A CLASS OF SPACES WITH WEAK NORMAL STRUCTURE

Brailey Sims

It has recently been shown that a Banach space enjoys the weak fixed point property if it is $\varepsilon_0$-inquadrate for some $\varepsilon_0 < 2$ and has WORTH; that is, if $x_n \rightharpoonup x$ then, $\|x_n - x\| - \|x_n + x\| \to 0$, for all $x$. We establish the stronger conclusion of weak normal structure under the substantially weaker assumption that the space has WORTH and is $\varepsilon_0$-inquadrate in every direction for some $\varepsilon_0 < 2$.

A Banach space $X$ is said to have the \textit{weak fixed point property} if whenever $C$ is a nonempty weak compact convex subset of $X$ and $T : C \to C$ is a \textit{nonexpansive mapping}; (that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$), then $T$ has a fixed point in $C$.

It is well known that if $X$ fails to have the weak fixed point property then it fails to have \textit{weak normal structure}; that is, $X$ contains a weak compact convex subset $C$ with more than one point which is \textit{diametral} in the sense that, for all $x \in C$

$$\sup\{\|y - x\| : y \in C\} = \text{diam } C := \sup\{\|y - z\| : y, z \in C\}.$$  

Further, if $X$ fails to have weak normal structure then there exists a sequence, $(x_n)$, satisfying;

(S1) \hspace{1cm} \lim_n \|x_n - x\| = \text{diam } C = 1, \hspace{1cm} \text{for all } x \in C.$$

That is, $(x_n)$ is a non-constant weak null sequence which is 'diameterising' for its closed convex hull. In particular, since $0 \in C$, we have $\|x_n\| \to 1$.

Details of these and related results may be found in the monograph by Goebel and Kirk [7] for example.
For the Banach space $X$ we define $\delta : [0, 2] \times X \setminus \{0\} \to \mathbb{R}$ by

$$
\delta(\varepsilon, x) := \inf \left\{ 1 - \left\| y + \frac{\varepsilon}{2\|x\|} x \right\| : \|y\| \leq 1 \text{ and } \left\| y + \frac{\varepsilon}{\|x\|} x \right\| \leq 1 \right\}.
$$

We refer to $\delta(\varepsilon, x)$ as the \textit{modulus of convexity in the direction} $x$. $X$ is \textit{uniformly convex in every direction} (UCED) if $\delta(\varepsilon, x) > 0$, for all $x \neq 0$ and all $\varepsilon > 0$ (Day, James and Swaminathan [2]).

The \textit{modulus of convexity} of $X$ is given by

$$
\delta(\varepsilon) := \inf_{x \neq 0} \delta(\varepsilon, x),
$$

and $X$ is \textit{uniformly convex} if $\delta(\varepsilon) > 0$, for all $\varepsilon > 0$.

Following Day, given $\varepsilon_0 \in (0, 2]$ we say $X$ is $\varepsilon_0$-\textit{inquadrate} if $\delta(\varepsilon_0) > 0$.

By analogue with this last definition, for $\varepsilon_0 \in (0, 2]$ we shall say $X$ is $\varepsilon_0$-\textit{inquadrate in every direction} if $\delta(\varepsilon_0, x) > 0$, for all $x \neq 0$.

It is readily verified that $X$ is $\varepsilon_0$-inquadrate in every direction if and only if whenever $\limsup_n \|x_n\| \leq 1$, $\limsup_n \|x_n + \lambda_n x\| \leq 1$, and $\|x_n + (\lambda_n/2)x\| \to 1$ we have $\limsup_n |\lambda_n| \|x\| \leq \varepsilon_0$.

Note there are also the weaker notions, of $X$ being $\varepsilon_0$-inquadrate in some subset of directions, and for each $x \neq 0$ there being an $\varepsilon_x \in (0, 2]$ with $\delta(\varepsilon_x, x) > 0$, however; these will not concern us.

Garkavi [5] showed that spaces which are UCED have weak normal structure and hence enjoy the weak fixed point property. An essentially similar argument establishes the result for spaces which are $\varepsilon_0$-inquadrate in every direction for some $\varepsilon_0 \in (0, 1)$.

To see this, suppose that $X$ fails weak normal structure and so contains a sequence $(x_n)$ satisfying (S1) and (S2). Choose $m$ so that $\|x_m\| > \varepsilon_0$, then putting $x = x_m$ we have $\|x_n\| \to 1$, $\|x_n - x\| \to 1$ and, since $0 \in C$, so $x/2 \in C$, $\|x_n - x/2\| \to 1$ contradicting the assumption that $X$ is $\varepsilon_0$-inquadrate in every direction.

In general the situation when $1 \leq \varepsilon_0 < 2$ remains unresolved, even in the $\varepsilon_0$-inquadrate case.

