PROPERTIES \((U\tilde{A}_2)^*\) AND \((W\tilde{A}_2)\) IN ORLICZ SEQUENCE SPACES AND SOME OF THEIR CONSEQUENCES

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Abstract. In this paper, we introduce a new geometric property \((U\tilde{A}_2)^*\) and we show that if a separable Banach space has this property, then both \(X\) and its dual \(X^*\) have the weak fixed point property. We also prove that a uniformly Gateaux differentiable Banach space has property \((U\tilde{A}_2)^*\) and that if \(X^*\) has property \((U\tilde{A}_2)^*\), then \(X\) has the \((UKK)\)-property. Criteria for Orlicz spaces to have the properties \((UA_\varepsilon 2)\), \((UA_\varepsilon 2)^*\) and \((NUS^*)\) are given.

Keywords and Phrases: Orlicz space, Property \((A_\varepsilon^2)\), Fixed point property, \((UKK)\)-property, Weak fixed point property, The weak Banach-Saks property.

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§ 1. INTRODUCTIONS

We will denote by \(\mathcal{N}\) and \(\mathcal{R}\) the sets of natural and real numbers, respectively. Let \(X\) be a Banach space and let \(S(X)\) and \(B(X)\) denote the unit sphere and the unit ball of \(X\), respectively.

Given any element \(x \in S(X)\) and any positive number \(\delta\), we define a \(w^*\)-slice by,

\[
S^*(x, \delta) = \{x^* \in B(X^*) : x^*(x) \geq 1 - \delta\}.
\]

Let \(A\) be a bounded subset of \(X\). Its Kuratowski measure of noncompactness, \(\alpha(A)\), is defined as the infimum of all numbers \(d > 0\) such that \(A\) may be covered by a finite family of sets with diameters smaller than \(d\).

A Banach space \(X\) is said to be \(NUS^*\) [14] (equivalently, its dual is \(UKK^*\), [17]) if for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(x \in S(X)\), then \(\alpha(S^*(x, \delta)) \leq \varepsilon\).

A Banach space \(X\) is said to have the weak Banach-Saks property whenever given any weak null sequence \(\{x_n\}\) in \(X\) there exists a subsequence \(\{z_n\}\) of \(\{x_n\}\) such that the sequence \(\frac{1}{k}(z_1 + z_2 + \cdots + z_k)\) converges strongly to zero.

A Banach space \(X\) is said to have property \((A_2)\) if there exists a number \(\Theta \in (0, 2)\) such that for each weak null sequence \(\{x_n\}\) in \(S(X)\), there are \(n_1, n_2 \in \mathcal{N}\) satisfying \(\|x_{n_1} + x_{n_2}\| < \Theta\). It is well known that if \(X\) has property \((A_2)\) then \(X\) has the weak Banach-Saks property (see [7]).

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A Banach space \( X \) is said to have property \((\widetilde{A}_2)\) if for each \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that for any \( t \in (0, \delta) \) and each weak null sequence \( \{x_n\} \) in \( S(X) \), there is \( k \in \mathcal{N} \) satisfying \( \|x_1 + tx_k\| < 1 + t\varepsilon \) (see [14] and [15]).

Now, we introduce the notions of the \((U\tilde{A}_2)\), \((U\tilde{A}_2)^*\) and \((W\tilde{A}_2)\) properties.

A Banach space \( X \) is said to have property \((U\tilde{A}_2)\) if for each \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that for each weak null sequence \( \{x_n\} \) in \( S(X) \), there is \( k \in \mathcal{N} \) satisfying \( \|x_1 + tx_k\| < 1 + t\varepsilon \) for all \( t \in (0, \delta) \).

The dual space \( X^* \) of a Banach space \( X \) is said to have property \((U\tilde{A}_2)^*\) if for each \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that for each weak* null sequence \( \{x^*_n\} \) of \( S(X^*) \), there is \( k \in \mathcal{N} \) satisfying \( \|x^*_1 + tx^*_k\| < 1 + t\varepsilon \) for all \( t \in (0, \delta) \).

Notice that for reflexive Banach spaces the properties \((U\tilde{A}_2)\) and \((U\tilde{A}_2)^*\) coincide.

Prus (see [15]) has proved that \( X \) is \( NUS^* \) if and only if \( X \) has property \((U\tilde{A}_2)\) and \( X \) contains no copy of \( l_1 \). He also proved that if \( X \) is \( NUS^* \), then \( X \) has the weak Banach-Saks property (see [14] and [15]).

A natural generalization of this notion is the following property \((W\tilde{A}_2)\) defined below.

We say a Banach space \( X \) has property \((W\tilde{A}_2)\) whenever it satisfies the condition from the definition of property \((U\tilde{A}_2)\) with ‘for some \( \varepsilon \in (0, 1) \)’ in place of ‘for every \( \varepsilon > 0 \)’.

Let \( C \) be a nonempty subset of \( X \). A mapping \( T : C \to C \) is said to be nonexpansive whenever the inequality \( \|Tx - Ty\| \leq \|x - y\| \) holds for every \( x, y \in C \).

