ON THE EXISTENCE OF SUPPORT MAPS WITH DENSE IMAGES

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Abstract

For a normed linear space $X$ we investigate conditions for the existence of support maps under which the image of $X$ is a dense subset of the dual. In the case of finite-dimensional spaces a complete answer is given. For more general spaces some sufficient conditions are obtained.

Throughout we will use $\| \cdot \|$ for the norm function of a normed linear space $X$, $X'$ for its dual space and $S(X)$ to denote its unit sphere \( \{ x \in X : \| x \| = 1 \} \).

We will be particularly interested in $S(X')$ regarded as a metric space under the metric $d(f, g) = \| f - g \|$ for all $f, g \in S(X')$.

Unless otherwise stated, by the interior, int $A$, or the boundary, bdry $A$, of a subset $A \subseteq S(X')$ we mean in the context of $(S(X'), d)$. Thus, for example, $f \in \text{int } A$ if there exists $r > 0$ such that $B_r(f) = \{ g \in S(X') : \| f - g \| < r \} \subseteq A$.

It is a simple consequence of the Hahn–Banach Theorem that we may define a set valued map $\mathcal{D}$ from $S(X)$ into the non-trivial subsets of $S(X')$ by

$$ f \in \mathcal{D}(x) \text{ if } f(x) = 1. $$

This map is frequently termed the duality map of $X$. When we want to emphasize the underlying space $X$ we will write $\mathcal{D}_X(x)$ in place of $\mathcal{D}(x)$.

A support map is a selector for $\mathcal{D}$, that is a function

$$ \phi : S(X) \rightarrow S(X') : x \mapsto \phi_x \in \mathcal{D}(x). $$

The important property of subreflexivity, as established by Bishop and Phelps (1961), states that for a Banach space $X$, $\bigcup_{x \in S(X)} \mathcal{D}(x)$ is a dense subset of $S(X')$. We will be interested in the geometry of spaces which have a support
map \( \phi \) with \( \phi(S(X)) = S(X') \). Such a support map will be referred to as having \textit{dense image.}

Not every Banach space has a support map with dense image, a fact amply demonstrated by the space \( l_2^\infty(\mathbb{R}) \).

Recalling that a Banach space \( X \) is \textit{smooth} at \( x \in S(X) \) if \( \mathcal{D}(x) \) is a singleton set, we see that subreflexivity establishes that for every smooth Banach space the unique support map has dense image. So a sufficient condition for a Banach space to have a support map with dense image would be the existence of a lower semi-continuous support map (norm to weak*, Cudia (1964)).

That the requirement of smoothness is over strong may be seen from the example of \( \mathbb{R}^3 \) equipped with norm the gauge of the "lens-shaped" set

\[
\{ x : \| x - (0, \frac{1}{2}, 0) \|_2 \leq 1 \ \text{and} \ \| x + (0, \frac{1}{2}, 0) \|_2 \leq 1 \}.
\]

In this space the selection of a support map with dense image follows from the existence of a function \( f : \mathbb{R} \to \mathbb{R} \) under which the image of an open neighbourhood is a dense subset of \( \mathbb{R} \). Accordingly we seek weaker conditions than smoothness which will ensure the existence of support maps with dense images.

The following equivalence is an obvious consequence of subreflexivity.

**Proposition 1.** A support map \( \phi \) of the Banach space \( X \) has dense image if and only if for each \( x \in S(X) \) and \( f \in \mathcal{D}(x) \) there exists a sequence \( \{ x_n \} \) of points in \( S(X) \) with \( \phi(x_n) \to f \).

As a consequence of this proposition we have:

If for any \( x \in S(X) \), \( \text{int} \mathcal{D}(x) \neq \emptyset \) and

\[
[\text{int} \mathcal{D}(x)] \cap \bigcup_{x \in S(X) \setminus \{ x \}} \mathcal{D}(y) = \emptyset,
\]

then \( X \) does not have a support map with dense image.

The next lemma shows that the second (underlined) condition is redundant.

**Lemma 2.** In the normed linear space \( X \), if \( f \in \text{int} \mathcal{D}(x) \) for some \( x \in S(X) \), then \( f \notin \mathcal{D}(y) \) for any \( y \in S(X) \setminus \{ x \} \).

**Proof.** Assume the contrary, that there exists \( y \in S(X) \setminus \{ x \} \) with \( f \in \mathcal{D}(y) \). Let \( Y \) be the two-dimensional subspace of \( X \) spanned by \( x \) and \( y \). Then \( f|_Y \in \mathcal{D}_Y(y) \) and further in \( S(Y') \), \( f|_Y \in \text{int} \mathcal{D}_Y(x) \) which clearly cannot be the case in a two-dimensional space unless \( x = y \), a contradiction.
Corollary 3. If the normed linear space $X$ has a support map with dense image, then $\text{int} \mathcal{D}(x) = \emptyset$ for all $x \in S(X)$.

