1. Verify axioms (M1)-(M4).

(M1): Obvious.

(M2): \( d(x_2, y) = 0 \) \( \iff \) \( |x_1 - y_1| = |x_2 - y_2| = 0 \) \( \iff \) \( x_1 = y_1, x_2 = y_2 \)
\( \iff \) \( x_2 = y_2. \)

(M3): Obvious since \( |x - y| = |y - x| \) for \( x, y \in \mathbb{R} \).

(M4): Let \( x, y, z \in \mathbb{R}^2 \). Then
\[
d(x, z) = |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|
\]
\[
= d(x, y) + d(y, z).
\]

2. (M1)-(M3) are clear. To prove (M4), let \( x, y, z \in \mathbb{R} \).

We want to prove
\[
\min \{|1, |x - z|\} \leq \min \{|1, |x - y|\} + \min \{|1, |y - z|\}.
\]

Now the RHS is equal to one of the numbers \( a = 2, b = 1 + |y - z|, c = 1 + |x - y|, d = |x - y| + |y - z| \). If
\[
|x - z| \leq 1, \text{ then } \text{ RHS} = |x - z| \leq a, b, c, d, \text{ so RHS} \leq \text{RHS}.
\]
If \( |x - z| > 1 \), then \( \text{LHS} = 1 \leq a, b, c, d \) and \( 1 < d \) since
\[
1 < |x - z| \leq |x - y| + |y - z|, \text{ so LHS} \leq \text{RHS}.
\]


(M2): \( a^*(x, y) = 0 \) \( \iff \) \( a(x, y) = 0 \) \( \iff \) \( x = y. \)

(M3): \( a^*(x, y) = \frac{a(x, y)}{1 + a(x, y)} = \frac{a(y, x)}{1 + a(x, y)} = a^*(y, x). \)

(M4): We first show that if \( a \geq 0, b \geq 0, c \geq 0 \) and
\( c \leq a + b, \text{ then } \frac{c}{1 + c} \leq \frac{a}{1 + a} + \frac{b}{1 + b}. \) Now
\[
\text{RHS} - \text{LHS} = \frac{a(1+b)(1+c)+b(1+a)(1+c)-c(1+a)(1+b)}{(1+a)(1+b)(1+c)}
\]
\[
\begin{align*}
= \frac{(a+b+c) + a \cdot b + a \cdot b \cdot c}{(1+a)(1+b)(1+c)} \geq 0.
\end{align*}
\]

Substituting \(a = d(x, y), b = d(y, z), c = d(x, z)\) gives (M4).

4. (i) (M1): \(d(x, y) = \max \{d^{(1)}(x_1, y_1), d^{(2)}(x_2, y_2)\} \geq 0.\)
(M2): \(d(x, y) = 0 \iff d^{(1)}(x_1, y_1) = d^{(2)}(x_2, y_2) = 0\)
(\(\Rightarrow\)) \(x_1 = y_1, x_2 = y_2 \iff x = y.\)
(M3): \(d(x, y) = \max \{d^{(1)}(x_1, y_1), d^{(2)}(x_2, y_2)\} \geq d(y, y).\)
(M4): let \(x, y, z \in X^{(1)} \times X^{(2)}\). Then
\[
\begin{align*}
d(x, z) &= \max \{d^{(1)}(x_1, z_1), d^{(2)}(x_2, z_2)\} \\
&\leq \max \{d^{(1)}(x_1, y_1) + d^{(1)}(y_1, z_1), d^{(2)}(x_2, y_2) + d^{(2)}(y_2, z_2)\} \\
&\leq \max \{d^{(1)}(x_1, y_1), d^{(2)}(x_2, y_2)\} \\
&\quad + \max \{d^{(1)}(y_1, z_1), d^{(2)}(y_2, z_2)\} \\
&= d(x, y) + d(y, z).
\end{align*}
\]

(ii) (M1): \(d(x, y) = d^{(1)}(x_1, y_1) + d^{(2)}(x_2, y_2) > 0.\)
(M2): \(d(x, y) = 0 \iff d^{(1)}(x_1, y_1) = d^{(2)}(x_2, y_2) = 0\)
(\(\Rightarrow\)) \(x_1 = y_1, x_2 = y_2 \iff x = y.\)
(M4): As above, follows from (ii). \(\Rightarrow\)
(M4): let \(x, y, z \in X^{(1)} \times X^{(2)}\). Then
\[
\begin{align*}
d(x, z) &= d^{(1)}(x_1, z_1) + d^{(2)}(x_2, z_2) \\
&\leq d^{(1)}(x_1, y_1) + d^{(1)}(y_1, z_1) + d^{(2)}(x_2, y_2) + d^{(2)}(y_2, z_2) \\
&= d(x, y) + d(y, z).
\end{align*}
\]

5. By the triangle inequality, we have
\[
d(x, z) \leq d(x, y) + d(y, z).
\]
Also by the triangle inequality we have
\[ d(z,y) \leq d(z,x) + d(x,y) \]
\[ \text{i.e.} \quad -d(x,y) \leq d(x,z) - d(z,y). \] (2)
Combining (1) and (2) we obtain the desired result.