We will say that \( X \) has the weak fixed point property \((WFPP\) for short) if every nonexpansive mapping \( T : K \to K \) from a nonempty weakly compact convex subset \( K \) of \( X \) into itself has a fixed point.

R. Browder, D. Gohde, W. A. Kirk (see [9]) and other authors have established many conditions of a geometric nature on the norm of \( X \) that guarantee the \( WFPP \). Uniform rotundity, uniform rotundity in every direction and normal structure are examples of such conditions.

To obtain a geometric property of a Banach space \( X \) that guarantees it has the weak fixed point property, García-Falset [7] introduced the coefficient \( R(X) \) defined by the formula:

\[
R(X) = \sup \left\{ \liminf_{n \to \infty} \|x_n + x\| : \{x_n\} \subset B(X), x_n \overset{w}{\to} 0, x \in B(X) \right\}.
\]

He proved in [7] that a Banach space \( X \) with \( R(X) < 2 \) has the weak fixed point property. This coefficient was also considered in [20].

A Banach space \( X \) with property \((W\tilde{A}_2)\) has \( R(X) < 2 \) (see Note 1 below). Therefore, a Banach space \( X \) with property \((W\tilde{A}_2)\) has the weak fixed point property.
We say that a norm \( \| \cdot \| \) on \( X \) is uniformly Fréchet differentiable (a UF-norm for short) if the limit
\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]
exists uniformly with respect to \( x \) and \( y \) in \( S(X) \).

Let \((G, \Sigma, \mu)\) be a measure space with a finite and non-atomic measure \( \mu \). Denote by \( L^0 \) the set of all \( \mu \)-equivalence classes of real-valued measurable functions defined on \( G \). Let \( l^0 \) stand for the space of all real sequences.

A map \( \Phi : \mathcal{R} \to [0, \infty) \) is said to be an \textit{Orlicz function} if it is even, convex, vanishes at 0, and it is not identically equal to 0.

An Orlicz function is called an \textit{N-function} if
\[
\lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty.
\]

By the \textit{Orlicz function space} \( L_\Phi \) we mean the space
\[
L_\Phi = \left\{ x \in L^0 : I_\Phi(cx) = \int_G \Phi(cx(t)) \, d\mu < \infty \text{ for some } c > 0 \right\}.
\]

Analogously, we define the \textit{Orlicz sequence space}
\[
l_\Phi = \left\{ x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \text{ for some } c > 0 \right\}.
\]

The spaces \( L_\Phi \) and \( l_\Phi \) are equipped with the so-called \textit{Luxemburg norm}
\[
\| x \| = \inf\{ \varepsilon > 0 : I_\Phi(x/\varepsilon) \leq 1 \}
\]
or with the equivalent one
\[
\| x \|_0 = \inf_{k > 0} \frac{1}{k} \left( 1 + I_\Phi(kx) \right),
\]
called the \textit{Orlicz} or the \textit{Amemiya norm}. It is well known that if \( \Phi \) is an \textit{N-function}, then for any \( x \neq 0 \) there exists a number \( k > 0 \) such that
\[
\| x \|_0 = \frac{1}{k} \left( 1 + I_\Phi(kx) \right).
\]
(see [1]).

To simplify notations, we put \( L_\Phi = (L_\Phi, \| \cdot \|) \), \( l_\Phi = (l_\Phi, \| \cdot \|) \), \( L^0_\Phi = (L_\Phi, \| \cdot \|_0) \), \( l^0_\Phi = (l_\Phi, \| \cdot \|_0) \).

For any Orlicz function \( \Phi \) we define its \textit{complementary function} \( \Psi : \mathcal{R} \to [0, \infty] \) by the formula
\[
\Psi(v) = \sup_{u > 0} \{ u |v| - \Phi(u) \},
\]
for every \( v \in \mathcal{R} \). The complementary function \( \Psi \) of an Orlicz function is also a convex function vanishing at zero.
For \( x \in L_0^0 \) (respectively \( l_0^0 \)) we denote by \( k(x) \) the set of those \( k > 0 \) such that \( \|x\|_0 = \frac{1}{k} (1 + I_{\Phi}(kx)) \). It is known (see [1], [2] and [19]) that \( k(x) = [k \ast (x), k \ast \ast (x)] \), whenever \( k \ast \ast (x) < \infty \), where,

\[
k\ast (x) = \inf\{\lambda > 0 : I_{\Phi}(p(\lambda|x|)) \geq 1\}, \quad k \ast \ast (x) = \sup\{\lambda > 0 : I_{\Phi}(p(\lambda|x|)) \leq 1\}
\]

and \( \Psi \) is the function complementary to \( \Phi \). In the case when \( k \ast \ast (x) = \infty \) and \( k \ast (x) < \infty \), we have \( k(x) = [k \ast (x), k \ast \ast (x)] \). When \( k \ast (x) = \infty \),

\[
\|x\|_0 = \lim_{k \to \infty} \frac{1}{k} (1 + I_{\Phi}(kx)) = \lim_{k \to \infty} \frac{1}{k} I_{\Phi}(kx).
\]

We say an Orlicz function \( \Phi \) satisfies the \( \Delta_2 \)-condition (\( \delta_2 \)-condition) if there exist constants \( k \geq 2 \) and \( u_0 > 0 \) such that \( \Phi(u_0) < \infty \) (respectively, \( \Phi(u_0) > 0 \)) and

\[
\Phi(2u) \leq k \Phi(u),
\]

for every \( |u| \geq u_0 \) (respectively, for every \( |u| \leq u_0 \)), (see [1], [11], [12], [14] and [16]).