We now develop some partial converses to Corollary 3.

Lemma 4. For the Banach space $X$, if $\text{int} \mathcal{D}(x) = \emptyset$ for all $x \in E$ a countable subset of $X$, then

$$\text{int} \left( \bigcup_{x \in E} \mathcal{D}(x) \right) = \emptyset.$$

Proof. Assume the contrary, then there exists $f_0 \in S(X')$ and $r > 0$ with $B_r(f_0) = \{f \in S(X'): \|f - f_0\| < r\} \subseteq \text{int} \left[ \bigcup_{x \in E} \mathcal{D}(x) \right]$. Now the closed subset $B_{r/2}[f_0] = \{f \in S(X'): \|f - f_0\| \leq \frac{1}{2}r\}$ is a complete metric space. However,

$$B_{r/2}[f_0] = \bigcup_{x \in E} (\mathcal{D}(x) \cap B_{r/2}[f_0])$$

and for each $x \in E$, $\mathcal{D}(x) \cap B_{r/2}[f_0]$ is nowhere dense, since $\mathcal{D}(x)$ is closed and in $B_{r/2}[f_0]$, $\text{int} (\mathcal{D}(x) \cap B_{r/2}[f_0]) = \emptyset$, contradicting the Baire Category Theorem.

For any normed linear space $X$ denote by $\lambda(X)$ the set of non-smooth points of the unit sphere $S(X)$ and let $\Delta = \bigcup \{\mathcal{D}(x): x \in S(X)\}$ and $\Lambda = \bigcup \{\mathcal{D}(x): x \in \lambda(X)\}$.

Lemma 5. Every support map of the Banach space $X$ has dense image in $S(X')\backslash \text{int} \Lambda$.

Proof. For $f \in S(X')\backslash \text{int} \Lambda$, either $f \in S(X')\backslash \Lambda$ or $f \in \text{bdry} \Lambda$. If $f$ belongs to the open set $S(X')\backslash \Lambda$, then by the subreflexivity of $X$ there exists a sequence $\{f_n\}$ of functionals in $\Delta \backslash \Lambda$ convergent to $f$. Now each $f_n \in \mathcal{D}(x_n)$ for some $x_n \in S(X)\backslash \lambda(X)$ in which case $\mathcal{D}(x_n)$ is the singleton set $\{\phi_{x_n}\}$ and so we have a sequence $\{x_n\}$ in $S(X)$ with $\phi_{x_n} \to f$.

On the other hand, if $f \in \text{bdry} \Lambda$, then by definition there exists a sequence $\{f_n\}$ of elements in $S(X')\backslash \Lambda$ with $f_n \to f$. From the first half of the proof we can choose an $x_n \in S(X)$ with $\|\phi_{x_n} - f_n\| < 1/n$ in which case

$$\|f - \phi_{x_n}\| \leq \|f - f_n\| + \|f_n - \phi_{x_n}\| \to 0$$

and again we have established the existence of a sequence $\{x_n\}$ in $S(X)$ with $\phi_{x_n} \to f$, thus establishing the result.

Corollary 6. Let $X$ be a Banach space and suppose $\Lambda$ is nowhere dense. Then every support map on $X$ has dense image.

Lemma 7. Let $X$ be a normed linear space. If $\Lambda$ has empty interior in the metric subspace $\Delta$, then every support map has dense image in $\Delta$. 
PROOF. If \( f \in \Delta \) then, for any \( \varepsilon > 0 \), \( B_\varepsilon(f) \) contains a point \( g \in \Delta \setminus A \).

Since \( g = \phi_x \) for some \( x \in S(X) \), \( \| f - \phi_x \| < \varepsilon \) so the image of \( \phi \) is dense in \( \Delta \).

**Theorem 8.** Let \( X \) be a Banach space with separable dual, then \( X \) has a support map with dense image if and only if \( \text{int} \, \mathcal{D}(x) = \emptyset \) for each \( x \in S(X) \).

**Proof.** Necessity has already been proved in Corollary 3.

To prove sufficiency, by Lemma 5, we need only ensure the image of \( \phi \) is dense in \( \text{int} \, \mathcal{A} \).

Since \( \text{int} \, \mathcal{A} \) is an open subset of \( S(X') \) we may choose \( \{ f_1, f_2, \ldots, f_n, \ldots \} \) to be a countable, dense subset of \( \text{int} \, \mathcal{A} \).

