A SUPPORT MAP CHARACTERIZATION OF THE
OPIAL CONDITIONS

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A Banach space [dual space] \( X \) satisfies the weak [weak*] Opial condition if whenever \( (x_n) \) converges weakly [weak*] to \( x_\infty \) and \( x_0 \neq x_\infty \) we have

\[
\lim_{n} \inf \|x_n - x_\infty\| < \lim_{n} \inf \|x_n - x_0\|.
\]

Zdzisław Opial [1967] introduced the weak condition to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a nonexpansive selfmapping of a closed convex subset to a fixed point. In particular he observed that a uniformly convex Banach space with a weak to weak* sequentially continuous support mapping satisfies the weak condition. A support mapping is a selector for the duality map

\[
D: X \to 2^{X^*}: x \mapsto \{f \in X^*: f(x) = \|f\| = \|x\|\}.
\]

Uniform convexity is not sufficient for the weak to weak* sequential continuity of the unique support mapping. Browder [1966], and independently Hayes and Sims in connection with operator numerical ranges, had observed that the uniformly convex space \( L_1[0, 1] \) does not have a weak to weak (= weak*) continuous support mapping, while all of the sequence spaces \( \ell_p \) \( (1 < p < \infty) \) do. Opial [1967] demonstrated that with the exception of \( p = 2 \) none of the spaces \( L_p[0, 1] \) have weak to weak continuous support mappings. Indeed, Fixman and Rao characterize \( L_p(\Omega, E, \mu) \) spaces with a weak to weak continuous support mapping as those spaces for which every element of \( \Sigma \) with finite positive measure contains an atom.

That uniform convexity is not necessary is shown by the example of \( \ell_1 \) with an equivalent smooth dual norm. That the unique support mapping is
weak to weak* sequentially continuous follows from the norm to weak* upper semi-continuity of a duality mapping and the fact that $l_1$ is a Schur space.

These early results were considerably improved by Gossez and Lami Dozo [1972]. In particular they show the following.

1. The assumption of uniform convexity is unnecessary for Opial's result: Any Banach space [dual space with a weak [weak*] to weak* sequentially continuous support mapping satisfies the weak [weak*] Opial condition. Indeed, their proof is easily adapted to show that a space has the weak [weak*] Opial condition if the Duality mapping is such that given any weak* - neighbourhood $N$ of zero, if $(x_n)$ converges weakly [weak*] to $x_\infty$ then eventually $D(x_n) \cap (D(x_\infty) + z) \neq \emptyset$.

2. The weak Opial condition implies the fixed point property for non-expansive self-maps of weak-compact convex sets. We give a direct proof [Van Dulst, 1982] which also applies in the weak* case.

Proposition 1: Let $X$ be a Banach space [dual space with a weak* - sequentially compact ball] satisfying the weak [weak*] Opial condition. If $C$ is a weak [weak*] - compact convex subset of $X$, then any non-expansive mapping $T: C \to C$ has a fixed point.

Proof: Choose $x_0 \in C$, then since $C$ is closed and convex, for any $n$ the mapping $(1 - \frac{1}{n})T + \frac{1}{n}x_0$ is a strict contraction on $C$ which by the Banach contraction mapping principle has a unique fixed point $x_n$ in $C$.

Using the boundedness of $C$ if follows that

$$\|x_n - Tx_n\| \to 0.$$ 

Passing to a subsequence if necessary we may also assume that $(x_n)$ converges weak [weak*] to a point $x_\infty$.

\[1\] For example; the dual of a separable space, or more generally the dual of any smoothable space.
Then,
\[ \liminf \|T_{x_n} - x_n\| = \liminf \|T_{x_n} - x_n\| \leq \liminf \|x_\infty - x_n\| \]
contradicting the weak [weak*] Opial condition unless \( T_{x_\infty} = x_\infty \) \( \square \)

Gossez and Lami Dozo [1972] in fact proved that the weak Opial condition implies normal structure thereby deducing the weak version of the above result via Kirk [1965].

Whether or not the weak* Opial condition implies normal structure for weak* compact convex sets remains an open question.

(3) Weak to weak* sequential continuity of a support mapping is not necessary for the weak Opial condition. For \( 1 < p < q < \infty \) the space \((l_p \oplus l_q^2)\) satisfies the weak Opial condition, but [Bruck, 1969] the unique support mapping is not weak to weak continuous.