6. We must verify (M1)-(M4).

(M1): Let \( x, z \in X \). Setting \( y = x \) in (M2') we obtain
\[ d(x,x) \leq 2d(x,z) \]. But \( d(x,x) = 0 \) by (M1'),
so \( d(x,z) \geq 0 \) for any \( x, z \in X \).

(M2): This is (M1').

(M3): Set \( z = x \) in (M2'). This gives \( d(x,y) \leq d(y,x) \)
for any \( x, y \in X \), and interchanging \( x \) and \( y \) we have
\[ d(y,x) \leq d(x,y) \]. Thus \( d(x,y) = d(y,x) \).

(M4): This is just (M2'), bearing in mind that
\( d \) has been shown to be symmetric.
1. Since \( \|x-y\| \) defines a metric on \( X \), the proof that 
\[ d(x,y) = \min \{ 1, \|x-y\| \} \] 
defines a metric on \( X \) is similar to the proof of Problem 1.2. For \( d \) to be induced 
by some norm \( \| \cdot \|' \), then by (n3) we have 
\[ d(\lambda x, \lambda y) = \| \lambda x - \lambda y \|' = \| x - y \|' \] 
for all \( x, y \in X \) and all scalars \( \lambda \). Let \( x \in X \) such that \( \|x\| = 1 \). Then 
\[ d(0,2x) = \min \{ 1, \|2x\|' \} = 1 \], but \( 2d(0,x) = 2 \).

2. \( \|x+y\| \leq \|x\| + \|y\| \) \] 
\[ \Rightarrow \|x+y\| - \|x\| \leq \|y\| \text{ for all } x, y \in X \] 
(by symmetry) \( \|y\| - \|x\| \leq \|y-x\| \).

Putting these two inequalities together we obtain 
\[ -\|x-y\| \leq \|x+y\| - \|x\| \leq \|x-y\| \] as required.

3. (i) By Taylor’s Theorem, \( f = p_n + R_n \) where 
\[ R_n(x) = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t) \, dt. \] 
\[ = \frac{1}{n!} \int_0^x (x-t)^n e^t \, dt. \]

Then \( d_\infty(f, p_n) = \|f - p_n\|_\infty \) 
\[ = \| R_n \|_\infty \] 
\[ = \sup_{0 \leq t \leq 1} \left| \frac{1}{n!} \int_0^x (x-t)^n e^t \, dt \right| \]
\[ \leq \sup_{0 \leq t \leq 1} \frac{1}{n!} x^{n+1} e^x = \frac{e}{n!}. \]
Since \( \| f \|_1 \leq \| f \|_\infty \) for all \( f \in C[0,1] \), we have
\[ d_1(f,g) \leq d_\infty(f,g) \] for all \( f,g \in C[0,1] \), and so also
\[ d_1(f, p_n) \leq \epsilon \frac{n}{n!} \]

(iii) Here \( R_n(x) = \frac{1}{n!} \int_0^x (x-t)^n \sin^n(t) \, dt \).

Thus
\[ d_\infty(f, p_n) = \| R_n \|_\infty \leq \frac{1}{n!} \sup_{0 \leq t \leq 1} \int_0^x (x-t)^n \sin^n(t) \, dt \]
\[ \leq \frac{1}{n!} n^{n+1} = \frac{1}{n!} \]
Thus also \( d_1(f, p_n) \leq \frac{1}{n!} \).

4.

\[ \| x \|_1 = |x_1| + |y_1| = 1 \]
\[ \| x \|_2 = \sqrt{x_1^2 + y_1^2} = 1 \]
\[ \| x \|_\infty = \max \{|x_1|, |y_1|\} = 1 \]

5. We show that \( \ell_2 \) is a vector space by showing that it is closed under addition and scalar multiplication.

Let \( x, y \in \ell_2 \). To show \( \sum_{n=1}^\infty (x_n + y_n)^2 \) converges.

By Minkowski's Inequality, we have for each \( N \),
\[ \sum_{n=1}^N (x_n + y_n)^2 \leq \left\{ \left( \sum_{n=1}^N x_n^2 \right)^{1/2} + \left( \sum_{n=1}^N y_n^2 \right)^{1/2} \right\}^2 \]
Denoting the LHS by $A_N$ and the RHS by $B_N$, we have that \( \{B_N\}_{N \geq 1} \) is an increasing, convergent sequence. Its limit is therefore an upper bound for the increasing sequence \( \{A_N\} \), which thus converges itself. But this means that the series \( \sum_{n=1}^{\infty} (\lambda n^2 + y_n)^2 \) converges, and hence \( x + y \in l_2 \).

