On Methods of Summability Based on Power Series

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Suppose $\chi$ is a real function of bounded variation in the interval $[0,1]$. The main result to be established is:

**Theorem A**

If

$$
\int_0^1 f(x) \chi(x) \, dx > 0 \quad \text{and} \quad \nu > 0
$$

then $\nu \neq 0$.

In the next section we prove three subsidiary theorems which have some bearing on Theorem A. Theorem 1 states necessary and sufficient conditions for $P$ to be regular; it is not new but its proof has been included for the sake of completeness. Theorems 2 and 3 show that (6) is a consequence of (3) when $\mu = \infty$. Theorem 4 states a necessary and sufficient condition for $p_0$ and $p_1$ to be equal when $\infty > p_0 > 0$ and (3) holds with a monotone $\chi$.

In § 3.4 Theorem A is linked with the theory of moment sequences; and in the final section some examples are given.

2. Subsidiary Theorems

**Theorem**

1. (i) If $\nu > p > p_0 > 0$, then a necessary and sufficient condition for $P$ to be regular is that $\sum_0^{\infty} p_0^n = \infty$.

(ii) If $p = \infty$, then $P$ is regular.

Proof of (i). Note first that, since $p_0 > 0$,

$$
\sum_0^{\infty} p_0^n \leq \lim_{x \to \infty} p(x) \sum p_0^n \quad \text{for each } x > 0,
$$

and hence that

$$
\lim_{x \to \infty} p(x) \sum_0^{\infty} p_0^n.
$$

Sufficiency. Suppose that $x \to \infty$ and let $m$ be any positive integer. Since, by hypothesis, $p(x) \to \infty$ as $x \to \infty$, we have

$$
\lim_{x \to \infty} \left| p(x) \right| \leq \lim_{x \to \infty} \left| \sum_0^{\infty} \frac{1}{p_0^n} \sum_0^{\infty} \frac{1}{p_0^n} \right| < \frac{1}{p_0^n} |a_n|, \quad x \to \infty,
$$

which tends to zero as $m \to \infty$. Hence $p(x) \to 0$ as $x \to \infty$, and an immediate consequence is that $P$ is regular.
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Necessity.—Suppose that $P$ is regular and let $m$ be an integer such that $p_m > 0$. Define a sequence $(s_n)$ as follows:

$$s_n = (n = m), \quad s_n = 1/p_n.$$

Then $s_n \to 0$ and therefore $p_n(s_n) = 1/p_n \to 0$ as $n \to \infty$. Hence

$$\sum_{n=1}^{\infty} p_n s_n = \lim_{n \to \infty} p_n s_n = 0.$$

Proof of (ii).—Suppose that $s_n \to 0$. Let $m$ be any positive integer and let $k$ be the first integer greater than $m$ such that $p_k > 0$. Then

$$\lim_{n \to \infty} |p_n(s_n)| = \lim_{n \to \infty} \sum_{k=m}^{n} p_k \left| s_k \right| \leq \sum_{k=m}^{n} p_k \left| s_k \right| = \sum_{k=m}^{k} p_k \left| s_k \right|,$$

and the final expression tends to zero as $m \to \infty$. Hence $P$ is regular.

**Theorem 2.**—If $p_n > A > 0, \quad (n > N),$

where

$$\omega > p_n \left| s_n \right| \left| d \phi(t) \right| > 0 \quad (n > 0),$$

and if $p_n = \infty$, then $p_n = \infty$.

Proof.—Since $p_n > A > 0$, $\int_0^1 \left| d \phi(t) \right|$ cannot be zero for all $n$ in $(0, 1)$. Further

$$p_n > \frac{\int_0^1 \left| d \phi(t) \right|}{(1 > n > A > N)}.$$

Since $p_n = \infty$, it follows that $p_n = \infty$.

**Theorem 3.**—Suppose that $p_n = p_n > 1 > n > N$, where $\omega$ is non-decreasing and bounded in $(0, 1)$, and that $\omega > p_n > 0$. Then in order that $p_n = p_n$ it is necessary and sufficient that $\chi(t) > \chi(t)$ whenever $t > 0$.

Proof. Sufficiency.—Let $n$ be any number in the open interval $(0, 1)$. Then

$$\int_0^1 \left| d \phi(t) \right| > 0 \quad \text{and} \quad \int_0^1 \left| d \phi(t) \right| > p_n \int_0^1 \left| d \phi(t) \right| \quad (n > N).$$

Consequently $p_n > p_n > u_n > p_n$, and it follows that $p_n = p_n$.

Necessity.—Suppose that $\omega$ is such that $1 > n > 0$ and $\chi(t) > \chi(t)$.

Then

$$\omega > \int_0^1 \left| d \phi(t) \right| < \int_0^1 \left| d \phi(t) \right| \quad (n > N).$$

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Since $\chi(t) > \chi(t)$ and $p_n > p_n$, it follows that $p_n > u_n > u_n$, which is only possible if $u_n = 0$, since $0 < p_n < \infty$. Hence the required result.

