and} \quad \quad B := \left\| \text{div } u_h - \frac{\phi_h}{\sqrt{\lambda}} \right\|_{0, \Omega}.
\]

Using Cauchy-Schwarz inequality, we have

\[
b((u - u_h, \phi - \phi_h), \eta) = \int_{\Omega} \left( \text{div } (u - u_h) - \frac{\phi - \phi_h}{\sqrt{\lambda}} \right) \eta dx \\
= -b((u_h, \phi_h), \eta) \leq \| \eta \|_{0, \Omega} \left\| \text{div } u_h - \frac{\phi_h}{\sqrt{\lambda}} \right\|_{0, \Omega}.
\]
and hence
\[ b((u - u_h, \phi - \phi_h), \eta) \leq \|\eta\|_{0, \Omega}B. \]

Since
\[ I \leq C \sup_{(v, \psi, \eta) \in V \times R \times R} \left\{ a((u - u_h, \phi - \phi_h), (v, \psi)) + b((u - u_h, \phi - \phi_h), \eta) + b((v, \psi), \xi - \xi_h) \right\}, \]

we have
\[ I \leq C \sup_{(v, \psi, \eta) \in V \times R \times R} A + b((u - u_h, \phi - \phi_h), \eta). \]

Then with the Galerkin orthogonality for arbitrary \( v_h \)
\[
A = 2\mu \int_{\Omega} \varepsilon(u - u_h) : \varepsilon(v) \, dx + \int_{\Omega} (\phi - \phi_h) \psi \, dx + \int_{\Omega} \left( \text{div} \left( v - \frac{\psi}{\sqrt{\lambda}} \right) \right) (\xi - \xi_h) \, dx
\]
\[
= 2\mu \int_{\Omega} \varepsilon(u - u_h) : \varepsilon(v - v_h) \, dx + \int_{\Omega} (\phi - \phi_h)(\psi - \psi_h) \, dx
\]
\[
+ \int_{\Omega} \left( \text{div} \left( v - v_h \right) - \frac{\psi - \psi_h}{\sqrt{\lambda}} \right) (\xi - \xi_h) \, dx
\]
\[
= \int_{\Omega} f \cdot (v - v_h) \, dx - 2\mu \int_{\Omega} \varepsilon(u_h) : \varepsilon(v - v_h) \, dx - \int_{\Omega} \phi_h(\psi - \psi_h) \, dx
\]
\[
- \int_{\Omega} \left( \text{div} \left( v - v_h \right) - \frac{\psi - \psi_h}{\sqrt{\lambda}} \right) \xi_h \, dx
\]
\[
= \sum_{T \in T_h} \left[ \int_{T} (f + \text{div} \, 2\mu \varepsilon(u_h)) \cdot (v - v_h) \, dx - \int_{\partial T} 2\mu(\varepsilon(u_h)n_T) \cdot (v - v_h) \, d\sigma
\]
\[
- \int_{T} \left( \phi_h - \frac{\xi_h}{\sqrt{\lambda}} \right) (\psi - \psi_h) \, dx - \int_{T} \frac{\xi_h}{\sqrt{\lambda}} \text{div} \, (v - v_h) \, dx \right],
\]
where \( n_T \) is the unit outward normal vector on \( \partial T \), and \( \varepsilon(u_h)n_T \) is interpreted as matrix-vector product, which produces a vector, and hence \( (\varepsilon(u_h)n_T) \cdot (v - v_h) \) is a scalar function. Thus using Cauchy-Schwarz inequality and trace theorem and choosing \( v_h \) an interpolant to \( v \), we obtain
\[
A \leq C \sum_{T \in T_h} \left[ h_T \|f + \text{div} \, 2\mu \varepsilon(u_h)\|_{0,T} \|v\|_{1,T} + \sum_{E \subset \partial T} h_E^{1/2} \|2\mu \varepsilon(u_h)n_T\|_{0,E} \|v\|_{1,T}
\]
\[
+ \left\| \phi_h - \frac{\xi_h}{\sqrt{\lambda}} \right\|_{0,T} \|\psi - \psi_h\|_{0,T} + \|\xi_h\|_{0,T} \|\text{div} \, (v - v_h)\|_{0,T} \right]
\]
\[
\leq C \sum_{T \in T_h} \left[ h_T \|f + \text{div} \, 2\mu \varepsilon(u_h)\|_{0,T} + \sum_{E \subset \partial T} h_E^{1/2} \|2\mu \varepsilon(u_h)n_T\|_{0,E}
\]
\[
+ \|\xi_h\|_{0,T} + \left\| \phi_h - \frac{\xi_h}{\sqrt{\lambda}} \right\|_{0,T} \right],
\]
where \([u]\) represents the jump of the function \(u\) across the interelement boundaries, and \(C\) now depends on \(v\) and \(\psi\). We have also used the fact that 
\[\|v - v_h\|_{0,T} \leq C\|v\|_{1,T}.\]

Let 
\[D = \left[ \sum_{T \in T_h} h_T \|f + \text{div} \ 2 \mu e(u_h)\|_{0,T} + \sum_{E \subset \partial T} h_E^{1/2} \|\text{II}(2 \mu e(u_h)n_T)\|_{0,E}\right] + \|\xi_h\|_{0,T} + \left\|\phi_h - \frac{\xi_h}{\sqrt{\lambda}}\right\|_{0,T}.\]

This shows that
\[I \lesssim B + D.\]

Thus the total error \(I\) is bounded by computable error \(B + D\).