Also, if \( x \in l_2 \) and \( \lambda \in \mathbb{R} \), then the series \( \sum_{n=1}^{\infty} (\lambda n^2 x_n)^2 \) is clearly convergent, so \( \lambda x \in l_2 \).

To show that \( \|x\|_2 = (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}} \) defines a norm on \( l_2 \). Axioms (n1), (n2) and (n3) are trivial to verify, so we just prove (n4) (the triangle inequality).

Let \( x, y \in l_2 \). By Minkowski's Inequality we have for each \( N \),

\[
\left( \sum_{n=1}^{N} (x_n + y_n)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{n=1}^{N} x_n^2 \right)^{\frac{1}{2}} + \left( \sum_{n=1}^{N} y_n^2 \right)^{\frac{1}{2}}.
\]

Letting \( N \to \infty \) on both sides we obtain

\[
\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.
\]
1. We saw in problem 2.3 that $d_\infty(\exp, p_n) \leq e/n!$.
   Thus $d_\infty(\exp, p_n) \to 0$ as $n \to \infty$, so $p_n \to \exp$ in $(C[0,1], d_\infty)$. Since $d_1(f, g) \leq d_\infty(f, g)$ for all $f, g \in C[0,1]$, it is also true that $p_n \to \exp$ in $(C[0,1], d_1)$.

2. By the triangle inequality we have
   $$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$
   and also
   $$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y).$$
   Thus
   $$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$$
   $\to 0$ as $n \to \infty$,
   which implies that $d(x_n, y_n) \to d(x, y)$ in $(R, d_1)$.

3. Let $x_n \to x$ in $(X, d)$ where $d$ is the discrete metric. By the definition of convergence, there exists $N$ such that $d(x_n, x) < \frac{1}{2}$ for $n > N$. But since $d$ takes only the values 0 and 1, this implies that $d(x_n, x) = 0$ hence $x_n = x$ for $n > N$. Thus there are at most $N+1$ distinct points in the sequence $\{x_n\}$.

4. Let $\{x_n\}$ be a Cauchy sequence in $(X, d)$. Then there exists $N$ such that $d(x_n, x_m) < \frac{1}{2}$ for $n, m > N$. Thus $x_n = x_m = x$, say, for $n, m > N$, so clearly $x_n \to x$. 
5. Example (one of many): In \((\mathbb{R}, d_1)\), \(\frac{1}{n} \to 0\) as \(n \to \infty\). But if \(d\) is the discrete metric on \(\mathbb{R}\), then \(d\left(\frac{1}{n}, 0\right) = 1\) for all \(n\), so \(\frac{1}{n} \not\to 0\) in \((\mathbb{R}, d)\).

6. Let \(\{x_n\}\) be a Cauchy sequence in \((X, d)\). The "only if" part is trivial, for if \(\{x_n\}\) is convergent, then \(\{x_n\}\) itself is a convergent subsequence. To prove the "if" part, let \(\{x_{n_k}\}, k = 1, 2, \ldots\) be a convergent subsequence with limit \(x\), and let \(\varepsilon > 0\). Since \(x_{n_k} \to x\), there exists \(K\) such that

\[
\forall k > K \Rightarrow d(x, x_{n_k}) < \varepsilon/2.
\]

Since \(\{x_n\}\) is Cauchy, there exists \(N\) such that

\[
n, m > N \Rightarrow d(x_n, x_m) < \varepsilon/2.
\]

Let \(N_1 = \max\left(n_K, N\right)\). Then if \(n > N_1\), choosing \(k\) such that \(k > K\) and \(n_K > N\), we have

\[
d(x, x_n) \leq d(x, x_{n_K}) + d(x_{n_K}, x_n)
\leq \varepsilon/2 + \varepsilon/2
= \varepsilon,
\]

so \(x_n \to x\) as \(n \to \infty\), proving \(\{x_n\}\) is itself convergent.

7. Consider \(\mathbb{R}\) with usual metric \(d_1\), and let \(x_n = y_n = n\). Then \(d(x_n, y_n) = 0\) for all \(n\).
so \( d(x_n, y_n)^2 \) is trivially convergent. However, \( \{x_n\} \) is clearly not a Cauchy sequence.

8. Let \( \{x_n\} = \{(x_{1n}, x_{2n})\} \) be a Cauchy sequence in \( X^{(1)} \times X^{(2)} \) under the metric \( d \) of problem 1.4(iii). Thus \( d(x_n, x_m) = \max \{d_1(x_{1n}, x_{1m}), d_2(x_{2n}, x_{2m})\} \rightarrow 0 \) as \( m, n \rightarrow \infty \). This implies that \( \{x_{1n}\} \) and \( \{x_{2n}\} \) are Cauchy sequences in the complete spaces \( (X^{(1)}, d_1) \) and \( (X^{(2)}, d_2) \) respectively, hence converge to limits \( x_1, x_2 \) respectively. Then letting \( x = (x_1, x_2) \) we have

\[
d(x_n, x) = \max \{d_1(x_{1n}, x_1), d_2(x_{2n}, x_2)\} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

hence \( x_n \rightarrow x \) in \( (X^{(1)} \times X^{(2)}, d) \), proving \( (X^{(1)} \times X^{(2)}, d) \) is complete.

