ERGODIC THEOREM AND STRONG CONVERGENCE
OF AVERAGED APPROXIMANTS FOR
NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. Let $C$ be a bounded closed convex subset of a uniformly convex Banach space $X$ and let $T$ be an asymptotically nonexpansive in the intermediate mapping from $C$ into itself. In this paper, we first provide an ergodic retraction theorem and a mean ergodic convergence theorem. Using this result, we show that the set $F(T)$ of fixed points of $T$ is a sunny, nonexpansive retract of $C$ if the norm of $X$ is uniformly Gâteaux differentiable. Moreover, we discuss the strong convergence of the sequence \{$x_n$\} defined by $x_n = a_n x + (1 - a_n)T(\mu)x_n$ for $n = 0, 1, 2, \ldots$, where $x \in C$, $\mu$ is a Banach limit on $l^\infty$ and $a_n$ is a real sequence in $(0, 1]$.

1. Introduction

Let $C$ be a nonempty subset of a Banach space $X$. A mapping $T : C \mapsto C$ is said to be

(a) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$.

(b) asymptotically nonexpansive [19] if there exists a sequence \{k$_n$\} such that
$$\limsup_{n \to \infty} k_n \leq 1 \text{ and } \|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for } x, y \in C \text{ and } n \in \mathbb{N}.$$  

(c) asymptotically nonexpansive in the intermediate if
$$\limsup_{n \to \infty} \sup_{x, y \in C} \left[ \|T^n x - T^n y\| - \|x - y\| \right] \leq 0.$$  

(d) asymptotically nonexpansive type [19] if for each $x$ in $C$,
$$\limsup_{n \to \infty} \sup_{y \in C} \left[ \|T^n x - T^n y\| - \|x - y\| \right] \leq 0.$$  

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It is easily seen that \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)\) and that both the inclusions are proper (cf. [19, p. 112]). We denote \(F(T)\) by the set of fixed points of \(T\).

Let \(C\) be a bounded closed convex subset of a Banach space \(X\). Let \(T\) be a nonexpansive mapping from \(C\) into itself and let \(x\) be an element of \(C\) and for each \(t\) with \(0 < t < 1\), let \(x_t\) be the unique point of \(C\) which satisfies \(x_t = tx + (1 - t)x_1\). Browder [5] showed that \(\{x_t\}\) converges strongly to the element of \(F(T)\) which is nearest to \(x\) in \(F(T)\) as \(t \downarrow 0\) in the case when \(X\) is a Hilbert space. Reich [30] extended Browder’s result to the case when \(X\) is a uniformly smooth Banach space and he showed that \(F(T)\) is a sunny, nonexpansive retract of \(C\), i.e., there is a nonexpansive retraction \(P\) from \(C\) onto \(F(T)\) such that \(P(px + t(x - Px)) = Px\) for each \(x \in C\) and \(t \geq 0\) with \(Px + t(x - Px) \in C\). Recently, using an idea of Browder [5], Shimizu and Takahashi [32] studied the convergence of another approximating sequence for an asymptotically nonexpansive mapping in a Hilbert space. This result was extended to a Banach space by Shioji and Takahashi [33].

On the other hand, Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\) and let \(T\) be a nonexpansive mapping of \(C\) into itself. If the set \(F(T)\) of fixed points of \(T\) is nonempty, then the Cesàro means

\[
S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

converge weakly as \(n \to \infty\) to a fixed point \(y\) of \(T\) for each \(x \in C\). In this case, putting \(y = Px\) for each \(x \in C\), \(P\) is a nonexpansive retraction of \(C\) onto \(F(T)\).

In recent years much effort has devoted to studying nonlinear ergodic theory for (asymptotically) nonexpansive mappings and semigroups. See [1-3, 15-18, 20-29, 34]. Most of the work was carried out in a uniformly convex Banach space \(X\) whose norm is either Frechet differentiable or satisfies Opial’s condition. In this paper, we first prove an ergodic retraction theorem and an mean ergodic convergence theorems for non-lipschitzian mapping in a uniformly convex Banach space without using the Frechet differentiable norm, which includes many known results as special cases. Using this result, we show that the set \(F(T)\) is a sunny, nonexpansive retract of \(C\) if the norm of \(X\) is uniformly Gâteaux differentiable. Moreover, we discuss the strong convergence of the sequence \(\{x_n\}\) defined by \(x_n = a_n + (1 - a_n)T(\mu)x_n\) for \(n = 0, 1, 2, \ldots\), where \(x \in C\), \(\mu\) is a Banach limit on \(l^\infty\) and \(a_n\) is a real sequence in \((0, 1]\).

2. Preliminaries and Notations

Let \(X\) be a Banach space. We recall that the modulus of convexity of \(X\) is the
function $\delta_X$ defined on $[0, 2]$ by
\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \text{ and } \|x - y\| \geq \epsilon \right\}.
\]
A Banach space $X$ is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. We need the following characterization of uniform convexity for a Banach space.

