From the outset we must be cautious to distinguish between incidental and isolated discoveries (though in hind-sight may seem quite profound) which did not gain general acceptance or contribute to the general advance of knowledge and those which were adopted and passed on to latter scholars. (Perhaps due to inherently poor communications, often aggravated by political/social factors, many seemingly significant ideas apparently went unnoticed, only to be re-discovered latter.)
$ax^2 = bx + c$, so that $a$, $b$, and $c$ are always positive. This avoids negative numbers standing alone and the subtraction of quantities that may be larger than the minuend. In this practice of using the separate forms, al-Khowârizmî follows Diophantus. Al-Khowârizmî recognizes that there can be two roots of quadratics, but gives only the real positive roots, which can be irrational. Some writers give both positive and negative roots.

One example of a quadratic treated by al-Khowârizmî reads as follows: “A square and ten of its roots are equal to nine and thirty dirhems, that is you add ten roots to one square, the sum is equal to nine and thirty.” He gives the solution thus: “Take half the number of roots, that is, in this case five, then multiply this by itself and the result is five and twenty. Add this to the nine and thirty, which gives sixty-four; take the square root, or eight, and subtract from it half the number of roots, namely five, and there remains three. This is the root.” The solution is exactly what the process of completing the square calls for.

Though the Arabs gave algebraic solutions of quadratic equations, they explained or justified their processes geometrically. Undoubtedly they were influenced by the Greek reliance upon geometrical algebra; while they arithmetized the processes, they must have believed that the proof had to be made geometrical. Thus to solve the equation, which is $x^2 + 10x = 39$, al-Khowârizmî gives the following geometrical method. Let $AB$ (Fig. 9.1) represent the value of the unknown $x$. Construct the square $ABCD$. Produce $DA$ to $H$ and $DC$ to $F$ so that $AH = CF = 5$, which is one-half of the coefficient of $x$. Complete the square on $DH$ and $DF$. Then the areas I, II, and III are $x^2$, $5x$, and $5x$, respectively. The sum of these is the left side of the equation. To both sides we now add area IV, which is 25. Hence the entire square is $39 + 25$ or 64 and its side must be 8. Then $AB$ or $AD$ is $8 - 5$ or 3. This is the value of $x$. The geometric argument rests on Proposition 4 of Book II of the Elements.

The Arabs solved some cubics algebraically and gave a geometrical explanation in the manner just illustrated for quadratics. This was done, for example, by Tâbit ibn Qorra (836-901), a pagan of Baghdad, who was also a physician, philosopher, and astronomer, and by the Egyptian al-Hasan ibn al-Haitham, known generally as Alhazen (c. 965–1039). As for the general
All learning in Latin—from translation into Latin (without proof of some Greek mathematics) to Boethius and Ptolemy.

Clearer sources of instruction were provided by the writings of Gerbert (Pope Sylvester II) in 1000, though nothing new was introduced.

None—the least—what little mathematics there was, rated highly in their teaching.

**Curriculum**

- Arithmetic (whole, fractions, half, 1/4 of a, +, x, ÷—all calculations done on forms of the abacus)
- Quadrivium
- Geometry (study of magnitudes and their relations)
- Astronomy (study of magnitudes in motion)
- Music (application of 1200 in the Pythagorean Tradition)

**Trivium**

- Rhetoric—communication (speaking/writing)
- Grammar—relationship between words and speech
- Dialectic—logical discussion to investigate truth

Fresh acceptance of the ideas of infinity and infinitesimals. God/1000?
middle ages

The entry of freeminded Greek and New Arabian Mathematics. Thus Arabic numerals, the Hindu mode of calculation including the use of fractions, \( \sqrt{2} \) was eventually adopted.

Understanding of this newly discovered learning gradually increased and dominated European thinking though little new was added. Algebra was still done with words, though the use of abbreviations eventually led to a clumsy undeveloped form of symbolism. Indeed, with few exceptions and thanks to the influx of knowledge in various disciplines, logic, natural history, etc., the work of understanding the practical knowledge of the artisan classes in some framing practical problems (surveying due to more ambitious building etc., projectile motion, etc.) led to a new "natural philosophy" based on the belief that mathematics was the language of nature (God).

Indeed these developments were more significant to later mathematics than the mathematics of the time.

Knowledge of statics, levers & optics regained.

Attempts made to understand projectile motion. Fractional hours & geometric figures & released rates.

Greece & Buridan (1300–1360) broke with Aristotle.

To introduce inertia, which when imparted to an object would maintain the motion in the absence of external resistance & why was gradually added to a falling body by natural gravity. (took as defo, sun & moon, planets, etc.)

Since celestial spheres suffer no resistances to their motion, God gives initial inertia so they continue!

Aristotle: only one force can act at any instant, when greatest offset, then lesser attends itself.

Huygens: resolved forces into components for a horizontally projected body.
Development of Science: Motivated largely by

Motion

Navigation (tables etc.)