The proof for the metric \( d \) of problem 1.4(iii) is exactly similar.
9. Let \( \{x_n\} \) be a Cauchy sequence in \( L^2 \), where
\[ x_n = (x_{n1}, x_{n2}, \ldots) \]; i.e., \( \|x_n - x_m\|_2 \to 0 \)
as \( m, n \to \infty \);

\[ \frac{\sum_{i=1}^{\infty} (x_{ni} - x_{mi})^2}{k} \to 0 \quad \text{as} \quad m, n \to \infty. \]

Thus, for each \( i \), \( \{x_{ni} : n = 1, 2, \ldots\} \) is Cauchy in \( \mathbb{R} \).

Let \( x_i = \lim_{n \to \infty} x_{ni} \) and \( x = (x_1, x_2, \ldots) \).

We claim that \( x \in L^2 \) and that \( x_n \to x \).

Let \( \varepsilon > 0 \). By the Cauchy property of \( \{x_n\} \), there exists \( N \) such that
\[ m, n > N \quad \Rightarrow \quad \left( \frac{\sum_{i=1}^{N} (x_{ni} - x_{mi})^2}{k} \right)^{\frac{1}{2}} < \varepsilon. \]

\[ \Rightarrow \quad \left( \frac{\sum_{i=1}^{k} (x_{ni} - x_{mi})^2}{k} \right)^{\frac{1}{2}} < \varepsilon \quad \text{for each} \quad k. \]

Letting \( n \to \infty \) we have that
\[ m > N \quad \Rightarrow \quad \left( \frac{\sum_{i=1}^{k} (x_{ni} - x_i)^2}{k} \right)^{\frac{1}{2}} \leq \varepsilon \quad \text{for each} \quad k. \] (*)

Then for \( m > N \) and each \( k \) we have by Minkowski's Inequality,
\[ \left( \frac{\sum_{i=1}^{k} x_i^2}{k} \right)^{\frac{1}{2}} \leq \left( \frac{\sum_{i=1}^{k} (x_{mi} - x_i)^2}{k} \right)^{\frac{1}{2}} + \left( \frac{\sum_{i=1}^{k} x_{mi}^2}{k} \right)^{\frac{1}{2}} \]

\[ \leq \varepsilon + \|x_m\|_2. \]

Thus the series \( \sum_{i=1}^{\infty} x_i^2 \) converges, so \( x \in L^2 \), and it follows from (*) that \( \|x_m - x\|_2 \leq \varepsilon \) for \( m > N \), proving \( x_m \to x \).
1. (i) \[ \begin{array}{c}
\end{array} \]

(ii) \[ \begin{array}{c}
\end{array} \]

2. (a) As usual, (M1)-(M3) are clear, and only the triangle inequality (M4) needs checking, i.e. for any \( x, y, z \in \mathbb{R}^2 \),

\[ d(x, z) \leq d(x, y) + d(y, z). \]

If \( x = z \), \( d(x, z) = 0 \) so the inequality is obvious. If either \( x = y \) or \( y = z \), then \( \text{LHS} = \text{RHS} \), so the inequality is again obvious. The only other possibility is that \( x, y, z \) are all distinct, in which case

\[ \text{LHS} = \|x - y\|_2 + \|y - z\|_2 \leq \|x\|_2 + 2\|y\|_2 + \|z\|_2 = \text{RHS}. \]

Geometric interpretation: The distance between two distinct points is the sum of their "usual" (Euclidean) distances from the origin.

Thus \( d(A, B) = AO + OB \).

(Email from A to B goes via the "mail exchange" 0, hence the name "Post Office metric").
3. Let $y \in A$. We want to show that $d(x, y) < 2r$. 

Let $z \in A \cap B_r(y)$, i.e. $z \in A$ and $d(z, y) < r$.

By the triangle inequality, $d(x, y) \leq d(x, z) + d(z, y)$.

Since $z, y \in A$ and $A$ has diameter $< r$, $d(z, y) < r$.

Thus $d(x, y) < 2r$; (i.e. $y \in B_{2r}(x)$).

4. (i) $x \in \text{Int } A \Rightarrow$ there exists $r > 0$ such that $B_r(x) \subseteq A$; since $A \subseteq B$ we have $B_r(x) \subseteq B$, so $x \in \text{Int } B$.