**Proposition 1** (cf. [36]). Let $p > 1$ and $r > 0$ be two real numbers. Then a Banach space $X$ is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, depending on $p$ and $r$, $g(0) = 0$, such that
\[
\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)
\]
for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$ and $B_r$ is the closed ball centered at the origin and with radius $r$.

Throughout this paper $X$ denotes a uniformly convex real Banach space, $C$ a non-empty bounded closed convex subset of $X$, and $T$ an asymptotically nonexpansive in the intermediate sense. Put
\[
c_n = \sup_{x, y \in C} (\|T^n x - T^y\| - \|x - y\|) \vee 0,
\]
we have
\[
\lim_{n \to \infty} c_n = 0. \tag{2.1}
\]
We denote by $\triangle^n$ the set $\{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ for $n \in \mathbb{N}$, the set of all nonnegative integers. For a subset $D$ of $X$, we denote by $\text{co} D$ and $\text{co} D$, the convex hull and convex closed hull of $D$ respectively.

Let $\mu$ be a continuous linear functional on $l^\infty$ and let $a = (a_0, a_1, \ldots) \in l^\infty$, we write $\mu(a)$ instead of $\mu(a)$. For $n \in \mathbb{N}$, we can define a point evaluation $\delta_n$ by $\delta_n(a) = a_n$ for each $a \in l^\infty$. A convex combination of point evaluations is called a finite mean on $\mathbb{N}$. Let $X^*$ be the dual space of $X$. The value of $y \in X^*$ at $x \in X$ will be denoted by $\langle x, y \rangle$. Since $X$ is reflexive, for any continuous linear functional $\mu$ and $x \in C$ there exists a unique element $T(\mu)x$ in $X$ such that
\[
\langle T(\mu)x, x^* \rangle = \mu(n)(T^n x, x^*)
\]
for all $x^* \in X^*$. We write $T(\mu)x$ by $\mu(n)(T^n x)$. Also, if $\mu$ is a finite mean on $\mathbb{N}$, say
\[
\mu = \sum_{i=1}^n a_i \delta_i, \quad (t_i \in \mathbb{N}, a_i \geq 0, i = 1, 2, \cdots, n, \sum_{i=1}^n a_i = 1),
\]
then
\[ T(\mu)x = \sum_{i=1}^{n} a_i T^n_i x. \]

Now, for each \( m \in \mathbb{N} \), we can defined bounded linear operator \( r_m \) in \( l^\infty \) by
\[ (r_m)(a_n) = (a_{n+m}). \]
We call \( \mu \) a Banach limit if \( \mu \) satisfies \( \|\mu\| = \mu(1) = 1 \) and \( \mu = r_n^* \mu \) for each \( n \in \mathbb{N} \), where \( r_n^* \) is the conjugate operator of \( r_n \). For a Banach limit, we know that
\[ \lim \inf_{n \to \infty} a_n \leq \mu(n)(a_n) \leq \lim \sup_{n \to \infty} a_n \quad \text{for all } (a_0, a_1, \ldots) \in l^\infty \quad (2.2) \]

The duality mapping \( J \) from \( X \) into \( X^* \) will be defined by
\[ J(x) = \{ y \in X^* : \langle x, y \rangle = \|x\|^2 = \|y\|^2 \}, \]
for each \( x \in X \). \( X \) is said to be smooth if for each \( x, y \in B_1 \), the limit
\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.3) \]
exists. The norm of \( X \) is said to be uniformly Gâteaux differentiable if for each \( y \in B_1 \), the limit (2.3) exists uniformly for \( x \in B_1 \). The norm of \( X \) is said to be uniformly Fréchet differentiable if for each \( x \in B_1 \), the limit (2.3) exists uniformly for \( y \in B_1 \). \( X \) is said to be uniformly smooth if (2.3) exists uniformly for \( x, y \in B_1 \).

It is well known that if \( X \) is smooth then the duality mapping is single-valued and norm to weak star continuous. In the case when the norm of \( X \) is uniformly Gâteaux differentiable, we know the following [35, Lemma 1]:

**Proposition 2.** Let \( C \) be a convex subset of a Banach space \( X \) whose norm is uniformly Gâteaux differentiable. Let \( \{x_n\} \) be a bounded subset of \( X \), let \( z \) be a point of \( C \) and let \( \mu \) be a Banach limit. Then
\[ \mu(n)\|x_n - z\|^2 = \min_{y \in C} \mu(n)\|x_n - y\|^2 \]
if and only if
\[ \mu(n)\langle y - z, J(x_n - z) \rangle \leq 0 \quad \text{for all } y \in C. \]