Now let \( \theta : n \mapsto (\theta_1(n), \theta_2(n)) \) be a 1–1 map from the set of natural numbers \( \mathbb{N} \) onto \( \mathbb{N} \times \mathbb{N} \), and inductively select \( x_n \) from

\[
\{ x \in \lambda(X); \{ x_1, x_2, \ldots, x_{n-1} \}; \mathcal{D}(x) \cap B_{\varepsilon_\theta(n)}(f_{\theta_1(n)}) \neq \emptyset \text{ where } r_n = \theta_1(n)^{-1} \}
\]

and \( \phi_{x_n} \) from \( \mathcal{D}(x_n) \cap B_{\varepsilon_\theta(n)}(f_{\theta_1(n)}) \).

Such a selection is possible since \( \text{int} \, \mathcal{A} \) is an open subset of \( \mathcal{A} \), and for any \( n \in \mathbb{N} \)

\[
\bigcup_{x \in \lambda(X) \setminus \{ x_1, x_2, \ldots, x_{n-1} \}} \mathcal{D}(x)
\]

is a dense subset of \( \mathcal{A} \) as \( \mathcal{D}(x) = \emptyset \), \( \mathcal{D}(x) \) is closed, and so \( \bigcup_{i=1}^{\infty} \mathcal{D}(x_i) \) is nowhere dense by Lemma 4.

It is clear from the above selection procedure that \( \{ \phi_{x_n}; n \in \mathbb{N} \} \) is dense in \( \text{int} \, \mathcal{A} \). Thus assigning \( \phi_{x_n} \) arbitrarily for \( x \in \lambda(X) \setminus \{ x_1, x_2, \ldots, x_{n-1} \} \) we arrive at a support map with dense image.

We now investigate some conditions under which \( \text{int} \, \mathcal{D}(x) = \emptyset \). From the convexity of the norm in the normed linear space \( X \) it follows that for any \( \alpha, y \in S(X) \) and \( \alpha \) real

\[
g^+(x; y) = \text{Limit}_{\alpha \to 0^+} \alpha^{-1}(\| x + \alpha y \| - 1) \quad \text{and}
\]

\[
g^-(x; y) = \text{Limit}_{\alpha \to 0^-} \alpha^{-1}(\| x + \alpha y \| - 1)
\]

exist.

It is well known that

\[
g^-(x; y) = \inf \{ \text{Re} \, f(y); f \in \mathcal{D}(x) \} \\
\leq \sup \{ \text{Re} \, f(y); f \in \mathcal{D}(x) \} \\
= g^+(x; y).
\]
The norm is differentiable at \( x \in S(X) \) in the direction \( y \) if \( g^- (x ; y) = g^+ (x ; y) \), in which case we will denote the common value of these two limits by \( g(x; y) \).

If the norm is differentiable at \( x \in S(X) \) in some direction \( y \in S(X) \setminus \{ x, -x \} \) we say the norm is differentiable at \( x \) in a non-radial direction, \( y \).

**Lemma 9.** In the normed linear space \( X \), if the norm is differentiable at \( x \in S(X) \) in a non-radial direction \( y \), then the real linear hull of \( D(x) \) is a proper subset of \( X' \).

**Proof.** It suffices to observe that \( z = y - g(x; y)x \) is a non-zero element of \( X \) for which \( \text{Re} f(z) = 0 \) for all \( f \in D(x) \), and so should the real linear hull of \( D(x) \) equal \( X' \) we would contradict the Hahn–Banach Theorem.

As a partial converse to this result we offer the following.

**Lemma 10.** If \( X \) is a finite-dimensional normed linear space and \( x \in S(X) \) is such that the real linear hull of \( D(x) \) is a proper subset of \( X' \), then the norm is differentiable at \( x \) in a non-radial direction.

**Proof.** Let \( D \) be the real linear hull of \( D(x) \) then \( D \) is a proper closed subspace of \( (X')_\mathbb{R} \) — the dual of \( X \) regarded as a linear space over \( \mathbb{R} \). So by the Hahn–Banach Theorem there exists \( F \in (X')_\mathbb{R} \) with \( \|F\| = 1 \) and \( F(D) = \{0\} \).

Form \( F' \) by \( F'(f) = F(f) - iF(if) \) for all \( f \in X' \) then \( F' \in X'' \) and so by the reflexivity of \( X \), \( F' = \hat{y} \) for some \( y \in S(X) \), where \( \hat{y}(f) = f(y) \). Clearly \( y \neq x, -x \) as \( f(x) = -f(-x) = 1 \) for all \( f \in D(x) \) while \( \text{Re} f(y) = \text{Re} \hat{y}(f) = F(f) = 0 \) for all \( f \in D(x) \). From this it also follows that \( g^-(x; y) = g^+(x; y) = 0 \) and so \( g(x; y) \) exists.