(ii) $x \in \text{Int } (A \cap B) \Rightarrow$ there exists $r > 0$ such that $B_r(x) \subseteq A \cap B \Rightarrow$ there exists $r > 0$ such that $B_r(x) \subseteq A$ and $B_r(x) \subseteq B$ $\Rightarrow$ $x \in (\text{Int } A) \cap \text{ (Int } B)$.

(iii) Since both $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have by (i) that $\text{Int } A \subseteq \text{Int } (A \cup B)$ and $\text{Int } B \subseteq \text{Int } (A \cup B)$, so $(\text{Int } A) \cup (\text{Int } B) \subseteq \text{Int } (A \cup B)$.

(iv). In $(\mathbb{R}, d_1)$, let $A = [0, 1]$, $B = [-1, 0]$. Then $0 \in \text{Int } (A \cup B) = (-1, 1)$, but neither $0 \in \text{Int } A$ nor $0 \in \text{Int } B$. 

(The only point whose distance from $(1,1)$ is less than $\frac{1}{2}$ is $(1,1)$ itself, since the distance of any other point from $(1,1)$ is at least $1 \|(1,1)\|_2 = \sqrt{2}$.)
5. Let \((x, d)\) be a metric space, \(x \in X\). To show \(X \setminus \{x\}\) is open. Given \(y \in X \setminus \{x\}\), let \(r = d(x, y)\). Then the open ball \(B_r(y)\) does not contain \(x\), hence lies within \(X \setminus \{x\}\). By Theorem 4.1(i), \(X \setminus \{x\}\) is open.

Now let \(\{x_1, \ldots, x_n\}\) be any finite set in \(X\). Then

\[
X \setminus \{x_1, \ldots, x_n\} = \bigcap_{i=1}^{n} X \setminus \{x_i\}.
\]

Thus \(X \setminus \{x_1, \ldots, x_n\}\) is a finite intersection of open sets by the first part, so by Theorem 4.2(iii) is open. This is the "hence" method of proof.

An "otherwise" method: let \(y \in X \setminus \{x_1, \ldots, x_n\}\) and let \(r_i = d(x_i, y), i = 1, \ldots, n\) and let \(r = \min r_i\). Then the open ball \(B_r(y)\) does not contain any of the points \(x_1, \ldots, x_n\), hence lies within \(X \setminus \{x_1, \ldots, x_n\}\), which is therefore open by Theorem 4.1(i).

6. The "only if" part is trivial, since if every subset of \(X\) is open, then in particular each singleton set is open. For the "if" part, observe that every subset is a union of singleton sets, and by Theorem 4.2(iv), any union of open sets is open.

7. (a) Let \((X, \|\cdot\|)\) be a normed linear space. Let \(B_r(x)\) be any open ball in \(X\). Then besides the point \(x\), \(B_r(x)\) contains the point \(\left(1 + \frac{1}{r^2\|x\|}\right)x\) since the distance between these two points is the norm of their difference, which is \(\frac{1}{r^2} < \frac{1}{r}\). Thus an open ball always contains more than one point, hence so
does any nonempty open set.

(6) Let \( x \in \mathbb{R}^2 \), \( x \neq 0 \). Then \( \|x\|_2 \neq 0 \) and the
distance of any other point from \( x \) is at least \( \|x\|_2 \). Letting \( r = \|x\|_2 \), we thus have that
\( B_r(x) = \{y \mid \|y-x\|_2 < r\} \), so that the singleton \( \{x\} \) is an open
ball and hence an open set. \( \{0\} \) is not an open set,
since an open ball around \( 0 \) in the Post Office metric
is the same as the corresponding open ball in the usual
metric, which is a nonempty circular disc.

It is thus immediate by (a) that no norm
on \( \mathbb{R}^2 \) induces the Post Office metric.

8. Referring to Theorem 4.1(i), it is sufficient to
show that each open ball in one metric,
with a given centre, contains an open ball in the
other metric with the same centre.

So let \( B_r(x) \) be any open ball in \( x \) with
respect to the metric \( d \). We want \( r' > 0 \) such that
\( B_{r'}(x) \subseteq B_r(x) \) (where \( B^* \) denotes open ball with
respect to \( d^* \)), i.e., if \( d^*(x,y) < r' \), then \( d(x,y) < r \).
Now \( d^*(x,y) < r' \Rightarrow d(x,y) < r'/r \), so setting
\( r' = r / (1+r) \), we get \( r' = r / (1+r) \).

Now let \( B_r^*(x) \) be any open ball in \( x \) with respect
to the metric \( d^* \). We want \( r' > 0 \) such that
\( B_r(x) \subseteq B_{r'}(x) \), i.e., if \( d(x,y) < r' \) then \( d^*(x,y) < r \).