Let \( C \) be a convex subset of \( X \), let \( K \) be a nonempty subset of \( C \) and let \( P \) be a retraction from \( C \) onto \( K \), i.e., \( Px = x \) for each \( x \in K \). A retraction \( P \) is said to be sunny if \( P(Px + t(x - Px)) = Px \) for each \( x \in C \) and \( t \geq 0 \) with \( Px + t(x - Px) \in C \). If the sunny retraction \( P \) is also nonexpansive, then \( K \) is said to be a sunny, nonexpansive retract of \( C \). Concerning sunny, nonexpansive retractions, we know the following [9, 29]:
**Proposition 3.** Let $C$ be a convex subset of a smooth space, let $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$. Then $P$ is sunny and nonexpansive if and only if
\[
\langle x - Px, J(y - Px) \rangle \leq 0 \quad \text{for all } x \in C \text{ and } y \in K.
\]
Hence there is at most one sunny, nonexpansive retraction from $C$ onto $K$.

3. Main Theorems

In this section, we will state our main Theorems and some remarks. The proof of Theorems will be given in the next section.

**Theorem 1.** Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, and let $T$ be an asymptotically nonexpansive in the intermediate sense mapping from $C$ to itself. Then, for any Banach limit $\mu$, the mapping $P$ defined by $Px = T(\mu)x$ is a retraction from $C$ onto $F(T)$ satisfying the following properties:

(i) $P$ is nonexpansive;
(ii) $PT = TP = P$;
(iii) $Px \in \cap_{n \geq m} \overline{T^n(x)}$ for all $x \in C$.

From Theorem 1, if there exists a unique retraction from $C$ onto $F(T)$ having properties (i) – (iii) of Theorem 1. Then $T(\mu) = T(\nu)$ for any Banach limits $\mu$ and $\nu$. By the proof of Theorem 2 of [16], we have following corollary.

**Corollary 1.** Let $X, C$ and $T$ be as in Theorem 1. Let $Q = \{q_{n,m}\}_{n,m \in \mathbb{N}}$ is a strongly regular matrix. Suppose that there exists a unique retraction from $C$ onto $F(T)$ having properties (i) – (iii) of Theorem 1. Then for every $x \in C$,
\[
w = \lim_{n \to \infty} \sum_{m=0}^{\infty} q_{n,m} T^{m+k} x = y \in F(T) \quad \text{uniformly in } m \in \mathbb{N}.
\]

Now, using Theorem 1, we shall give a new approximating sequence for an nonlipschitzian mapping.

Let $\{a_n\}$ be a real sequence such that
\[
0 < a_n \leq 1, \quad \lim_{n \to \infty} a_n = 0.
\]

Let $x$ be an element of $C$ and let $\mu$ be a Banach limit, and let $x_n$ be the unique point of $C$ which satisfies
\[
x_n = a_n x + (1 - a_n) T(\mu) x_n \quad (3.1)
\]

We remark that (3.1) is well defined since the mapping $T_n$ from $C$ into itself defined by $T_n u = a_n x + (1 - a_n) T(\mu) u$ satisfies $\|T_n u - T_n v\| \leq (1 - a_n) \|u - v\|$ for each $u, v \in C$. 

Theorem 2. Let $C$ be a bounded convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let $T$ be an asymptotically nonexpansive in the intermediate sense mapping from $C$ into itself. Then $F(T)$ is a sunny, nonexpansive retract of $C$.

Theorem 3. Let $C$ be a bounded convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let $T$ be an asymptotically nonexpansive in the intermediate sense mapping from $C$ into itself and let $P$ be the sunny, nonexpansive retract from $C$ onto $F(T)$. Let $x$ be an element of $C$ and let $\{x_n\}$ be sequence of $C$ which satisfies (3.1). Then $\{x_n\}$ converges strongly to $Px$.

4. Proof of Theorems

To simplify, in the following, for each $\varepsilon \in (0, 1]$, we define

$$a(\varepsilon) = \frac{\varepsilon^2}{10R} \delta_X\left(\frac{\varepsilon}{R}\right)$$

and

$$\mathcal{N}_\varepsilon = \{n_\varepsilon \in \mathcal{N} : c_{n+n_\varepsilon} < a(\varepsilon) \text{ for each } n \in \mathcal{N}\},$$

where $\delta_X$ is the modulus of convexity of the norm, $d = 2 \sup \{\|x\| : x \in C\}$, and $R = 4d + 1$. Noting that from (2.1), $\mathcal{N}_\varepsilon$ is nonempty for each $\varepsilon > 0$, and if $n_\varepsilon \in \mathcal{N}_\varepsilon$, then $n + n_\varepsilon \in \mathcal{N}_\varepsilon$ for each $n \in \mathcal{N}$.

The following lemma shall play a crucial role in the proof of our main theorems.