**Lemma 11.** If, in the normed linear space \( X \), \( x \in S(X) \) has \( \text{int} D(x) \neq \emptyset \), then \( X' \) is the real linear hull of \( D(x) \).

**Proof.** Choose \( f \in \text{int} D(x) \), then, for \( g \in X' \) either \( g = kf \) for some \( k \in \mathbb{R} \) or \( \{ f \} \subset \langle \langle f, g \rangle \rangle \) where \( \langle f, g \rangle \rangle \) is the real linear hull of \( \{ f, g \} \). So there exists \( f' \in \langle \langle D(x) \rangle \rangle \cap \langle f, g \rangle \rangle \) and further \( f' \neq kf \) \( (k \in \mathbb{R}) \) since \( |k| = 1 \) so \( k = \pm 1 \) but if \( f' = -f \) then \( 0 = \frac{1}{2} f + \frac{1}{2} f' \in D(x) \) which is impossible. Thus \( f, f' \) form a basis of \( \langle f, g \rangle \rangle \) and so \( g \) is a real linear combination of \( f \) and \( f' \) as required.

**Lemma 12.** Let \( X \) be a normed linear space. If the norm is differentiable at \( x \in S(X) \) in a non-radial direction, then \( \text{int} D(x) = \emptyset \).

**Proof.** The result follows from Lemmas 9 and 11.

Whether this requirement of differentiability is also a necessary condition
is not known. Since in general the converse of Lemma 11 may be untrue, a
reversal of the above line of reasoning cannot be attempted. However in the
case of finite-dimensional spaces we have the following result.

**Lemma 13.** Let $X$ be a normed linear space of finite dimension $n$. If $x \in S(X)$ is such that the real linear hull of $D(x)$ is $X'$, then $\text{int } D(x) \neq \emptyset$.

**Proof.** Let $f_1, f_2, \ldots, f_n \in D(x)$ have $X'$ as their real linear hull. Form $f = \sum_{k=1}^{n} (1/n)f_k \in D(x)$ by its convexity. From the continuity of the natural projections $\pi_k : X' \to \langle f_k \rangle$ we can choose an $\varepsilon > 0$ so that, if $g = \sum_{k=1}^{n} \mu_k f_k$ $\mu_k \in \mathcal{R}$ has $\|g - f\| < \varepsilon$ then $|\mu_k - 1/n| < 1/2n$ and so $\mu_k > 0$ for each $k$. Form $g' = g/\sum_{k=1}^{n} \mu_k$ is a convex combination of the $\{f_k\}$ and so belongs to $D(x)$. Consequently $g'$ has norm 1, whence $\sum_{k=1}^{n} \mu_k = 1$ and so $g = g' \in D(x)$. That is $\{g \in S(X); \|g - f\| < \varepsilon\} \subseteq D(x)$ and so $f \in \text{int } D(x)$.

Combining this partial converse to Lemma 11 with Lemma 10 and Theorem 8 we arrive at the following characterization in finite-dimensional spaces.

**Theorem 14.** Let $X$ be a finite-dimensional normed linear space. Then the norm is differentiable at $x \in S(X)$ in a non-radial direction if and only if $\text{int } D(x) = \emptyset$. Therefore $X$ has a support map with dense image if and only if at each point of $S(X)$ the norm is differentiable in a non-radial direction.

**Proof.** Lemmas 10, 12 and 13 establish the first equivalence, while the second equivalence follows from the first and Theorem 8.

**Theorem 15.** Let $X$ be a Banach space with $\lambda(X)$ finite. If at each $x \in \lambda(X)$ the norm is differentiable in some non-radial direction, then every support map has dense image.

**Proof.** Applying Lemma 12 then Lemma 4 shows that $\lambda$ is nowhere dense. Hence the conclusion follows from Corollary 6.

**Theorem 16.** Let $X$ be a reflexive space and $\lambda(X)$ countable. If at each $x \in \lambda(X)$ the norm is differentiable in some non-radial direction, then every support map has dense image.

**Proof.** Since $X$ is reflexive, $S(X') = \Delta$, so the result follows from the successive application of Lemmas 12, 4 and 7.

**Theorem 17.** Let $X$ be a Banach space with separable dual. If at each $x \in S(X)$ the norm is differentiable in some non-radial direction, then $X$ has a support map with dense image.
PROOF. The conclusion follows from Lemma 12 and Theorem 8.

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References


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