Observing that \( d^*(x,y) \leq d(x,y) \), we can just take \( r' = r \).
1. (i) Let $y$ be any cluster point of $B_r(x_0)$. Thus for each $\varepsilon > 0$ there exists $x \in B_r(x_0)$ such that $d(x,y) < \varepsilon$. Then $d(x_0,y) \leq d(x_0,x) + d(x,y) < r + \varepsilon$. Since $\varepsilon$ can be arbitrarily small, we deduce that $d(x_0,y) \leq r$; i.e., $y \in B_r(x_0)$, proving $B_r(x_0)$ is closed.

(ii) Let $(X,d)$ be a discrete metric space. Then for $x_0 \in X$, $B_r(x_0) = \{ x_0 \}$ which is closed, hence $\overline{B_r(x_0)} = \{ x_0 \}$. However, $B_r(x_0) = X$, so provided $X$ has more than one point, we have $\overline{B_r(x_0)} \neq B_r(x_0)$.

2. (i) Since every cluster point of $A$ is a cluster point of $B$ if $A \subseteq B$, we have $A' \subseteq B'$, and hence

$$\overline{A} = A \cup A' \subseteq B \cup B' = \overline{B}.$$ 

(ii) The inclusion $\subseteq$ follows by (i), since $A \subseteq A \cup B = \overline{A \cup B}$ and similarly $B \subseteq A \cup B$, so $\overline{A \cup B} \subseteq A \cup B$.

For the reverse inclusion, note that $\overline{A \cup B}$ is a closed set (by Thm. 5.4 (iii)) containing $A \cup B$, and hence contains $A \cup B$, the smallest closed set containing $A \cup B$ by Thm. 5.5.

(iii) Either $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so by (i)

$$\overline{A \cap B} \subseteq \overline{A}$$

and $\overline{A \cap B} \subseteq \overline{B}$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Or: $\overline{A \cap B}$ is a closed set (by Thm. 5.4 (iii)) containing $A \cap B$, hence contains $\overline{A \cap B}$ by Thm. 5.5.

The reverse inclusion does not hold; e.g. let $A = (-1,0)$, $B = (0,1)$ in $\mathbb{R}$. Then $\overline{A \cap B} = \emptyset = \emptyset$, but $\overline{A \cap B} = [-1,0] \cap [0,1] = \{0\}$.

The inclusion does hold if either $A \subseteq B$ or $B \subseteq A$. 
3. (i) If \( x \in \text{Int} \ A \), then some open ball \( B_r(x) \) lies within \( A \). Thus \( x \notin \overline{X \setminus A} \), since neither \( x \in X \setminus A \) nor is \( x \) a cluster point of \( X \setminus A \) (otherwise \( B_r(x) \) would contain some point of \( X \setminus A \)). Thus \( x \notin \text{bdry} \ A \), so \( \text{Int} \ A \cap \text{bdry} \ A = \emptyset \).

(ii). The inclusion \( \subseteq \) is immediate since both \( \text{Int} \ A \subseteq \overline{A} \) and \( \text{bdry} \ A \subseteq \overline{A} \) from their definitions. To show that every point of \( \overline{A} \) belongs to either \( \text{Int} \ A \) or \( \text{bdry} \ A \): let \( x \in \overline{A} \) such that \( x \notin \text{Int} \ A \). Since \( x \notin \text{Int} \ A \), every open ball \( B_r(x) \) contains a point of \( X \setminus A \) so \( x \in X \setminus A \). Thus \( x \in \overline{A} \cap X \setminus A = \text{bdry} \ A \).

4. (i) \( \Rightarrow \) (ii): let \( Y \) be any closed superset of \( A \). Then \( Y \) contains \( \overline{A} = X \) by Thm. 5.5, so \( Y = X \).

(ii) \( \Rightarrow \) (iii): let \( U \) be an open set such that \( U \cap A = \emptyset \). Thus \( A \subseteq X \setminus U \) so \( \overline{A} \subseteq X \setminus U \). But \( \overline{A} = X \) and \( X \setminus U = X \setminus U \) (since \( X \setminus U \) is closed). Thus \( X = X \setminus U \), hence \( U = \emptyset \).

(iii) \( \Rightarrow \) (iv): Immediate (in fact (iv) is just the contrapositive of (iii)).

(iv) \( \Rightarrow \) (i): let \( x \in X \), and let \( B_r(x) \) be any open ball centred at \( x \). Then \( B_r(x) \) is a nonempty open set so there exists \( y \in A \cap B_r(x) \). Thus \( x \in \overline{A} \), so \( \overline{A} = X \).

5.
6. Let $X$ be a normed linear space, and let $U$ be an open subset of $X$ such that $U \neq \emptyset, X$. We show that $U$ cannot be closed.

**Notation:** For $a, b \in X$, $[a, b]$ denotes the "closed line segment" joining $a$ and $b$, i.e.

$$[a, b] = \{ a + t(b-a) : 0 \leq t \leq 1 \}.$$  

Choose $x \in U$, $y \notin U$.