Karlovitz [1976] explored other connections between the Opial conditions and the space's geometry, establishing a relationship with approximate symmetry in the Birkhoff-James notion of orthogonality.

The purpose of this note is to provide the following characterization of the weak [weak*] Opial condition in terms of support mappings.

**Theorem 2:** The Banach space [dual space] \( X \) satisfies the weak [weak*] Opial condition if and only if whenever \( (x_n) \) converges weakly [weak*] to a non-zero limit \( x_\infty \), there exists a \( \delta > 0 \) such that eventually \( D(x_n)x_\infty \subset [\delta, \infty) \).

**Proof:** \( (\Rightarrow) \) Assume this were not the case, then by passing to subsequences we can find \( (x_n) \) converging weakly [weak*] to \( x_\infty \), \( \|x_n\| \geq \|x_\infty\| > 0 \) and \( f_n \in D(x_n) \) such that \( \lim_{n\to\infty} f_n(x_\infty) \leq 0 \).
But

$$\lim_n \inf \|x_n\|^2 = \lim_n \inf \|x_n - 0\|^2$$

$$> \lim_n \inf \|x_n - x_\infty\|^2$$

$$\geq \lim_n \inf f_n(x_n - x_\infty)$$

$$= \lim_n \inf (\|x_n\|^2 - f_n(x_\infty))$$

$$= \lim_n \inf \|x_n\|^2 - \lim f_n(x_\infty),$$

whence $\lim f_n(x_\infty) > 0$, a contradiction.

(* a modification of the proof in Gossez and Lami Dozo [1972].)

Using the integral representation for the convex function

$$t \mapsto \frac{1}{2}\|x + ty\|^2$$

[Roberts and Varberg, 1973, 12 Theorem A] we have

$$\frac{1}{2}\|x + y\|^2 = \frac{1}{2}\|x\|^2 + \int_0^1 g^+(x + ty; y) \ dt$$

where

$$g^+(u; y) = \lim_{h \to 0^+} \frac{\|u + hy\|^2 - \frac{1}{2}\|u\|^2}{h}$$

To establish the weak [weak*] Opial condition it suffices to show

that if $y_n$ converges weakly [weak*] to $y_\infty \neq 0$ then

$$\lim_n \inf \frac{1}{2}\|y_n\|^2 > \lim_n \inf \frac{1}{2}\|y_n - y_\infty\|^2.$$ 

Now,

$$\frac{1}{2}\|y_n\|^2 = \frac{1}{2}\|y_n - y_\infty\|^2 + \int_0^1 g^+(y_n - y_\infty + ty_\infty; y_\infty) \ dt$$

So

$$\lim_n \inf \frac{1}{2}\|y_n\|^2 \geq \lim_n \inf \frac{1}{2}\|y_n - y_\infty\|^2$$

$$+ \lim_n \inf \int_0^1 g^+(y_n - y_\infty + ty_\infty; y_\infty) \ dt.$$ 

By Fatou's lemma [Halmos, 1950] it is therefore sufficient to prove

for each $t \in (0, 1)$ that

$$\lim_n \inf g^+(y_n - y_\infty + ty_\infty; y_\infty) > 0.$$
But,
\[ g^+(y_n - y_\infty + ty_\infty; y_\infty) = \text{Max}\{f(y_\infty) : f \in D(y_n - y_\infty + ty_\infty)\} \]
[Barbu and Precupanu, 1978, §2.1 example 2° and Proposition 2.3] and
\[ y_n - y_\infty + ty_\infty \text{ converges weakly [weak*] to } ty_\infty \neq 0, \text{ so for } n \text{ sufficiently large and some } \delta > 0 \text{ we have } f(ty_\infty) > \delta \text{ for all } f \in D(y_n - y_\infty + ty_\infty) \]

Remarks:

(1) Using the weak* - neighbourhood \( \{g \in X^* : g(x_\infty) > -\|x_\infty\|^2\} \) of 0 in \( X^* \)

it is easily seen that the condition of the theorem is satisfied if the Duality mapping is sequentially weak [weak*] to weak* upper semi-continuous.

(2) From the details of the proof we see that if for some selection of \( f_n \) from \( D(x_n) \) we have \( \liminf_n f_n(x_\infty) > 0 \), where \( x_n \) converges weak [weak*] to \( x_\infty \neq 0 \), then the same is true for all selections.

REFERENCES