In the following, we show that the part \(B\) of the computable error \(B + D\) converges asymptotically to zero when \(h \to 0\). To show this, we introduce another quasi-projection operator \(Q_h^* : L^2(\Omega) \to S_h\),
\[\int_{\Omega} Q_h^* v \eta_h \, dx = \int_{\Omega} v \eta_h \, dx, \quad \eta_h \in M_h.\]

This operator is uniquely defined due to the biorthogality relation between \(S_h\) and \(M_h\). Since \(Q_h^* v \in S_h\) and \(Q_h w \in M_h\), for \(v, w \in L^2(\Omega)\), we have
\[\int_{\Omega} Q_h^* v Q_h w \, dx = \int_{\Omega} w Q_h^* v \, dx = \int_{\Omega} v Q_h w \, dx.\]

Moreover, we have \(Q_h^* v_h = v_h\) for \(v_h \in S_h\), and hence \(Q_h^*\) is an projection onto \(S_h\). Therefore, \(Q_h^*\) is stable in the \(L^2\)-norm and has the following approximation property in the \(L^2\)-norm:
\[\|v - Q_h^* v\|_{0,\Omega} \leq h \|v\|_{1,\Omega}, \quad v \in H^1(\Omega).\]

That means Lemmas 8 and 9 can be reproduced for \(Q_h^*\). Then using \(\phi_h = \sqrt{\lambda} Q_h^* \text{div} \ u_h\), we have for \(0 < r \leq 1,\)
\[B = \left\|\text{div} \ u_h - \frac{\phi_h}{\sqrt{\lambda}}\right\|_{0,\Omega} \leq \left\|\text{div} \ u_h - \text{div} \ u + \text{div} \ u - Q_h^* \text{div} \ u_h\right\|_{0,\Omega}\]
\[\leq C h^r \left\|\text{div} \ u\right\|_{r,\Omega},\]

where we have assumed that the exact solution \(u \in H^{1+r}(\Omega)\) with \(r > 0\). This shows that \(B\) converges to zero asymptotically when \(h \to 0\).

The displacement solution is computed by inverting the reduced system after statically condensing out \(\phi_h\) and \(\xi_h\). Then the solutions \(\phi_h\) and \(\xi_h\) can be efficiently computed just by inverting a diagonal matrix. This gives an efficient implementation of the error estimator.
7 Numerical examples

In this section, we demonstrate the performance of our new formulation for linear elasticity with some numerical examples. The material parameters are given in compatible units. Both following examples do not have pure Dirichlet boundary condition, and therefore, we do not need to satisfy the zero mean condition for functions in $M_h$ and $S_h$. However, in case of pure Dirichlet boundary condition, both trial and test functions in $M_h$ and $S_h$ should satisfy the zero mean condition. That means the solutions $\xi_h \in M_h$ and $\phi_h \in S_h$ are determined up to an additive constant. We then search for solutions $\xi_h \in M_h$ and $\phi_h \in S_h$ whose values at one arbitrary vertex are prescribed a priori.

**Example 1: Cook’s membrane** Our first problem is the popular benchmark problem called Cook’s membrane problem. The Cook’s membrane is a two-dimensional tapered panel

$$\Omega := \text{conv}\{(0, 0), (48, 44), (48, 60), (0, 44)\},$$

where $\text{conv}\{\xi\}$ represents the convex hull of the set $\xi$. The left boundary of the panel is clamped in both directions and the right boundary is subjected to an in-plane shear load in the positive $y$-direction as shown in the left picture of Figure 1. We consider the material parameters $E = 250$ and $\nu = 0.49999$. We compute the vertical tip displacement at the top-right corner of the membrane using uniform refinement of the initial mesh given in the right picture of Figure 1. In the right picture of Figure 1, we show the convergence of the numerical results with respect to the number of elements using three finite element formulations for linear elasticity. We can see the good convergence behaviour of the two mixed formulations, whereas the standard displacement formulation shows the locking effect. The new mixed formulation shows even better accuracy than the mini finite element in the coarse mesh. The mini finite element proposed in [2] is applied to the linear elasticity.

**Example 2: Rectangular beam** In this second example, we consider a linear elastic beam of rectangular size subjected to a couple at one end, as shown in Figure 2. Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At the point $(0, 0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x, y) = \frac{2f(1 - \nu^2)}{El} x \left( \frac{l}{2} - y \right), \quad \text{and} \quad v(x, y) = \frac{f(1 - \nu^2)}{El} \left[ x^2 + \frac{\nu}{1 - \nu y} (y - l) \right].$$

We set $L = 10, l = 2, E = 1500, \nu = 0.4999$, and $f = 3000$. We have shown the setting of the problem in Figure 2, and the discretization errors with respect to the number of elements are presented in Figure 3. As can be seen from Figure 3, the standard approach locks completely, whereas we get very good numerical approximations with the new approach and mini-element.
Figure 1: Problem setting and initial mesh (left) and vertical tip displacement at $T$ versus number of elements using different formulations (right).

Figure 2: The rectangular beam with initial mesh and problem setting.

Figure 3: Error plot versus number of elements, $L^2$-norm (left) and $H^1$-norm (right), rectangular beam.

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References