Lemma 4.1. Let $x$ be a element of $C$ and let $\lambda$ be a finite mean on $\mathcal{N}$ and let $\varepsilon_i \in (0, 1](i = 1, 2)$ be positive numbers. Then there exists $n_{\varepsilon_2} \in \mathcal{N}$, where $n_{\varepsilon_2}$ is independent of $\varepsilon_1$, such that

$$\|T^l T(\lambda) T^{n_\varepsilon} x - T(\lambda) T^{l+n_\varepsilon} x\| < \varepsilon_1 + \varepsilon_2$$

for all $n \geq n_{\varepsilon_2}$ and $l \in \mathcal{N}_{\varepsilon_1}$.

Proof. We shall prove the Lemma by mathematical induction.

If $\lambda = \delta_{m_1}, m_1 \in \mathcal{N}$, then the assertion is clear. Now suppose that the assertion holds for such $\lambda = \sum_{i=1}^{k-1} a_i \delta_{m_i}, (m_i \in \mathcal{N}, (a_1, a_2, \cdots, a_{k-1}) \in \Delta^{k-1})$. Let

$$\lambda = \sum_{i=1}^{k} a_i \delta_{m_i}, \quad (m_i \in \mathcal{N}, (a_1, a_2, \cdots, a_k) \in \Delta^k).$$

Defining

$$\mu = \frac{1}{1 - a_k} \sum_{i=1}^{k-1} a_i \delta_{m_i},$$
we claim that
\[
\lim_{n \to \infty} \|T(\mu)T_n^m x - T^{n+m_k} x\| \quad \text{exists.} \quad (4.4)
\]

Let \(\varepsilon > 0\), from assumption of induction there exists \(n_1 \in \mathcal{N}\) such that
\[
c_n < \frac{1}{3}\varepsilon,
\]
and
\[
\|T(l)T(\mu)T_n^m x - T(\mu)T^{n+l} x\| < \frac{1}{3}\varepsilon
\]
for all \(n \geq n_1\) and \(l \geq n_1\). It follows that, for all \(n \geq n_1\) and \(l \geq n_1\),
\[
\|T(\mu)T^{n+l} x - T^{n+l+m_k} x\| \leq \|T(\mu)T^{n+l} x - T^l T(\mu)T_n^m x\|
\]
\[
+ \|T^l T(\mu)T_n^m x - T^{n+l+m_k} x\|
\]
\[
\leq \|T(\mu)T_n^m x - T^{n+m_k} x\| + \varepsilon.
\]

For fixed \(n \geq n_1\), taking \(l \to \infty\), we get
\[
\limsup_{l \to \infty} \|T(\mu)T^l x - T^{l+m_k} x\| \leq \|T(\mu)T_n^m x - T^{n+m_k} x\| + \varepsilon,
\]
and hence
\[
\limsup_{l \to \infty} \|T(\mu)T^l x - T^{l+m_k} x\| \leq \liminf_{n \to \infty} \|T(\mu)T_n^m x - T^{n+m_k} x\| + \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, this implies (4.4) holds.

Put
\[
r = \lim_{n \to \infty} \|T(\mu)T_n^m x - T^{n+m_k} x\|.
\]

By assumption of induction again, for given \(\varepsilon_2 > 0\), there exists \(n_2 = n_2(\lambda, \varepsilon_2)\) such that
\[
\left|\|T(\mu)T_n^m x - T^{n+m_k} x\| - r\right| < \frac{1}{2}a(\varepsilon_2), \quad (4.5)
\]
and
\[
\|T^l T(\mu)T_n^m x - T(\mu)T^{n+l} x\| < \frac{1}{2}a(\varepsilon_2), \quad (4.6)
\]
for all \( l, n \geq n_2 \). Now, we put \( n_{e_2} = 2n_2 \in \mathcal{N} \). Since for \( n \geq n_{e} \),

\[
\|T^lT(\mu)T^n x - T(\mu)T^{l+n} x\| \leq \|T^lT(\mu)T^n x - T^{l+n_2}T(\mu)T^{n-n_2} x\| + \|T^{l+n_2}T(\mu)T^{n-n_2} x - T(\mu)T^{l+n} x\|
\]

\[
\leq c_1 + \frac{1}{2} a(\varepsilon_2)
\]

\[
+ \|T(\mu)T^n x - T^{n_2}T(\mu)T^{n-n_2}\|
\]

\[
\leq c_1 + a(\varepsilon_2)
\]

it then follows from (4.2) and (4.5) that

\[
\|T^lT(\mu)T^n x - T(\mu)T^{l+n} x\| < a(\varepsilon_1) + a(\varepsilon_2) \tag{4.7}
\]

for each \( l \in \mathcal{N}_{\varepsilon_1} \) and \( n \geq n_{e_2} \). Put

\[
x = (1 - a_n)(T^lT(\lambda)T^n x - T(\mu)T^{n+l} x)
\]

and

\[
y = a_n(T^{n+l+m_k} x - T^lT(\lambda)T^n x).
\]

