ON A CONNECTION BETWEEN THE NUMERICAL RANGE AND SPECTRUM OF AN OPERATOR ON A HILBERT SPACE

BRAILEY SIMS

For a complex Hilbert space $H$ we denote by $B(H)$ the algebra of continuous linear operators on $H$. For $T \in B(H)$, $T^*$ denotes the adjoint operator. The numerical range of $T$, $W(T)$, is defined as

$$W(T) = \{(Tx, x) : x \in H, \|x\| = 1\},$$

and

$$v(T) = \sup \{ |\lambda| : \lambda \in W(T) \}$$

is the numerical radius of $T$. $W(T)$ is a convex subset of the complex plane whose closure contains the spectrum of $T$, $\sigma(T)$. The set of eigenvalues of $T$ is denoted by $\rho \sigma(T)$ and the set of approximate eigenvalues by $\pi \sigma(T)$. $\sigma_0(T)$ is the convex hull of $\sigma(T)$.

A point $\lambda \in W(T)$ is a bare point of $W(T)$ if $\lambda$ lies on the perimeter of a closed circular disc containing $W(T)$. We say $W(T)$ has a corner with vertex $\lambda$ if $\lambda \in W(T)$ and $W(T)$ is contained in a half-cone with vertex $\lambda$ and angle less than $\pi$.

We aim to relate the vertices of corners of $W(T)$ to points in $\sigma(T)$. The starting point is the following lemma first suggested to me by A. M. Sinclair.

**Lemma 1.** For a complex Hilbert space $H$ and $T \in B(H)$, if $1 = v(T) \in W(T)$, then $1 \in \rho \sigma(U)$ where $U = \frac{1}{2}[T + T^*]$.

**Proof.** $1 = \sup \text{Re} W(T) = \sup W(U) \leq v(U) = \|U\| \leq \frac{1}{2}(v(T) + v(T^*)) = 1$; so $\|U\| = 1$. Now for some $x \in H$, $\|x\| = 1$, we have

$$1 = (Tx, x) = \text{Re} (Tx, x) = (Ux, x) \leq \|Ux\| \|x\| \leq 1;$$

so, by the rotundity of $H$, $Ux = x$.

**Lemma 2.** For a complex Hilbert space $H$ and $T \in B(H)$, if $\lambda \in W(T)$ is a bare point of $W(T)$, then $(e^{-i\theta} T + e^{i\theta} T^*) x = (e^{-i\theta} \lambda + e^{i\theta} \lambda) x$ for some $x \in H$, $\|x\| = 1$, and $0, 0 \leq \theta < 2\pi$.

**Proof.** Since $\lambda$ is a bare point of $W(T)$ there exists $r > 0$ and $x \in C$ such that $W(T) \subseteq D = \{z \in C : |z - \lambda| \leq r\}$ and $\lambda \in W(T) \cap \text{bdry} D$. Let $\lambda - x = re^{i\theta}$, $0 \leq \theta < 2\pi$ and set $T_1 = r^{-1} e^{-i\theta} (T - xI)$. Then $W(T_1)$ is contained in the unit disc and if $x \in H$, $\|x\| = 1$, is such that $\lambda = (Tx, x)$, we have

$$1 = (T_1 x, x) = v(T_1) \in W(T_1);$$

Received 29 March, 1972; revised 21 August, 1972.

[J. LONDON MATH. SOC. (2), 8 (1974), 57-59]
so, by Lemma 1, \( \frac{1}{2}[T_i + T_i^*]x = x \). Therefore
\[
\frac{1}{2}[r^{-1} e^{-i\theta}(T - \alpha I) + r^{-1} e^{i\theta}(T^* - \bar{\alpha}I)]x = x
\]
or
\[
\frac{1}{2}[e^{-i\theta} T + e^{i\theta} T^*]x = r x + \frac{1}{2}(e^{i\theta} \lambda - r + e^{i\theta} \lambda + r)x
\]
\[= \frac{1}{2}(e^{i\theta} \lambda + e^{i\theta} \lambda)x.
\]
This last lemma is similar to a result by B. A. Mirnai for compact operators [4; sledstvie 1], and from it our first main result follows.

**THEOREM 1.** For a complex Hilbert space \( H \) and \( T \in B(H) \), if \( \lambda \in W(T) \) is the vertex of a corner of \( \overline{W(T)} \), then \( \lambda \in \rho \sigma(T) \).