We say an Orlicz function \( \Phi \) satisfies the \( \nabla_2 \)-condition (respectively, \( \bar{\delta}_2 \)-condition) if its complementary function \( \Psi \) satisfies the \( \Delta_2 \)-condition (respectively, \( \delta_2 \)-condition).

An Orlicz function \( \Phi \) is said to be uniform convex in \([0, u_0]\), if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\Phi\left(\frac{u + v}{2}\right) \leq (1 - \delta) \frac{\Phi(u) + \Phi(v)}{2}
\]

for all \( u, v \in [0, u_0] \) satisfying \( |u - v| \geq \varepsilon \max\{u, v\} \).

We say an Orlicz function \( \Phi \) is strictly convex in \( \mathbb{R} \) if for any \( u, v \in \mathbb{R}, u \neq v \), and \( \alpha \in (0, 1) \) we have

\[
\Phi(\alpha u + (1 - \alpha)v) < \alpha \Phi(u) + (1 - \alpha)\Phi(v).
\]

For more details on Orlicz functions and Orlicz spaces we refer to [1], [11], [12], [14] and [18].

\section{General Results}

We begin with the following observation. \textbf{Note 1.} Property \((W\tilde{A}_2)\) of a Banach space \( X \) implies that \( R(X) < 2 \).

\textbf{Proof.} Take any weak null sequence \( \{x_n\} \) in \( S(X) \) and \( x \in S(X) \). Then we have that the sequence \( \{x, x_1, x_2, \ldots\} \subset S(X) \) is weakly null. So, by property \((W\tilde{A}_2)\), for some \( \varepsilon > 0 \) and \( \delta \) which we may take to be in \((0,1)\) we can find a \( k_1 \) such that \( \|x + \delta x_{k_1}\| \leq 1 + \delta \varepsilon \). Consider next the weak null sequence \( \{x, x_{k_1+1}, x_{k_1+2}, \ldots\} \). There is a \( k_2 > k_1 \) such that \( \|x + \delta x_{k_2}\| \leq 1 + \delta \varepsilon \). In this way we can inductively construct a sequence

\[
k_1 < k_2 < \ldots < k_l < \ldots
\]
of natural numbers such that \( \| x + \delta x_k \| \leq 1 + \delta \varepsilon \) for all \( l \in \mathbb{N} \). Therefore, \( \| x + x_{k_l} \| = \| x + \delta x_k + (1 - \delta)x_k \| \leq 1 + \delta \varepsilon + (1 - \delta) = \eta(\varepsilon) \in (1, 2) \). Since \( \eta(\varepsilon) \) is independent of \( x \in S(X) \) and independent of the weakly convergent sequence \( \{ x_n \} \) in \( S(X) \), the proof is complete.

**Theorem 1.** If \( \| \cdot \| \) is a UF-norm in a Banach space \( X \), then \( X \) has property \((U\tilde{A}_2)\).

**Proof.** Since \( \| \cdot \| \) is a UF-norm in \( X \), it follows that \( X \) is Gateaux differentiable; that is, \( X \) is smooth. Let \( f_x \in S(X^*) \) denote the unique supporting functional at \( x \in S(X) \). It is known that the norm \( \| \cdot \| \) is uniformly Fréchet differentiable on the space \( X \) if and only if
\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} = f_x(y)
\]
eexists uniformly with respect to \( x, y \in S(X) \).

Now, for any \( \varepsilon > 0 \) and each weak null sequence \( \{ x_n \} \) in \( S(X) \), there exists \( n_0 \in \mathbb{N} \) such that
\[
| f_x(x_n) | < \frac{\varepsilon}{2}
\]
for all \( n \geq n_0 \). Since the norm \( \| \cdot \| \) is (by assumption) UF on \( X \), there exists a \( \delta > 0 \) such that
\[
\left| \frac{\| x + tx_{n_0} \| - \| x \|}{t} - f_x(x_{n_0}) \right| < \frac{\varepsilon}{2}
\]
whenever \( |t| < \delta \), whence
\[
\| x + tx_{n_0} \| - \| x \| < \frac{t\varepsilon}{2} + | f_x(x_{n_0}) | t < t\varepsilon
\]
uniformly with respect to \( x \in S(X) \). This means that \( X \) has property \((U\tilde{A}_2)\), as required.

**Theorem 2.** Suppose that a Banach space \( X \) has property \((W\tilde{A}_2)\). Then \( X \) has the weak Banach-Saks property and the weak fixed point property.