Projectile astronomical

While the New Science based itself on "experiment" these were usually seen as subordinate (except Galileo) being suggested need to confirm laws (which could usually be reasoned from other ground also) expressed mathematically which was then the basis for a chain of deductions or theorems.
Renaissance

Specific mathematical discoveries during this period were scanty
and were rapidly enunciated by the
Mathematics
of the late 14th & 15th. None-the-less it was a preparation
for the rapid flowering, perhaps the most important
contribution being a gradual improvement in notation.
However it was the successful application of ancient
mathematics to the newly developing Physical sciences
and the attendant new views points & new problems which
these posed, that set the stage for 16th development.

Some developments

More complicated irrational numbers were considered & treated in the trades of the
Hindus. Negative numbers, though known, and
occasionally used, were not the least not generally accepted as
numbers. Complex numbers were also blundered into
and treated formally as numbers (e.g. Cardan) though
they were not accepted as "real" numbers.

Decimal notation (out of need for compact tables) was also
introduced by Vieta et al & strongly advocated by
Stevin: 5.912. 5090123
Continued fractions & inf products also considered
(e.g. Bombelli, Wallis et al.)

Logarithms: Stifel & R. & L. & T. Dr. &
A. P. D. 123 et.

Napier / Briggs v independent Bürgi
Tartaglia (Cardan): solution of cubic eqns, all coefficients considered.

Symbols & Notation

\( P + eP \), \( m e \) - by \( P + \sqrt{e} \) - in use (from German merchants) from about 1580.

Recorde (1510-58)

Harriot

\( f \) not till Descartes)
Cardan:
\[ -\sqrt{7+\sqrt{14}} \]
\[ \sqrt{7+\sqrt{14}} \]
\[ \sqrt{7-\sqrt{14}} \]
\[ \sqrt{7-\sqrt{14}} \]
all that follows
\[ a \times b = c \]
\[ a \]
\[ b \]
\[ c \]
A symbol for the unknown, though used by Diophantus, was slow to be introduced.

[Cardan (1501-1576)]

\[ x^2 = 4x + 32 \]

\[ \text{quadratic after 4 values p: 32 thing} \]

Case i: German

Hence algebra, the "Cossic art".

Once symbols were adopted, different ones were used for the various powers:

\[ x^3 + x^2 + x = C P Z P R \]

Case ii: (no = thing)

Gradually exponents used.

[Chequed (1484) 8^3, 7^1m = 8x^3, 7x^{-1}]

[Bombelli (1526-1573) 1 + 3x + 6x^2 + x^3 = 1, 3, 6, 1]

[Stevin (1548-1620) = 1^2 + 3^1 + 6^2 + 1]

Also used 4 5 for root + cube root, etc.

This Descartes
\[ 1 + 3x + 6x^2 + x^3 \]

(Vieta also used and was, with Descartes, adopted by Gauss.)
The Growth of Symbolism and Notation in Renaissance Europe

In the 15th century the use of \( p \) and \( m \) as abbreviations for + and − was common.

By the 16th century + and − were in use, having been adopted from German merchants.

\( = \) introduced by Recorde (1510-1558).

\( >, < \) by Harriot (1560-1621).

Parentheses from about 1560 onwards.

\( \sqrt{\text{−}} \) not till Descartes, 1596-1650.

Previously \( R \) had been used.

\[ \sqrt{7 + \sqrt{14}} \text{ as } \sqrt[2]{7} \sqrt[2]{4} \text{, if } A, B, C \]

were such expressions he would set out \( A \times B = C \) as:

\[ A \cdot B = C \]

Decimals, successfully introduced by Vieta and strongly advocated by Stevin (1548-1620): 5.912 written as \( 5\overset{\circ}{5}9\overset{\circ}{5}1\overset{\circ}{5}2\overset{\circ}{5} \).

Although used by Diophantus (Greek, ~ 250 A.D.) a symbol for the unknown was slow to be introduced. Initially it was an abbreviation for "thing" (\( R \) from \( \text{res} \) in Latin; c from coss in German, hence "Cossic Art".) Initially different symbols used for various powers.

\[ x^3 + x^2 + x \text{ as } C \cdot p \cdot 2 \cdot p \cdot R \]

\( \text{cubus xenus res} \)

Chuquet (1484) introduced exponents: Writing

\[ 8x^3 \cdot 5x - 1 \text{ as } 8^3 \cdot 7^1m. \]

However Cardan still wrote

\[ x^2 - 4x + 32 \text{ as } q\text{dratu aeqtur 4 rebus p: 32 } \]

thing

Bombelli (1526-1573), following Chuquet wrote

\[ 1 + 3x + 6x^2 + x^3 \text{ as } 1 \cdot p \cdot 3 \overset{\circ}{3} \cdot p \cdot 6 \overset{\circ}{3} \cdot p \cdot 1 \overset{\circ}{3} \]

and Stevin (1548-1620) used \( \overset{\circ}{3} + 3 \overset{\circ}{3} + 6 \overset{\circ}{3} + \overset{\circ}{3} \); he also used fractional exponents denoted by \( \overset{\circ}{3}, \overset{\circ}{3} \) etc.

Viète (Vieta) (1540-1603) was probably the first to use symbols purposely. He introduced \( x, y, z \) for unknowns and also allowed variable coefficients denoted by \( a, b, c \) etc.

Thus by the time of Descartes it was "natural" to write

\[ 1 + 3x + 6xx + x^3 \]

(and sometimes \( x^2 \), though this was not fully adopted till Gauss).