**Proof.** Since \( \lambda \) is the vertex of a corner of \( \overline{W(T)} \), \( \lambda \) is a bare point of \( \overline{W(T)} \), and in fact we can find at least \( r_1, r_2 > 0 \) and \( \alpha_1, \alpha_2 \in \mathbb{C}, \alpha_1 \neq t \alpha_2 \) for any \( t \in \mathbb{R} \), such that \( \overline{W(T)} \subseteq D_j = \{z \in \mathbb{C} : |z - \alpha_j| \leq r_j \} \) and \( \lambda \in \overline{W(T)} \cap D_j \) for \( j = 1, 2 \). So from the proof of Lemma 2 there exist \( \theta_1, \theta_2 \in (0, 2\pi), 0 < |\theta_1 - \theta_2| < \pi \), such that
\[
\frac{1}{2} [e^{-i\theta_j} T + e^{i\theta_j} T^*]x = \frac{1}{2}(e^{-i\theta_j} \lambda + e^{i\theta_j} \lambda)x
\]
or
\[
\frac{1}{2} [e^{-2i\theta_j} T + T^*]x = \frac{1}{2}(e^{-2i\theta_j} \lambda + \lambda)x, \quad j = 1, 2.
\]
Subtracting these two equations gives
\[
\frac{1}{2}(e^{-2i\theta_1} - e^{-2i\theta_2}) T x = \frac{1}{2}(e^{-2i\theta_1} - e^{2i\theta_2}) \lambda x
\]
and so, since \( \theta_1 \neq \theta_2 \), \( T x = \lambda x \).

**COROLLARY 1.1.** For a complex Hilbert space \( H \) and compact operator \( T \in B(H) \), if \( 0 \neq \lambda \in W(T) \) is the vertex of a corner of \( \overline{W(T)} \), then \( \lambda \in \rho \sigma(T) \).

**Proof.** Since \( \lambda \) is the vertex of a corner of \( \overline{W(T)} \), \( \lambda \) is a non-zero exposed point of \( \overline{W(T)} \) and so, by [1; Theorem 1], \( \lambda \in W(T) \) and the result now follows from Theorem 1.

**COROLLARY 1.2.** For a complex Hilbert space \( H \) and \( T \in B(H) \), if \( W(T) \) is a closed polygon then \( \text{co } \sigma(T) = W(T) \).

**Proof.** Let the vertices of the convex polygon \( W(T) \) be \( \{ \lambda_i \} \). Then, by Theorem 1, \( \lambda_i \in \rho \sigma(T) \) for all \( i \); so
\[
\text{co } \sigma(T) \supseteq W(T) \quad \text{but} \quad \text{co } \sigma(T) \subseteq \overline{W(T)} = W(T).
\]

**COROLLARY 1.3.** A closed bounded polygon with \( m \) vertices is the numerical range of an operator on \( n \)-dimensional Hilbert space if and only if \( m \leq n \).
Proof. Let the numerical range of $T$ be the closed polygon with vertices $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then by Theorem 1 each $\lambda_i$ is an eigenvalue of $T$ and there are at most $n$ of them. Conversely, let $\lambda_1, \ldots, \lambda_m (m \leq n)$ be the vertices of a closed polygon $P$. Then the normal operator represented by the diagonal matrix

$$a_{ij} = \lambda_i \delta_{ij} \quad 1 \leq i \leq m$$

$$= 0 \quad m < i \leq n$$

has $W(T) = \text{co} \sigma(T) = P$.

We now consider the case when $\lambda$ is the vertex of a corner of $\overline{W(T)}$ but $\lambda \in \overline{W(T)} \setminus W(T)$.

Theorem 2. For complex Hilbert space $H$ and $T \in \mathcal{B}(H)$, if $\lambda \in \overline{W(T)}$ is the vertex of a corner of $\overline{W(T)}$ then $\lambda \in \pi \sigma(T)$.

Proof. By a construction of S. K. Berberian [2] and a result of Berberian and G. H. Orland [3] we can embed $H$ in a larger Hilbert space $K$ and extend $T$ to $[T] \in \mathcal{B}(K)$ such that $\overline{W(T)} = W([T])$ and $\pi \sigma(T) = \rho \sigma([T])$. The result now follows by applying Theorem 1 to $[T]$, since $\lambda \in W([T])$ is the vertex of a corner of $\overline{W([T])} = W([T])$.

Corollary 2.1. For a complex Hilbert space $H$ and $T \in \mathcal{B}(H)$, if $\overline{W(T)}$ is a closed polygon, then $\text{co} \sigma(T) = \overline{W(T)}$.

Proof. Let $\{\lambda_i\}$ be the vertices of $\overline{W(T)}$. Then, by Theorem 2, $\lambda_i \in \pi \sigma(T)$ for all $i$; so $\text{co} \sigma(T) \supseteq \overline{W(T)}$.

A result corresponding to Theorem 2 is not generally valid in a Banach algebra without further restrictions. B. Schmidt [5, 6] has shown that if $\lambda$ is the vertex of a corner of $V(B, T)$ with angle less than $\pi/2$ then $\lambda \in \sigma(T)$ and that this is best possible.

The author wishes to express his gratitude to the referee, who suggested a number of improvements to the original manuscript. It has also been brought to the author's notice that a similar result to that of Theorem 2 has been obtained by S. Hilderbrandt (Math. Annalen 163 (1966), pp. 230–247).

References


The University of New England,
Armidale, N.S.W. 2351.