It then follows from (4.4), (4.5), and (4.6) that, for \( l \in \mathcal{N}_{\varepsilon_1} \) and \( n \geq n_{e_2} \),

\[
\|x\| \leq (1 - a_n)(\|T^lT(\lambda)T^n x - T^lT(\mu)T^n x\|
\]

\[
+ \|T^lT(\mu)T^n x - T(\mu)T^n x\|
\]

\[
\leq (1 - a_n)(a(\varepsilon_1) + a(\varepsilon_2) + c_1 + \|T(\lambda)T^n x - T(\mu)T^n x\|)
\]

\[
\leq a_n(1 - a_n)r + 2a(\varepsilon_1) + 2a(\varepsilon_2) (\leq R),
\]

\[
\|y\| \leq a_n(c_1 + \|T^{n+m_k} x - T(\lambda)T^n x\|)
\]

\[
\leq a_n(1 - a_n)r + a(\varepsilon_1) + a(\varepsilon_2) (\leq R),
\]

and

\[
\|x - y\| = \|T^lT(\lambda)T^n x - T(\lambda)T^{l+n} x\|
\]

Suppose that

\[
\|x - y\| \geq \varepsilon_1 + \varepsilon_2
\]

for some \( l \in \mathcal{N}_{\varepsilon_1} \) and \( n \geq n_{e_2} \). Then we shall give the contradiction in following two cases.
Case I. If \( 4a_n(1 - a_n) r \leq \max\{\varepsilon_1, \varepsilon_2\} \), then

\[
\|x - y\| \leq \|x\| + \|y\| \leq 2a_n(1 - a_n) r + 3a(\varepsilon_1) + 3a(\varepsilon_2) < \varepsilon_1 + \varepsilon_2.
\]

This is a contradiction.

Case II. If \( 4a_n(1 - a_n) r > \max\{\varepsilon_1, \varepsilon_2\} \), then we have

\[
\|a_n x + (1 - a_n) y\| \leq (a_n(1 - a_n) r + 2a(\varepsilon_1) + 2a(\varepsilon_2))(1 - 2a_n(1 - a_n) \delta(\frac{\varepsilon_1 + \varepsilon_2}{R}))
\]

by Lemma in [14]. And hence

\[
a_n(1 - a_n) \|T(\mu)T_n x - T_n x\| \\
\leq a_n(1 - a_n) r + 2a(\varepsilon_1) + 2a(\varepsilon_2) - 2a_n^2(1 - a_n)^2 r \delta(\frac{\varepsilon_1 + \varepsilon_2}{R}).
\]

It then follows (4.5) that

\[
0 \leq 2a(\varepsilon_1) + 3a(\varepsilon_2) - 2a_n^2(1 - a_n)^2 r \delta(\frac{\varepsilon_1 + \varepsilon_2}{R}).
\]

If \( \varepsilon_1 \geq \varepsilon_2 \), then \( a(\varepsilon_1) \geq a(\varepsilon_2) \), \( 4a_n(1 - a_n) r > \varepsilon_1 \), and \( a_n(1 - a_n) > \frac{\varepsilon_1}{R} \). It follows that

\[
0 < 5a(\varepsilon_1) - \frac{\varepsilon_1^2}{2R} \delta(\frac{\varepsilon_1}{R}),
\]

this contradicts (4.1). If \( \varepsilon_1 < \varepsilon_2 \), then we also have a contradiction in the same way. This completes the proof. \( \square \)

Since \( N \) is commutative semigroup, there exists a net \( \{\lambda_\alpha : \alpha \in A\} \) of finite means on \( N \) such that

\[
\lim_{\alpha \in A} \|\lambda_\alpha - r_n^* \lambda_\alpha\| = 0 \quad (4.8)
\]

for every \( n \in N \), where \( A \) is a directed set (see [12]).

For each \( \varepsilon > 0 \) and \( l \in N \), we set

\[
F_\varepsilon(T^l) = \{ x \in C : \|T^l x - x\| \leq \varepsilon \}.
\]
Lemma 4.2. For each $0 < \epsilon < 1$, there exist $\delta > 0$ and $l_0 \in \mathbb{N}$ such that

$$coF_\delta(T_i^l)) \subset F_i(T_i^l))$$

for each $l \geq l_0$.