We shall use the following easily verified lemma in the proof of Theorem A.

**Lemma.**—If $P$ is regular and $p_n = p_n$, for $n > N$, then $P \geq Q$.

3. **Proof of Theorem A**

Suppose that condition (a) holds with $N = 0$. In view of the lemma this involves no loss in generality.

Let $p = p_n = p_n$ and let $(s_n)$ be any sequence such that $\sum s_n(w_n)$ is convergent whenever $|w| < p$. Suppose that $0 < n < p$. The equality in (a) yields

$$\sum_{n=0}^{\infty} |s_n w_n| = \sum_{n=0}^{\infty} |s_n w_n| \int_0^1 \left| d \phi(t) \right| \sum_{n=0}^{\infty} |s_n w_n| \left| d \phi(t) \right|,$$

the inversion being legitimate since

$$\int_0^1 \left| d \phi(t) \right| \sum_{n=0}^{\infty} |s_n w_n| \left| d \phi(t) \right| = \sum_{n=0}^{\infty} |s_n w_n| \int_0^1 \left| d \phi(t) \right| \sum_{n=0}^{\infty} |s_n w_n| \left| d \phi(t) \right| < \infty.$$

Hence

$$\frac{p_n}{p_n} = \frac{\int_0^1 \left| d \phi(t) \right|}{\int_0^1 \left| d \phi(t) \right|}.$$

Similarly, using the inequality in (a), we obtain

$$\frac{p_n}{p_n} = \frac{\int_0^1 \left| d \phi(t) \right|}{\int_0^1 \left| d \phi(t) \right|}.$$

Further, in view of hypothesis (b) and Theorem 1,

$$\frac{p_n}{p_n} = \infty \quad \text{as} \quad n \to \infty.$$

It follows from (2), (3) and (4) that, for $0 < n < p$,

$$\lim_{n \to \infty} \frac{p_n}{p_n} = \frac{\int_0^1 \left| d \phi(t) \right|}{\int_0^1 \left| d \phi(t) \right|}.$$

Consequently $s_n \to 0$ (P) whenever $s_n \to 0$ (Q); and hence $P \geq Q$. 

* Monthly, 1951, No. 78-79.
4. Moment Sequences

Given any function $\phi$ of bounded variation in $[0, 1]$ we define an associated normalized function $\Phi$ as follows:

$$
\Phi(t) = \left\{ \begin{array}{ll}
0 & (t = 0) \\
\frac{1}{t} \{ \phi(t) - \phi(t - \epsilon) \} & (0 < t < 1) \\
\phi(1) - \phi(0) & (t = 1)
\end{array} \right.
$$

Let $\alpha$ be a real function of bounded variation in $[0, 1]$, then (Widder 1946, Theorem 8a)

$$
\int_0^\alpha \phi(t) \, dt = \int_0^\alpha \Phi(t) \, dt \quad (\alpha > 0),
$$

and (Widder 1946, Theorem 88 and Holson 1927, § 247)

$$
\int_0^\alpha |\phi(t)| > \int_0^\alpha |\Phi(t)| \quad (\alpha > 0).
$$

Further, it is known (Hardy 1949, Theorem 203) that if $\beta$ is a function of bounded variation in $[0, 1]$ such that

$$
\int_0^\alpha \phi(t) \, dt = \int_0^\alpha \beta(t) \, dt \quad (\alpha > 0, \ 1, \ldots, 3),
$$

then $\alpha(t) = \beta(t)$ for $0 < t < 1$.

A sequence $(\alpha_n)$ is said to be an $m$-sequence (moment sequence) if

$$
\alpha_n = \int_0^1 \phi(t) \, dt \quad (\alpha > 0)
$$

where $\phi$ is a real function of bounded variation in $[0, 1]$; if, in addition,

$$
\alpha_n > \int_0^\alpha \phi(t) \, dt \quad (1 > \alpha > 0, \ n = N, N + 1, \ldots),
$$

we shall call $(\alpha_n)$ an $m$-sequence. In view of the introductory remarks in this section the definition of $m$-sequences is unambiguous.

We can now re-rewrite Theorem $A$ as follows:

**Theorem $A'$:** If $\alpha_n = \phi_n \phi(x) \quad (x > N)$, where $(\alpha_n)$ is an $m$-sequence, and if $\phi_n \phi_n > 0$ and $P$ is regular, then $P \geq Q$.

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We conclude this section with some useful results concerning $m$-sequences.

A sequence $(\alpha_n)$ is said to be totally monotone if

$$
\delta_n = \sum_{i=0}^{\infty} (-1)^{i+1} \alpha_{n+i} > 0 \quad (n = 0, 1, \ldots; \ \delta_0 = 1, \ldots, \).
$$

It is known (Hardy 1949, § 11.8) that a necessary and sufficient condition for $(\alpha_n)$ to be totally monotone is that

$$
\delta_n = \int_{\alpha_{n+1}}^{\alpha_{n+1}} \phi(t) \quad (\alpha > 0),
$$

where $\phi$ is non-decreasing and bounded in $[0, 1]$.