**Proof.** Since \( X \) has the property \((W\tilde{A}_2)\), there exist \( \varepsilon \in (0, 1) \) and \( \delta > 0 \) such that for any \( t \in [0, \delta] \) and any weak null sequence \( \{ x_n \} \) in \( B(X) \) there exists \( k \in \mathbb{N}, k > 1 \), such that \( \| x_1 + tx_k \| < 1 + \varepsilon \delta \). Hence
\[
\| x_1 + x_k \| = \| x_1 + \delta x_k + (1 - \delta)x_k \|
\leq \| x_1 + \delta x_k \| + (1 - \delta) \leq 1 + \varepsilon \delta + 1 - \delta = 2 - \delta(1 - \varepsilon),
\]
which means that a Banach space with property \((W\tilde{A}_2)\) has property \((A_2)\). Consequently, a Banach space with property \((W\tilde{A}_2)\) has the weak Banach-Saks property.

Moreover, we have by the above estimate that \( R(X) \leq 2 - \delta(1 - \varepsilon) < 2 \), so \( X \) enjoys the weak fixed point property (see [7]).
Let us recall that for a Banach space $X$ with basis $\{x_i\}$, the basis constant of the space is the number $M = \sup \|P_n\|$, where $P_n$ are the projections defined by $P_n(x) = \sum_{i=1}^{n} a_i x_i$, where $x = \sum_{i=1}^{\infty} a_i x_i$.

**Theorem 3.** Let $X$ be a separable Banach space. If its dual space $X^*$ has property $(U \tilde{A}_2)^*$, then $X$ has the $(U \tilde{K}K)$-property.

**Proof.** Let $\{x_n\}$ be a sequence in $S(X)$ with $\text{sep}(\{x_n\}) := \inf_{m \neq n} \|x_m - x_n\| > \varepsilon$ and $x_n \overset{w^*}{\to} x \in B(X)$. Deleting at most one element of the sequence, we can assume that $\text{sep}(\{x_n - x\}) > \varepsilon$. For any $\varepsilon > 0$ let $M = 1 + \varepsilon$. By the Bessaga-Pelczynski selection principle, there exists a subsequence $\{z_n\}$ of the sequence $\{x_n - x, x\}$ with $z_1 = x$, being a basic sequence with the basis constant less than or equal to $M$ (see [5], p. 46).

Let us consider the sequence $\{z_n^*\}$ of the Hahn-Banach extensions of the coefficient functionals of the basic sequence $\{\tilde{z}_n\}$ and put $X_0 = \overline{\text{Span}}\{z_n : n = 1, 2, \ldots\}$. Then we can prove that $\langle z_n^*, z \rangle \to 0$ for any $z \in X_0$ as $n \to \infty$. Namely, for any $z \in X_0$ we have $z = \sum_{i=1}^{\infty} z_i^*(z)z_i$, whence

$$|\langle z_n^*, z \rangle| = \|z_n^*(z)z_n\| = \left\| \sum_{i=n}^{\infty} z_i^*(z)z_i - \sum_{i=n+1}^{\infty} z_i^*(z)z_i \right\| \leq \left\| \sum_{i=n}^{\infty} z_i^*(z)z_i \right\| + \left\| \sum_{i=n+1}^{\infty} z_i^*(z)z_i \right\| \to 0.$$ 

Since $X$ is separable, we can assume that $z_n^* \overset{w^*}{\to} z^*$ as $n \to \infty$.

Let us now take any $\varepsilon_2 \in (0, 1)$. Since $X^*$ has property $(WU \tilde{A}_2)^*$, there exists $0 < \delta_2 \leq 1$ and $k \in N$, $k > 1$, such that for any $t \in (0, \delta_2)$

$$(1) \quad \left\| \frac{z_k^*}{\|z_k^*\|} + t \frac{(z_k^* - z^*)}{\|z_k^* - z^*\|} \right\| < 1 + t \varepsilon_2.$$ 

It is easy to see that:

(2) For all $k \in N$, $\langle z_k^*, z_k \rangle = 0$ and $\langle z_k^*, z_k \rangle = \|z_k\|$. In particular $\langle z_k^*, x \rangle = 0$,

(3) For all $k \geq 2$, $\|x + z_k\| = 1$ and $\langle z_k^*, x \rangle = 0$,

(4) For all $k \in N$, $\|z_k^* - z^*\| \leq 4M$ and $\|z_k^*\| \leq M$.

Since $\text{sep}(\{x_n\}) > \varepsilon$ we can assume that $\|z_n\| \geq \frac{\varepsilon}{2}$ for $n \geq 2$. Let $k > 1$ be a natural number for which (1) holds for all $t \in (0, \delta_2)$. Then by conditions (2)-(4) and the fact that $z_1 = x$, we obtain

$$\|x\| = \langle z_1^*, x \rangle = \|z_1^*\| \langle \frac{z_1^*}{\|z_1^*\|}, x \rangle = \|z_1^*\| \langle \frac{z_1^*}{\|z_1^*\|}, x + z_k \rangle$$
\[ = \|z_1^*\| \left[ \left( \frac{z_1^*}{\|z_1^*\|} , x + z_k \right) + t \left( \frac{z_k^* - z^*}{\|z_k^* - z^*\|} , x + z_k \right) - t \frac{\|z_k^*\|}{\|z_k^* - z^*\|} \right] \]