Two other 'classical' conditions known to be sufficient for weak normal structure are:

1. \textit{The condition of Opial}, whenever $x_n \rightharpoonup 0$ and $x \neq 0$ we have

$$
\limsup_n \|x_n\| < \limsup_n \|x_n - x\|.
$$

The condition was introduced by Opial [10], and shown to imply weak normal structure by Gossez and Lami Dozo [8]. The condition is unchanged if both lim sups are
replaced by limifs. We say $X$ satisfies the non-strict Opial condition if the condition holds with strict inequality replaced by `$\leq$'.

(2) $\varepsilon_0$-Uniform Radon-Reisz ($\varepsilon_0$-URR), for some $\varepsilon_0 \in (0, 1)$; there exist $\delta > 0$ so that whenever $x_n \rightharpoonup x$, with $\|x_n\| \leq 1$ and $\text{sep}(x_n) := \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon_0$ we have $\|x\| < 1 - \delta$. When the condition holds for all $\varepsilon_0 > 0$ we say $X$ is URR. The condition is essentially due to Huff [9], and was shown to imply weak normal structure by van Dulst and Sims [4].

Gossez and Lami Dozo [8] showed that Opial's condition follows from the non-strict version in the presence of uniform convexity, however a careful reading of their argument establishes the following.

**Proposition 1.** If $X$ is UCED and satisfies the non-strict Opial condition then $X$ satisfies the Opial condition.

**Proof:** Suppose $X$ fails the Opial condition, then there exists a sequence $x_n \rightharpoonup 0$ and $x \neq 0$ with

$$\liminf_n \|x_n\| \neq \liminf_n \|x_n - x\|.$$ 

By the non-strict Opial condition we must have equality, and without loss of generality we may assume that $\|x_n\| \rightarrow 1$. Let $y_n := x_n - x$, then $\|x_n\|, \|y_n\| \rightarrow 1$ and $x_n - y_n = x$. Thus, by UCED we must have that

$$\liminf_n \|x_n - x/2\| = \liminf_n \|(x_n + y_n)/2\| < 1 = \liminf_n \|x_n\|,$$

contradicting the non-strict Opial condition.

In Sims [12], the notion of weak orthogonality (WORTH); if $x_n \rightharpoonup 0$ then for all $x \in X$ we have

$$\|x_n - x\| - \|x_n + x\| \rightarrow 0,$$

was introduced (also see Rosenthal [11]), and it was asked whether spaces with WORTH have the weak fixed point property. WORTH generalises the lattice theoretic notion of 'weak orthogonality' introduced by Borwein and Sims [1] and shown to be sufficient for the weak fixed point property in Sims [12].

**Proposition 2.** The non-strict Opial condition is entailed by WORTH.

**Proof:** If $x_n \rightharpoonup 0$ then for any $x \in X$ we have

$$\limsup_n \|x_n\| \leq \frac{1}{2} \left( \limsup_n \|x_n - x\| + \limsup_n \|x_n + x\| \right)$$

$$= \limsup_n \|x_n - x\|,$$

as $\lim_n \|x_n - x\| - \|x_n + x\| = 0$, by WORTH.

Combining this with proposition 1 we obtain the following.
**Corollary 3.** A Banach space which has UCED and WORTH satisfies the Opial condition.

Recently García Falset [6] working through the intermediate notion of the ACM-property has shown that spaces which are $\varepsilon_0$-inquadrate for some $\varepsilon_0 < 2$ and have WORTH have the weak fixed point property.

We give a direct and elementary proof that the stronger conclusion of weak normal structure follows from the substantially weaker premises of WORTH and $\varepsilon_0$-inquadrate in every direction for some $\varepsilon_0 < 2$. That $\varepsilon_0$-inquadrate in every direction is genuinely weaker than $\varepsilon_0$-inquadrate follows since spaces which are $\varepsilon_0$-inquadrate, for an $\varepsilon_0 < 2$ are necessarily superreflexive (van Dulst [3]) while every separable Banach space has an equivalent norm which is UCED [2], and hence $\varepsilon_0$-inquadrate in every direction for $0 < \varepsilon_0 < 2$.

**Definition:** We say a Banach space $X$ has property (k) if there exists $k \in [0, 1)$ such that whenever $x_n \rightharpoonup 0$, $\|x_n\| \to 1$ and $\liminf_n \|x_n - x\| < 1$ we have $\|x\| \leq k$. Note: By considering subsequences we see that the property remains unaltered if in the definition we replace $\liminf$ by $\limsup$.

Property (k) is an interesting condition which clearly exposes the uniformity in Opial's condition. Indeed Opial's condition corresponds to property (k) with $k = 0$.

**Proposition 4.** If $X$ has property (k) then $X$ has weak normal structure.

**Proof:** Suppose $X$ fails weak normal structure, then there is a sequence $(x_n)$ satisfying (S1) and (S2). Choosing $m$ sufficiently large so that $\|x_m\| > k$ (see the remark following S2) and taking $x := x_m$ we have that property (k) is contradicted by the sequence $(x_n)$.