Define $t_0 = \sup \{ t \geq 0 : [x, t(y-x)] \subseteq U \}$. Then $0 \leq t_0 \leq 1$. Define $z = x + t_0(y-x)$. Intuitively it is obvious that $z \notin U$, since no open ball around $z$ lies within $U$. We prove this rigorously.

If $t_0 = 1$, then $z = y$ and $y \notin U$, so we suppose $t_0 < 1$. Let $r > 0$ be arbitrary, $r \leq 1-t_0$. Let $r' = \min \{ r, r/\|y-x\| \}$.

By definition of $t_0$, the interval $[z, z + r'(y-x)]$ does not lie within $U$, so there exists $r_0$, $0 < r_0 < r'$, such that $w = z + r_0(y-x) \notin U$. Then

$$d(w, z) = \| w - z \| = r_0 \| y-x \| < r.$$  

Thus the open ball $B_r(z)$ does not lie within $U$, since $U$ is open and $r$ was arbitrary, we conclude that $z \notin U$.

However, $z$ is clearly a cluster point of $U$, since if $x_n = x + (t_0 - \frac{1}{n})(y-x)$, then $x_n \in U$ and

$$d(x_n, z) = \| x_n - z \| = \| x_n - y \|/n \to 0 \text{ as } n \to \infty.$$  

Thus $U$ is not closed. (Here we have used the fact that $t_0 > 0$; if $t_0 = 0$, then $z = x \in U$, which contradicts $z \notin U$.)
7. Since \((X,d)\) is complete, there exists \(a \in X\) such that \(a_n \to a\). Thus \(a\) is a cluster point of \(A\) which is closed, hence \(a \in A\).
1. We use Thm. 6.1. Let \( x \in X \), and let \( x_n \to x \). Thus by definition \( d(x_n, x) \to 0 \). Now
\[
\begin{align*}
d(x_n, x) & \leq d(x_n, x_0) + d(x_0, x) \\
d(x_0, x) & \leq d(x_0, x_n) + d(x_n, x)
\end{align*}
\]
so \( d(x_n, x_0) - d(x_0, x) \leq d(x_n, x_0) \to 0 \). Thus
\[
d(x_n, x_0) \to d(x, x_0) \text{; i.e. } f(x_n) \to f(x) \text{ as required.}
\]

2. Observing that the mapping \( F \) is linear, by Thm. 6.3 we need only show that it is bounded as a mapping from the normed linear space \((C[a, b], \| \cdot \|_\infty)\) to the normed linear space \((\mathbb{R}, | \cdot |)\).

Now, for any \( f \in C[a, b] \),
\[
|F(f)| = |f(x_0)| \\
\leq \sup_{x \in [a, b]} |f(x)| = \| f \|_\infty,
\]
so \( F \) is bounded, hence continuous on \((C[a, b], \| \cdot \|_\infty)\).

However, \( F \) is not bounded on \((C[a, b], \| \cdot \|_1)\). E.g., if \( x_0 = a \), define \( f_n \in C[a, b] \) by
\[
f_n(x) = \begin{cases} 
-x^2 + (n+n^2a), & a \leq x \leq a+n \\
0, & a+n \leq x \leq b
\end{cases}
\]
Then \( \| f_n \|_1 = \int_a^b |f_n| = \frac{1}{2} \text{ for all } n \).

But \( |F(f_n)| = |f_n(a)| = n \).
Thus there cannot exist a constant \( M > 0 \) such that
\[ |f(f)(y)| \leq M \|y\| \quad \text{for all} \quad y \in C[0, b], \quad \text{so} \quad F \text{ is not bounded hence not continuous on} \quad (C[0, b], \|\cdot\|). \]

3. We use Thm. 6.1. Let \( x_n \to x \) in \( X \). Then \( f(x_n) \to f(x) \)
   in \( Y \) since \( f \) is continuous; then \( g(f(x_n)) \to g(f(x)) \)
   in \( Z \) since \( g \) is continuous; i.e. \( (g \circ f)(x_n) \to (g \circ f)(x) \)
   as required.

4. The sequence \( \{x_n\} \) is clearly Cauchy in \( (0, \infty) \),
   since \( |x_n - x_m| \to 0 \) as \( m, n \to \infty \). However,
   the sequence \( \{f(x_n)\} = \{n^2\} \) is clearly not Cauchy.
   (We are assuming \( f : x \mapsto x^2 \) is continuous on \( (0, \infty) \)).

5. Let \( A \) be any open subset of \( \mathbb{R} \). Then a constant
   mapping \( f : x \mapsto c \) is continuous and \( f(A) = \{c\} \),
   which is not an open subset of \( \mathbb{R} \), thus \( f \) is not open.
   (\( \{c\} \) is not open by problem 4.7(a)).