\[ = \|z_1^*\| \left[ \frac{z_1^*}{\|z_1^*\|} + t \frac{z_k^* - z^*}{\|z_k^* - z^*\|} , x + z_k \right] - \frac{t\|z_k^*\|}{\|z_k^* - z^*\|} \]

\[ \leq \|z_1^*\| \left[ \frac{z_1^*}{\|z_1^*\|} + t \frac{z_k^* - z^*}{\|z_k^* - z^*\|} \right] - \frac{t\|z_k^*\|}{\|z_k^* - z^*\|} \]

\[ \leq M \left[ (1 + t\varepsilon_2) - \frac{t\varepsilon}{2\|z_k^* - z^*\|} \right] \leq M \left[ (1 + t\varepsilon_2) - \frac{t\varepsilon}{8M} \right]. \]

So, we have \( \|x\| \leq M \left[ (1 + t\varepsilon_2) - \frac{t\varepsilon}{8M} \right]. \) Using \( M = 1 + \varepsilon_1, \) and taking the limit as \( \varepsilon_1 \to 0, \) we obtain

\[ \|x\| \leq 1 + t(\varepsilon_2 - \frac{\varepsilon}{8}). \]

Now taking \( \varepsilon_2 = \frac{\varepsilon}{16}, \) and \( t = \frac{\delta_2}{2}, \) we get

\[ \|x\| \leq 1 - \frac{\delta_2\varepsilon}{32}, \]

completing the proof.

Remark 1. It is worth noticing that separability of \( X \) in the last theorem is only necessary to ensure that \( w^*- \) compact subsets of \( X \) are \( w^*- \) sequentially compact. We can relax the assumption of separability of \( X, \) requiring for example that \( X \) admits an equivalent smooth norm (see [10]).

The next result follows directly from our Theorems 2 and 3.

Corollary 1. Let \( X \) be a separable Banach space. If its dual space \( X^* \) has property \((UA_2)^*\), then both \( X \) and \( X^* \) have the weak fixed point property.

§ 3. THE CASE OF ORLICZ SPACES

Corollary 2. Let \( X \) be the Orlicz space \( L_M \) or \( L_M^0 \). Then the following statements are equivalent:

(1) \( X \) is uniformly smooth;

(2) \( X \) is nearly uniformly smooth;

(3) \( X \) is \((NUS^*)\);

(4) \( X \) has property \((UA_2)\);

(5) \( \Psi \in \Delta_2, \Psi \) is strictly convex on the whole real line and \( \Phi \) is uniformly convex outside a neighborhood of zero.

Proof. This follows from our Theorem 3 and Theorem 3.15 in [1].
Lemma 1. Suppose $\Phi \in \delta_2$. Then for any $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that,

$$I_\Phi(x + ty) - I_\Phi(x) < t\varepsilon,$$

whenever $I_\Phi(x) \leq L$, $I_\Phi(y) \leq \delta$ and $t \in (0, 1)$.

**Proof.** Since $\Phi \in \delta_2$, for any $\varepsilon > 0$ and $L > 0$ there exists $\delta \in (0, 1)$ such that,

$$I_\Phi(x + y) - I_\Phi(x) < \varepsilon$$

whenever $I_\Phi(x) \leq L$ and $I_\Phi(y) \leq \delta$ (see [4]). So for any $t \in (0, \delta)$, we have,

$$I_\Phi(x + ty) = I_\Phi(tx + ty + (1 - t)x)$$

$$\leq tI_\Phi(x + y) + (1 - t)I_\Phi(x)$$

$$\leq t(I_\Phi(x) + \varepsilon) + (1 - t)I_\Phi(x) = I_\Phi(x) + t\varepsilon,$$

whenever $I_\Phi(x) \leq L$ and $I_\Phi(y) \leq \delta$.

Lemma 2. Suppose $\Phi \in \delta_2$. Then for any $\varepsilon > 0$ and $u_0 > 0$ there exists $\delta > 0$ such that

$$\Phi(tu) \leq t\varepsilon \Phi(u),$$

whenever $|u| \leq u_0$ and $t \in (0, \delta)$.

**Proof.** Suppose that $\Phi \in \delta_2$. Then for any $u_0 > 0$ there exists $\theta \in (0, 1)$ such that

$$\Phi\left(\frac{u}{2}\right) \leq \frac{\theta}{2} \Phi (u)$$

whenever $|u| \leq u_0$ (see [1] and [16]). Take $n \in \mathcal{N}$ such that $\theta^n \leq \varepsilon$. Then for $\delta = \frac{1}{2^n}$, we have

$$\Phi(\delta u) = \Phi\left(\frac{u}{2^n}\right) \leq \left(\frac{\theta}{2}\right)^n \Phi (u) \leq \delta \varepsilon \Phi (u),$$

whenever $|u| \leq u_0$.

Hence, for any $t \in (0, \delta)$, we have

$$\Phi(tu) = \Phi\left(\frac{t \delta u}{\delta}\right) \leq \frac{t}{\delta} \delta \varepsilon \Phi (u) = t\varepsilon \Phi (u),$$

whenever $|u| \leq u_0$, which finishes the proof.