Proof. Since $X$ is uniformly convex, by [7, Theorem 1.1], for given $\epsilon > 0$ we can choose a positive integer $p$ such that for each $M \subset C$,

$$coM \subset co_pM + B_{\epsilon/4}, \quad (4.9)$$

where $co_pM$ denotes the set of sums $\lambda_1x_1 + \cdots + \lambda_px_p$ with $(\lambda_1, \ldots , \lambda_p) \in \Delta^p$ and $x_i \in M, 1 \leq i \leq p$. We first claim that

$$co_2F_a(\frac{\epsilon}{4})(T_i^l) \subset F_i(T_i^l), \quad (4.10)$$

for each $l \in G_a(\frac{\epsilon}{4})$, where $a(\frac{\epsilon}{4})$ and $G_a(\frac{\epsilon}{4})$ are defined in (3.1) and (3.2). In fact, let $x_0, x_1 \in F_a(\frac{\epsilon}{4})(T_i^l)$ and $x_t = tx_0 + (1 - t)x_1$ for some $0 < t < 1$. Put $x = (1 - t)(T_i^l x_t - x_1)$ and $y = t(x_0 - T_i^l x_t)$. Then we have

$$\|x\| \leq (1 - t)(\|T_i^l x_t - T_i^l x_1\| + \|T_i^l x_1 - x_1\|)$$

$$\leq t(1 - t)\|x_0 - x_1\| + 2(1 - t)a(\frac{\epsilon}{4}) (\leq R)$$

$$\|y\| \leq t(1 - t)\|x_0 - x_1\| + 2ta(\frac{\epsilon}{4}) (\leq R)$$

and

$$\|x - y\| = \|T_i^l x_t - x_t\|$$

We show the claim in the following two cases.

Case I. If $t(1 - t)\|x_0 - x_1\| \leq \frac{\epsilon}{10}$, then

$$\|T_i^l x_t - x_t\| = \|x - y\| \leq \|x\| + \|y\|$$

$$\leq 2t(1 - t)\|x_0 - x_1\| + 2a(\frac{\epsilon}{4})$$

$$< \frac{\epsilon}{4}.$$  

Case II. If $t(1 - t)\|x_0 - x_1\| > \frac{\epsilon}{10}$, then $t(1 - t) > \frac{\epsilon}{5R}$. Therefore we have

$$\|tx + (1 - t)y\| \leq t(1 - t)\|x_0 - x_1\| + 2a(\frac{\epsilon}{4}) + (1 - 2t(1 - t)\delta_X \frac{\|x - y\|}{R})$$

$$\leq t(1 - t)\|x_0 - x_1\| + 2a(\frac{\epsilon}{4}) - 2t^2(1 - t)^2\|x_0 - x_1\||\delta_X \frac{\|x - y\|}{R})$$

$$\leq t(1 - t)\|x_0 - x_1\| + 2a(\frac{\epsilon}{4}) - \frac{\epsilon^2}{15R}\delta_X \frac{\|x - y\|}{R})$$
That is
\[ \delta_X \left( \frac{\|x - y\|}{R} \right) \leq \frac{30R}{\epsilon^2} a \left( \frac{\epsilon}{4} \right) < \delta_X \left( \frac{\epsilon}{4R} \right). \]

It follows that
\[ \|T^t x_t - x_t\| \leq \frac{\epsilon}{4}. \]

This shows (4.10) holds. By induction, we also have
\[ \text{co} F_\delta(T^l) \subset F_{\epsilon/4}(T^l) \quad (4.11) \]

for \( \delta = a^{(p-1)}(\epsilon/4) \) and \( l \in G_{a^{(p-1)}(\epsilon/4)}. \) From (4.9) and (4.11), we get
\[ \text{co} F_\delta(T^l) \subset F_{\epsilon/4}(T^l) + B_{\epsilon/4}. \]

But
\[ C \cap (F_{\epsilon/4}(T^l) + B_{\epsilon/4}) \subset F_{\epsilon}(T^l) \]

because
\[ \|T^l x - x\| \leq \|x - y\| + \|y - T^l y\| + \|T^l y - T^l x\| \leq 2\|x - y\| + \|y - T^l y\| + c_l. \]

This completes the proof. \( \square \)

**Lemma 4.3.** For each \( 0 < \epsilon < 1 \) and \( l \in N_{\epsilon/4}, \) there exist \( \alpha \in A \) and \( n_\alpha \in N \) such that
\[ T(\lambda_\alpha)T^{n_\alpha+n_\alpha} x \subset F_{\epsilon}(T^l) \quad \text{for all} \quad n \in N. \]

**Proof.** For \( l \in N_{\epsilon/4}, \) from (4.8), there exists \( \alpha \in A \) such that
\[ \|\lambda_\alpha - r^*_l \lambda_\alpha\| < \frac{\epsilon}{R}. \]

By Lemma 4.1, there is an \( n_\alpha \in N \) such that
\[ \|T^l T(\lambda_\alpha)T^{n_\alpha+n_\alpha} x - T(\lambda_\alpha)T^{l+n_\alpha} x\| < \frac{\epsilon}{2} \]

for all \( n \in N. \) It follows that
\[ \|T^l T(\lambda_\alpha)T^{n_\alpha+n_\alpha} x - T(\lambda_\alpha)T^{l+n_\alpha} x\| \leq \|T^l T(\lambda_\alpha)T^{n_\alpha+n_\alpha} x - T(\lambda_\alpha)T^{l+n_\alpha} x\| + \|T(\lambda_\alpha)T^{l+n_\alpha} x - T(\lambda_\alpha)T^{n_\alpha+n_\alpha} x\| \leq \frac{\epsilon}{2} + d \|\lambda_\alpha - r^*_l \lambda_\alpha\| < \epsilon. \]