6. Define \( f : (-\pi/2, \pi/2) \to \mathbb{R} \) by \( f(x) = \tan x \).
   Then \( f \) is continuous, one-one, and onto \( \mathbb{R} \).
7. The verification that \( d'' \) is indeed a metric is routine.

To show that \( d', d'' \) are equivalent metrics, it is sufficient to show that if \( y_n \to y \) in \( Y \) in one metric, then \( y_n \to y \) in the other metric. (Proof: this condition is (by Thm. 6.1) equivalent to continuity of the identity mappings \( I : (Y, d') \to (Y, d'') \) and \( I : (Y, d'') \to (Y, d') \).

If \( U \) is any open set in \( (Y, d'') \), then by continuity of the first mapping, \( I^{-1}(U) = U \) is open in \( (Y, d') \) — Thm. 6.2. Similarly, any open set \( U \) in \( (Y, d') \) is open in \( (Y, d'') \), hence \( d', d'' \) give rise to the same open sets, and are thus equivalent.

So suppose \( y_n \to y \) in \( (Y, d') \); \( \Rightarrow f^{-1}(y_n) \to f^{-1}(y) \) in \( (X, d) \) since \( f^{-1} \) is continuous \( \Rightarrow d(f^{-1}(y_n), f^{-1}(y)) \to 0 \Rightarrow d''(y_n, y) \to 0 \Rightarrow y_n \to y \) in \( (Y, d'') \).

Conversely, suppose \( y_n \to y \) in \( (Y, d'') \); \( \Rightarrow d''(y_n, y) \to 0 \Rightarrow d(f^{-1}(y_n), f^{-1}(y)) \to 0 \Rightarrow f^{-1}(y_n) \to f^{-1}(y) \) in \( (X, d) \Rightarrow y_n = f(f^{-1}(y_n)) \to f(f^{-1}(y)) = y \) in \( (Y, d') \) since \( f \) is continuous.

8. For \( x = (x_1, \ldots, x_n) \in \ell_1^n \), we have

\[
\| T(x) \|_1 = \sum_{j=1}^{m} \| \sum_{i=1}^{n} t_{ji} x_i \|_1 \\
\leq \sum_{j=1}^{m} \sum_{i=1}^{n} | t_{ji} | | x_i | \\
\leq \sum_{j=1}^{m} \| \sum_{i=1}^{n} t_{ji} | x_i | \|_1 \\
\leq \sum_{j=1}^{m} \max_{i=1}^{n} | t_{ji} | \sum_{i=1}^{n} | x_i | \\
= \left( \sum_{j=1}^{m} \max_{i=1}^{n} | t_{ji} | \right) \| x \|_1,
\]

so \( T \) is bounded from \( \ell_1^n \) to \( \ell_1^m \).
9. It follows from Thm. 6.2, on considering complements, that a mapping is continuous if and only if the inverse image of every closed set is a closed set. Now by problem 9.5, $\{0\}^c$ is a closed set in any normed linear space. Thus if $T$ is a continuous linear mapping, then $T^{-1}(\{0\}^c)$ is closed.

10. The condition $\|T(x)\| \leq M\|x\|$ implies that $T$ is bounded. Hence continuous. We require $T^{-1}$ to exist and be continuous. Now $T(x) = T(y) \Rightarrow \|T(x)-T(y)\| = \|T(x-y)\| = 0$, so $m\|x-y\| \leq \|T(x-y)\| = 0$, hence $\|x-y\| = 0$, i.e. $x = y$. Thus $T$ is one-one; i.e. $y \in T(x), \ y = T(x)$, then $m\|x\| \leq \|T(x)\|$, i.e. $m\|T^{-1}(y)\| \leq \|y\|$; i.e. $\|T^{-1}(y)\| \leq m^{-1}\|y\|$ proving $T^{-1}$ is bounded hence continuous.

11. (i) Let $\sim$ denote "is homeomorphic to". For any metric space $(X, d)$, the identity mapping $I: (X, d) \to (X, d)$ is clearly a homeomorphism, so $(X, d) \sim (X, d)$. i.e. $\sim$ is reflexive.

(ii) If $(X, d) \sim (Y, d')$, then there exists a homeomorphism $f: (X, d) \to (Y, d')$; then clearly $f^{-1}: (Y, d') \to (X, d)$ is a homeomorphism, so $(Y, d') \sim (X, d)$; i.e. $\sim$ is symmetric.

(iii) If $(X, d) \sim (Y, d')$ and $(Y, d') \sim (Z, d'')$ under homeomorphisms $f: (X, d) \to (Y, d')$ and $g: (Y, d') \to (Z, d'')$, then clearly $g \circ f: (X, d) \to (Z, d'')$ is a homeomorphism, so $(X, d) \sim (Z, d'')$; i.e. $\sim$ is transitive.

Thus $\sim$ is an equivalence relation.