From here on we will make use of the following parameter for an Orlicz function $\Phi$:

$$m(\Phi) = \sup \left\{ n \in \mathbb{N} : \sum_{i=1}^{n} \Psi(A) < 1 \right\}$$

where $A := \lim_{u \to \infty} (\Phi(u)/u)$ and $\Psi$ is the function complementary to $\Phi$ in the sense of Young.

For any $x \in l^0_\Phi$, put $N(x) = \{i \in N : x(i) \neq 0\}$ and define $D(l^0_\Phi) = \{x = (x(i)) \in B(l^0_\Phi) : N(x) \text{ is finite}\}$. 
Lemma 4. Let $\Phi$ be an Orlicz function with $\Phi \in \delta_2$, $m(\Phi) \leq 1$ and $\Phi \in \overline{\delta}_2$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every weak null sequence $\{x_n\}$ in $B(l^0_\Phi)$ and every $x \in D(l^0_\Phi)$ there is a natural number $k > 1$ such that
\[
\|x + tx_n\|^0 \leq 1 + t\varepsilon,
\]
whenever $t \in (0, \delta)$.

**Proof.** Case I. Assume that $\lim_{u \to \infty} \frac{\Phi(u)}{u} = +\infty$. Let $\varepsilon > 0$ be given. By $\Phi \in \overline{\delta}_2$, the set $Q = \{k_x : \frac{1}{2} \leq \|x\|_0 \leq 1 \text{ and } \|x\|^0 = \frac{1}{k_x} (1 + I_\Phi(k_x x))\}$ is bounded; that is, there exists $k > 1$ such that $1 \leq k_x \leq k$ whenever $\frac{1}{2} \leq \|x\|_0 \leq 1$ (see [1]). By Lemma 2, we know that there exists $\delta \in (0, 1)$ such that
\[
\Phi(tu) \leq t\delta \Phi(u)
\]
whenever $t \in (0, \delta)$ and $|u| \leq \Phi^{-1}(k)$. By Lemma 1, there exists $\theta > 0$ such that
\[
|I_\Phi(x + ty) - I_\Phi(x)| < t\varepsilon,
\]
whenever $I_\Phi(x) \leq L$, $I_\Phi(y) \leq \theta$ and $t \in (0, 1)$.

Fix $t \in (0, \frac{\delta}{2})$ and let $\{x_n\}$ be an arbitrary weak null sequence in $S(l^0_\Phi)$. For any $x \in D(l^0_\Phi)$, take $i_0 \in \mathcal{N}$ such that $x(i) = 0$ for $i > i_0$. Since $x_n \overset{w}{\to} 0$, we conclude that $x_n \to 0$ coordinatewise, and so there exists $n_0 \in \mathcal{N}$ such that $\sum_{i=1}^{i_0} \Phi(x_n(i)) < \theta$ for all $n \geq n_0$. Hence, we get for $l \geq 1$ satisfying $\|x\| = \frac{1}{l} (1 + I_\Phi(lx))$ that:
\[
\|x + tx_n\|^0 \leq \frac{1}{l} [1 + I_\Phi(l(x + tx_n))]
\]
\[
= \frac{1}{l} \left[ 1 + \sum_{i=1}^{i_0} \Phi(l(x(i) + tx_n(i))) + \sum_{i=i_0+1}^{\infty} \Phi(ltx_n(i)) \right]
\]
\[
\leq \frac{1}{l} \left[ 1 + \sum_{i=1}^{i_0} \Phi(lx(i)) + t\varepsilon + \sum_{i=i_0+1}^{\infty} \Phi(ltx_n(i)) \right]
\]
\[
\leq \frac{1}{l} \left[ 1 + \sum_{i=1}^{i_0} \Phi(lx(i)) + t\varepsilon + ltx \sum_{i=i_0+1}^{\infty} \Phi(x_n(i)) \right]
\]
\[
\leq \frac{1}{l} \left[ 1 + \sum_{i=1}^{i_0} \Phi(lx(i)) \right] + 2t\varepsilon \leq 1 + 2t\varepsilon. \quad \square
\]

Case II.

Assume that $\lim_{u \to \infty} \frac{\Phi(u)}{u} = A < \infty$. Let $\{x_n\}$ be a weak null sequence in $S(l^0_\Phi)$ and $x$ be in $D(l^0_\Phi)$. Put,
\[
y_m = \left( \frac{1}{A}, \frac{1}{m}, 0, 0, \ldots \right),
\]
where $m := m(\Phi)$. Since $x_n \overset{w}{\to} 0$, we may assume without loss of generality that $x_n(i) = 0$ for $i = 1, 2$ (because weak convergence to zero in $l^0_\Phi$ implies
coordinatewise convergence to zero). By the condition $m(\Phi) \leq 1$, we know that there exists $k_m > 0$ such that
\[ \|y_m\|_0 = \frac{1}{k_m} (1 + I_\Phi(k_m y_m)) \quad \forall m \in \mathbb{N}. \]

It is clear that the sequence $\{k_m\}$ is bounded. Hence, by virtue of Lemma 2,
\[ \|x + tx_n\|_0 \leq \|y_m + tx_n\|_0 \leq \frac{1}{k_n} \left(1 + \sum_{i=1}^\infty \Phi(k_m y_m(i)) + \sum_{i=3}^\infty \Phi(k_m tx_n(i))\right) \]
\[ \leq \|y_m\|_0 + t\varepsilon \sum_{i=3}^\infty \Phi(x_n(i)) \]
\[ \leq \|y_m\|_0 + t\varepsilon \]
Passing to the limit as $m$ tends to $\infty$, we obtain that
\[ \|x + tx_n\|_0 \leq 1 + t\varepsilon, \]
as required.