This completes the proof. \( \square \)
Lemma 4.4. Let \( \mu \) be a Banach limit, and \( x \in C \). Then
\[
T(\mu)x \in F(T) \bigcap \bigcap_{m \in \mathcal{N}} \text{co}\{T^n x : n \geq m\}.
\]

proof. We only need to prove that \( T(\mu)x \) is the fixed point of \( T \). Let \( \varepsilon > 0 \), then we can choose \( l_0 \in \mathbb{N} \) such that \( F_{2\varepsilon}(T^l) \subset F_{\varepsilon}(T^l) \) for all \( l \geq l_0 \).

By Lemma 4.2, there exists an \( \delta > 0 \) and \( l_1 \geq l_0 \) such that \( coF_{\delta} \subset F_{\varepsilon}(T^l) \) for all \( l \geq l_1 \).

It follows that \( coF_{\delta}(T^l) \subset F_{\varepsilon}(T^l) \) for all \( l \geq l_1 \).

By Lemma 4.3, there exist \( l_2 \geq l_1 \) and for each \( l \geq l_2 \), there exist \( \alpha \in A \) and \( n_\alpha \in \mathbb{N} \) such that
\[
T(\lambda_\alpha)T^{n_\alpha}x \subset F_{\delta}(T^l)
\]
for all \( n \in \mathcal{N} \). It follows that
\[
T(\mu)x = \mu_n(T(\lambda_\alpha)T^{n_\alpha}x) \subset coF_{\delta}(T^l) \subset F_{\varepsilon}(T^l)
\]
This implies that \( T(T(\mu)x) \to T(\mu)x \) strongly as \( l \to \infty \). Since \( T^N \) is continuous for some \( n \in \mathcal{N} \), we have \( T^N(T(\mu)x) = \lim_{l \to \infty} T^N T^l T(\mu)x = T(\mu)x \). This implies that \( T(T(\mu)x) = T^{1+N}(T(\mu)x) \to T(\mu)x \) as \( l \to \infty \). That is \( T(\mu)x \in F(T) \). This completes the proof. \( \Box \)

Now we can give the proof of Theorem 1.

Proof of Theorem 1. Let \( \mu \) be a Banach limit, for \( x \in C \), put \( Px = T(\mu)x \). It then follows from Lemma 4.4 that \( P \) is a retraction from \( C \) onto \( F(T) \) and \( Px \in \cap_m \text{co}\{T^n x : n \geq m\} \) for all \( x \in C \). For \( x, y \in C \) and \( m \in \mathcal{N} \), we have
\[
\|Px - Py\| = \|\mu(n)T^{n+m}x - \mu(n)T^{n+m}y\| \leq \|x - y\| + c_m(x).
\]
Which proves (i). Finally, since \( Px \in F(T) \), \( TPx = Px \) is obvious. That \( PTx = Px \) follows from the following reasoning:
\[
PTx = T(\mu)Tx = \mu(n)T^n Tx = \mu(n)T^{n+1}x = T(u)x = Px.
\]

To continue the proof of Theorem 3, we also need some Lemmas.
Lemma 4.5. [11]. Let \( X \) be a real Banach space, then for all \( x, y \in X \)
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle
\]
for all \( j(x + y) \in J(x + y) \).

We now turn to the proofs of Theorem 2 and Theorem 3. In the rest of this section, let \( x \in C \) and \( \{a_n\}, \{x_n\} \text{ and } \mu \) be as in (3.1).

Lemma 4.6. Let \( x_{n_i} \) be a subsequence of \( \{x_n\} \) and \( \mu \) be a Banach limit. Then there exists the unique element \( z \) of \( C \) satisfying
\[
\mu_i\|x_{n_i} - z\|^2 = \min_{y \in C} \mu_i\|x_{n_i} - y\|^2
\]
and the point \( z \) is a fixed point of \( T \).