We now turn to conditions sufficient for property (k), and hence also for weak normal structure.

**Proposition 5.** If $X$ is $\varepsilon_0$-URR, for some $\varepsilon_0 \in (0,1)$ then $X$ has property (k).

**Proof:** Suppose $x_n \rightharpoonup x$, $\|x_n\| \to 1$ and $\limsup_n \|x_n - x\| \leq 1$. Choose $m$ so that $\|x_m\| > \varepsilon_0$, then, since $\liminf_n \|x_n - x_m\| \geq \|x_m\|$, we may extract a subsequence, which we continue to denote by $(x_n)$, with $\|x_n - x_m\| > \varepsilon_0$ for all $n$. Continuing in this way we obtain a subsequence, still denoted by $(x_n)$, with $\text{sep}(x_n) > \varepsilon_0$. But, then $x_n - x \rightharpoonup x$ is a sequence in the unit ball with a separation constant greater than $\varepsilon_0$ and so $\|x\| \leq 1 - \delta$, where $\delta$ is given by the definition of $\varepsilon_0$-URR. Thus $X$ has property (k) with $k = 1 - \delta$.

**Proposition 6.** If $X$ is $\varepsilon_0$-inquadrate in every direction for some $\varepsilon_0 \in (0,1)$ and satisfies the non-strict Opial condition then $X$ has property (k).
Weak normal structure

PROOF: Suppose $x_n \to 0$, $\|x_n\| \to 1$ and $\limsup_n \|x_n - x\| \leq 1$. If $x = 0$ there is nothing to prove, so we assume that $x \neq 0$. Then $x_n$ and $y_n := x_n - x$ are two sequences in the unit ball with $x_n - y_n = x$ a fixed direction, so

$$
\left\|\frac{x_n + y_n}{2}\right\| \leq 1 - \delta(\|x\|, x) < 1, \quad \text{if } \delta(\|x\|, x) > 0.
$$

But, then $\limsup_n \|x_n - x/2\| < 1 = \limsup_n \|x_n\|$, contradicting the non-strict Opial condition. Thus we must have $\delta(\|x\|, x) = 0$ and this requires $\|x\| \leq \epsilon_0$. So $X$ has property $(k)$ with $k = \epsilon_0$.

The case when $X$ is $\epsilon_0$-inquadrate in every direction for an $\epsilon_0 \in [1, 2)$ is handled by the following proposition which in conjunction with proposition 3 yields our main result.

**Proposition 7.** If $X$ is $\epsilon_0$-inquadrate in every direction for some $\epsilon_0 \in (0, 2)$ and has WORTH then $X$ has property $(k)$.

**Proof:** Suppose $x_n \to 0$, $\|x_n\| \to 1$ and $\limsup_n \|x_n - x\| \leq 1$. Let $a_n := x_n - x$ and $b_n := x_n + x$. Then by WORTH $\|a_n\| - \|b_n\| \to 0$, so $\limsup_n \|a_n\| - \limsup_n \|x_n - x\| \leq 1$, and $b_n - a_n = 2x$. Therefore we have

$$
\limsup_n \|x_n\| = \limsup_n \|(a_n + b_n)/2\| \leq 1 - \delta(2\|x\|, x) < 1,
$$

a contradiction, unless $2\|x\| \leq \epsilon_0$. Thus $X$ has property $(k)$ with $k = \epsilon_0/2$.

If $X$ is $\epsilon_0$-inquadrate then the calculations of the previous proof allow some flexibility. If we measure the ‘degree of WORTHwhileness’ of a Banach space $X$ by

$$
w := \sup\{\lambda : \liminf_n \|x_n + x\| \leq \liminf_n \|x_n - x\|, \text{ whenever } x_n \to 0 \text{ and } x \in X\},
$$

(so $X$ has WORTH if and only if $w = 1$) then we can adapt the above calculations to verify the following.

**Proposition 8.** $X$ has property $(k)$ if

$$
w > \max\{\epsilon_0/2, 1 - \delta(\epsilon_0)\}
$$

for some positive $\epsilon_0$.

We close by noting that many spaces have WORTH, including all Banach lattices which are ‘weakly orthogonal’ in the sense introduced by Borwein and Sims [1]. In particular for $0 < \alpha < 1$ the space $\ell_2$ with the equivalent norm $\|x\| := \max\{\alpha \|x\|_2, \|x\|_\infty\}$

[5] Weak normal structure 527
has WORTH and hence satisfies the non-strict Opial condition, but fails to satisfy the Opial condition for any $\alpha$.

However, many important spaces do not enjoy WORTH, for example with the exception of $p = 2$ all the spaces $L_p[0, 1]$ fail to satisfy the non-strict Opial condition (see the details of the example given in Opial [10]) and hence also fail to have WORTH. They do however enjoy property (k); for example, when $p > 2$ it follows from Clarkson’s inequality that we may take $k = (1 - 2^{-p})^{1/p}$.

**REFERENCES**