**Theorem 4.** Let $\Phi$ be an $N$-function and $X = l_0^\Phi$ fail the Schur property. Then the following statements are equivalent:

1. $X$ has property $(U \tilde{A}_2)$;
2. $X$ has property $(W \tilde{A}_2)$;
3. $R(X) < 2$;
4. $\Phi \in \delta_2$, $m(\Phi) \leq 1$ and $\Phi \in \tilde{\delta}_2$.

**Proof.** That (1) implies (2) is clear and by Note 1, (2) implies (3).

To see that (3) implies (4), suppose that $\Phi \notin \delta_2$, then for any $\varepsilon > 0$ there exists $x \in S(l_0^\Phi)$ such that
\[ 1 - \varepsilon \leq \left\| \sum_{i=n}^\infty x(i) e_i \right\|_0 \leq 1 \]
for all $n \in \mathcal{N}$. Take a sequence $\{n_i\}$ in $\mathcal{N}$ with $n_1 < n_2 < \cdots$ such that
\[ \left\| \sum_{j=n_{i+1}}^{n_i+1} x(j) e_j \right\|_0 \geq 1 - 2\varepsilon \quad \text{for all } i \in \mathcal{N}. \]

Put $x_i = \sum_{j=n_{i+1}}^{n_i+1} x(j) e_j$. Since $\Phi$ is an $N$-function,
\[ \lim_{\lambda \to 0} \max_{i \in \mathcal{N}} \frac{I_\Phi(\lambda x_i)}{\lambda} \leq \lim_{\lambda \to 0} \frac{I_\Phi(\lambda x)}{\lambda} = 0, \]
so we have that \( x_i \xrightarrow{\ell_q} 0 \). Notice that every singular functional vanishes on any \( x_i \). In consequence \( x_i \xrightarrow{w} 0 \).

But \( \lim \inf_{i \to \infty} \|x_i + x\|^0 \geq \lim \inf_{i \to \infty} \|x_i\|^0 \geq 2(1 - 2\varepsilon) \). By the arbitrariness of \( \varepsilon > 0 \), we get \( R(l_\Phi^0) = 2 \). Thus, we have proved that if \( \Phi \notin \delta_2 \), then (3) does not hold.

Now we need to prove the necessity of the condition \( m(\Phi) \leq 1 \) for \( R(X) < 2 \). Let us assume that \( m(\Phi) \geq 2 \) and for each \( n \in \mathbb{N} \) define

\[
x_n = \left( 0, \ldots, 0, \frac{1}{A}, 0, \ldots \right),
\]

where \( \frac{1}{A} \) is in the \( n \)’th place and \( A := \lim_{u \to \infty} \frac{\Phi(u)}{u} \). Then \( \|x_n\|^0 = 1 \), because \( m(\Phi) \leq 2 \) yields \( k^*(x_n) = \infty \), and so from our earlier discussion \( \|x_n\|^0 = \lim_{k \to \infty} \frac{(I_\Phi(kx_n))/k}{k^*} \). Since \( l_\Phi^0 \) fails the Schur property, we have the equality \( \lim_{u \to \infty} \frac{(\Phi(u))/u}{A} = 0 \). Consequently,

\[
\lim_{\lambda \to 0} \left( \sup_n \frac{I_\Phi(\lambda x_n)}{\lambda} \right) = \lim_{\lambda \to 0} \frac{\Phi(\lambda)}{\lambda} = 0.
\]

Therefore, by virtue of lemma 2.3 in [3] (also see, Theorem 1.69 in [1]) and \( \Phi \notin \delta_2 \), we conclude that \( \{x_n\} \) is a weak null sequence (also see the proof of Theorem 2.3 in [6]). Moreover,

\[
\|x_n + x_1\|^0 = 2A \cdot \frac{1}{A} = 2,
\]

so \( R(l_\Phi^0) = 2 \), which establishes the necessity of the condition \( m(\Phi) \leq 1 \) for \( R(l_\Phi^0) < 2 \).

Suppose that \( \Phi \notin \delta_2 \). Then the Kottman constant \( K(l_\Phi^0) = \sup \{d_x : x \in S(l_\Phi^0)\} = 2 \) (see [1] and [18]). Hence for any \( \varepsilon > 0 \) there exists \( x \in S(l_\Phi^0) \) such that \( d_x > 2 - \varepsilon \). Furthermore, we have \( d_{x,k} \geq d_x > 2 - \varepsilon \) for all \( k > 1 \).