Proof. Let \( f \) be a real valued function on \( C \) defined by
\[
f(y) = \mu_i\|x_{n_i} - y\|^2 \text{ for each } y \in C.
\]
Then we know from [31] that \( f \) is continuous and convex and satisfies \( \lim_{\|y\| \to \infty} f(y) = \infty \). Therefore there exists a unique \( z \in C \) such that \( f(z) = \min\{f(y) : y \in C\} \). Now, we show that \( z \) is a fixed point of \( T \). By the proof of Lemma 4.4 it is enough to show that \( \lim_{l \to \infty} T^l z = z \). To this end, from Property 1 we have, for each \( l \in \mathbb{N} \),
\[
\|x_{n_i} - \frac{T^l z + z}{2}\|^2 \leq \frac{1}{2}\|x_{n_i} - T^l z\|^2 + \frac{1}{2}\|x_{n_i} - z\|^2 - \frac{1}{4}g(\|T^l z - z\|).
\]
That is
\[
g(\|T^l z - z\|) \leq 2(f(T^l z) - f(z)).
\]
Since we have from Lemma 4.4 and (3.2) that
\[
\|x_{n_i} - T^l z\| \leq a_n\|x - T^l z\| + (1 - a_n)\|T(\mu)x_{n_i} - T^l z\|
\]
\[
\leq a_n\|x - T^l z\| + (1 - a_n)(c_l + \|T(\mu)x - z\|)
\]
\[
\leq a_n\|x - T^l z\| + \|x - z\| + c_l + \|x_{n_i} - z\|
\]
It follows that
\[
g(\|T^l z - z\|) \leq \mu_i(c_l + \|x_{n_i} - z\|)^2 - \mu_i\|x_{n_i} - z\|^2
\]
\[
\leq c_l\mu_i(c_l + 2\|x_{n_i} - z\|)
\]
This implies that \( T^l z \to z \) strongly. This completes the proof. \( \square \)
Lemma 4.7. Suppose that the norm of $X$ is uniformly Gâteaux differentiable. Then
\[ \langle x_n - x, J(x_n - z) \rangle \leq 0 \]
for all $n \in \mathbb{N}$ and $z \in F(T)$.

Proof. Let $z \in F(T)$. Since $x_n - x = \frac{1-a_n}{a_n}(T(\mu)x_n - x_n)$, we have
\[
\langle x_n - x, J(x_n - z) \rangle = \frac{1-a_n}{a_n} \langle T(\mu)x_n - x_n, J(x_n - z) \rangle \\
= \frac{1-a_n}{a_n} (\langle (T(\mu)x_n - z), J(x_n - z) \rangle + \langle z - x_n, J(x_n - z) \rangle) \\
\leq \frac{1-a_n}{a_n} (\|T(\mu)x_n - z\| \|x_n - z\| - \|x_n - z\|^2) \\
\leq 0.
\]

Lemma 4.8. Suppose that the norm of $X$ is uniformly Gâteaux differentiable. Then the set \( \{x_n : n \in \mathbb{N}\} \) is a relative compact subset of $C$ and each strong limit point of \( \{x_n\} \) is fixed point.

Proof. Let \( \{x_{n_i}\} \) be a subsequence of \( \{x_n\} \), it then follow from Lemma 4.7 that there is unique element $z$ of $F(T)$ satisfying (4.15). By Lemma 4.8, we get $\langle x_{n_i} - x, J(x_{n_i} - z) \rangle \leq 0$. This inequality and Proposition 2 yield
\[
\mu_i \|x_{n_i} - z\|^2 \leq \mu_i \langle x - z, J(x_{n_i} - z) \rangle \leq 0.
\]

By (2.2), there exists a subsequence of \( \{x_{n_i}\} \) converging strongly to $z$. This completes the proof. \[\square\]

Proof of Theorem 2. Put $a_n = \frac{1}{n}$. First we shall show that \( \{x_n\} \) converges strongly to an element of $F(T)$. By Lemma 4.8, we know that \( \{x_n : n \geq 1\} \) is a relative compact subset of $C$. Let \( \{x_{n_i}\} \) and \( \{x_{m_i}\} \) be subsequences of \( \{x_n\} \) converging strongly to $y$ and $z$ of $F(T)$, respectively. We shall show that $y = z$. From Lemma 4.7, we have $\langle y - x, J(y - z) \rangle \leq 0$ and $\langle z - x, J(z - y) \rangle \leq 0$. So we get $\|y - z\|^2 \leq 0$, i.e., $y = z$. So \( \{x_n\} \) converges strongly to an element of $F(T)$. Hence we can define a mapping $P$ from $C$ onto $F(T)$ by $Px = \lim_{n \to \infty} x_n$. Using Lemma 4.7 again, we have $\langle Px - x, J(Px - z) \rangle \leq 0$ for all $x \in C$ and $z \in F(T)$. Therefore $P$ is the sunny, nonexpansive retraction by Proposition 4. \[\square\]
Proof of Theorem 3. Let \( \{x_n\} \) be a subsequence of \( \{x_n\} \) converging strong to an element \( y \) of \( F(T) \). We shall show \( y = Px \). By Lemma 4.7, we have \( \langle x_n - x, J(x_n - Px) \rangle \leq 0 \). So we get \( \langle y - x, J(y - Px) \rangle \leq 0 \). Hence we get
\[
\|y - Px\|^2 \leq \langle x - Px, J(y - Px) \rangle \leq 0
\]
by Proposition 4. This completes the proof. \( \square \)

References

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