Hence $(\alpha_n)$ is an $m$-sequence if and only if $\mu_n = \delta_n - \lambda_n$ where $(\lambda_n)$ and $(\delta_n)$ are totally monotone.

The following propositions are easily verified:

1. If $(\mu_n)$ and $(\lambda_n)$ are $m$-sequences, then so also are $(\mu_n + \lambda_n)$ and $(\mu_n - \lambda_n)$ (Cf. Hardy 1949, Theorem 210).

2. If $\mu_n = \lambda_n - \delta_n$ where $(\alpha_n)$ and $(\beta_n)$ are totally monotone, and if

$$
\alpha_n > \beta_n \quad (\beta > 0, \ n = N, N + 1, \ldots),
$$

then $(\mu_n)$ is an $m$-sequence.

3. Any $m$-sequence which converges to a positive limit is an $m$-sequence.

IV. If both $(\mu_n)$ and $(1/\mu_n)$ are positive $m$-sequences, then they are $m$-sequences.

Note that III is a consequence of II, and IV a consequence of III.

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5. Examples

Let

$$
\phi(x) = \left\{ \begin{array}{ll}
\frac{\alpha + \epsilon}{\pi} & (\alpha > -1) \\
(\alpha + 1)^{-1} & (\alpha = -1)
\end{array} \right.
$$

so that, for $|x| < 1$,

$$
\sum_{n=1}^{\infty} \frac{\phi(x-n)}{x} = \frac{1 - x^{n+1}}{1 - x^{n+1}} \quad (\alpha > -1)
$$

Denote the power series summability method associated with the sequence $(\phi_n)$ by $A_\alpha$. $A_\alpha$ is then the ordinary Abel method.
For \( a > -1 \), the radius of convergence of \( \sum p_n x^n \) is unity and, by
Theorem I (II), \( A_a \) is regular.

Let \( \mu > \lambda > -1 \). Then
\[
\frac{\lambda^2}{\mu^2} \int_0^1 \frac{(\mu + 1)}{(\mu - \lambda)^{1/2}(\lambda + 1)^{1/2}} \rho^2 (1 - \rho)^{\mu + \lambda - 1} d\rho,
\]
so that the sequence \( \{p_n/\mu^2\} \) is totally monotone. Further
\[
\frac{\lambda^2}{\mu^2} = \frac{1}{\mu + 1} \frac{\mu + \mu + 1}{\mu + \mu + 1},
\]
since \( (1/p_n^2) \) is totally monotone and \( \{n(n+1)/(n+1)\} \) is an \( n \)-sequence
(Hardy 1949, 264) which converges to unity, it follows, in view of
propositions I and III, that \( \{p_n/\mu^2\} \) is an \( n \)-sequence.

Hence, by Theorem \( A' \),
\[
A_\lambda \supsetneq A_\mu \quad (\mu > \lambda > -1).
\]

Denote by \( A'_\lambda \) the power series method associated with the sequence
\( \{n+1/(n+1)\} \). It is known that, for \( a > -1 \), both \( \{p_n/(n+1)^2\} \) and \( \{n(n+1)/(n+1)\} \)
are \( n \)-sequences (Hardy 1949, Theorem 211). Hence, by proposition
IV and Theorem \( A' \),
\[
A'_\lambda \supsetneq A_\mu \quad (a > -1).
\]

Let
\[
p_n = \frac{1}{(a+1)(a+2) \ldots (a+n)} \quad (a > -1, \ n = 1, 2, \ldots),
\]
and denote by \( B \) the method associated with the sequence \( \{p_n\} \). \( B \) is then the
covex exponential method.

The series \( \sum q_n x^n \) is convergent for all \( x \) and hence, by Theorem I (II),
\( B_a \) is regular for \( a > -1 \). Since
\[
\frac{p_n}{q_n} = \frac{\lambda^2}{\mu^2} \quad (\lambda > -1, \ mu > -1),
\]
it follows that
\[
B_\lambda \supsetneq B_\mu \quad (\mu > \lambda > -1).
\]

Finally, denote by \( B'_\lambda \) the method associated with the sequence
\( \{1/(n+1)^{3/2}\} \), where \( a \) is any real number. As before we obtain
\[
B'_a \supsetneq B_a \quad (a > -1),
\]
from which it follows that \( B'_\lambda \supsetneq B_\mu \) when \( \mu > \lambda \) and \( \lambda > -1 \). However

* See Borwein 1957, where the result \( A_\lambda \supsetneq A_\mu \quad (a > \lambda > -1) \) is proved.

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The restriction \( \lambda > -1 \) is unnecessary in this case. Since \( B_\mu \) is regular
for all \( a \) and the sequence \( \{(n+1)^{-3/2}\} \) is totally monotone whenever
\( \mu > -1 \) (Hardy 1949, 265), we have, by Theorem \( A' \),
\[
B'_\lambda \supsetneq B_\mu \quad (\mu > \lambda).
\]

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**References to Literature**


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