Put,

\[
x_1 = (x(1), 0, x(2), 0, x(3), 0, x(4), 0, x(5), 0, x(6), 0, \ldots),
\]

\[
x_2 = (0, x(1), 0, 0, 0, x(2), 0, 0, 0, 0, 0, 0, x(3), 0, 0, \ldots),
\]

\[
x_3 = (0, 0, 0, x(1), 0, 0, 0, 0, 0, 0, 0, 0, x(2), 0, 0, 0, \ldots),
\]

\[
\ldots,
\]

so the supports of the \( x_n \) are pairwise disjoint and for any \( n \in \mathbb{N} \) the non-zero coordinates of \( x_n \) are precisely the coordinates of \( x \).

Then, \( \|x_n\|^0 = 1 \), for any \( n \in \mathcal{N} \), \( x_n \xrightarrow{w} 0 \) and for any \( k > 1 \) we have

\[
\frac{1}{k} \left( 1 + I_\Phi \left( \frac{k(x_n + x_1)}{d_x} \right) \right) \geq \frac{1}{k} \left( 1 + I_\Phi \left( \frac{k(x_n + x_1)}{d_{x,k}} \right) \right)
\]

\[
= \frac{1}{k} \left( 1 + I_\Phi \left( \frac{kx}{d_{x,k}} \right) \right) + I_\Phi \left( \frac{kx}{d_{x,k}} \right) = \frac{1}{k} (1 + k - \frac{k-1}{2}) = 1.
\]
So, we get $\left\| \frac{x_n + x_1}{d_n} \right\|^0 \geq 1$; that is, $\liminf_{n \to \infty} \left\| x_n + x_1 \right\|^0 \geq d - \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get $R(l^0_\Phi) = 2$. Therefore, we have proved that $\Phi \not\in \delta_2$ implies that (3) does not hold.

(4) $\Rightarrow$ (1). By Lemma 4, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every weak null sequence $\{x_n\}$ in $B(l^0_\Phi)$ and any $x \in D(l^0_\Phi)$, there exists a number $m > 1$ such that

$$\left\| x + tx_m \right\|^0 \leq 1 + \frac{t\varepsilon}{2},$$

whenever $t \in (0, \delta)$.

Let $t \in (0, \delta)$ be given arbitrarily. For any weak null sequence $\{x_n\}$ in $B(l^0_\Phi)$, we only need to consider the case when $N(x_1)$ is infinite. Take $i_0$ large enough so that $\left\| \sum_{i=1}^{i_0} x_1(i) e_i \right\|^0 \leq \frac{t\varepsilon}{2}$. Then there exists $m \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{i_0} x_1(i) e_i + tx_m \right\|^0 \leq 1 + \frac{t\varepsilon}{2}.$$

Hence,

$$\left\| x_1 + tx_m \right\|^0 \leq \left\| \sum_{i=1}^{i_0} x_1(i) e_i + tx_m \right\|^0 + \frac{t\varepsilon}{2} \leq 1 + \frac{t\varepsilon}{2} + \frac{t\varepsilon}{2} = 1 + t\varepsilon.$$

**Corollary 3.** Let $\Phi$ be any Orlicz function and $X = l^0_\Phi$. Then the following statements are equivalent:

(1) $X$ is $(NUS^*)$;

(2) $X$ is nearly uniformly smooth;

(3) $\Phi \in \delta_2$, $\Phi \in \overline{\delta}_2$ and $m(\Phi) \leq 1$.

**Proof.** (3) $\Rightarrow$ (1). If $\Phi \in \delta_2$, $\Phi \in \overline{\delta}_2$ and $m(\Phi) \leq 1$, by Theorem 4, $l^0_\Phi$ has property $(U\tilde{A}_2)$. Moreover, $l^0_\Phi$ is then $B$-convex (see [1]), so $l^0_\Phi$ contains no copy of $\ell_1$. Since a Banach space $X$ has $(NUS^*)$ if and only if has property $(U\tilde{A}_2)$ and contains no copy of $\ell_1$ (see [15]), condition (3) implies condition (1).

Again by our Theorem 4 and the result from [15] that we just mentioned, we have that (1) $\Rightarrow$ (2), because condition (1) implies reflexivity of $l^0_\Phi$ and we therefore also have (2) $\Rightarrow$ (3).

The following theorem can be proved in a similar way as for $X = l^0_\Phi$, so we omit its proof.

**Theorem 5.** For any Orlicz function $\Phi$ and $X = l_\Phi$ the following statements are equivalent:

(1) $X$ has property $(U\tilde{A}_2)$;

(2) $X$ has property $(W\tilde{A}_2)$;
(3) \( R(X) < 2 \);
(4) \( \Phi \in \delta_2 \) and \( \Phi \in \overline{\delta}_2 \).

**Corollary 4.** Let \( \Phi \) and \( X \) be as in Theorem 5. The following statements are equivalent:

(1) \( X \) is nearly uniformly smooth;
(2) \( X \) is \((NUS^*)\);
(3) \( \Phi \in \delta_2 \) and \( \Phi \in \overline{\delta}_2 \).

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