Generalized Hausdorff and Weighted Mean Matrices as Operators on $l_p$

DAVID BORWEIN AND XIAOPENG GAO*

Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

Submitted by Bruce Berndt

Received April 27, 1992

Two theorems are proved. Theorem 1 establishes sufficient conditions for a generalized Hausdorff matrix $H(a, b)$ either to be in $B(l_p)$ or not to be in $B(l_p)$. Theorem 2 shows, inter alia, that if $1 < p < \infty$, $a_i > 0$, $A_n := a_0 + a_1 + \cdots + a_n$, and $A_n/n = \gamma > 0$, then the weighted mean matrix $M_n$ with weights $a_i$ is in $B(l_p)$ if and only if $\gamma < p$. There are two examples about cases when the conditions of the theorems are not satisfied. A short proof of the fact that weighted mean matrices are special generalized Hausdorff matrices is also given.

1. INTRODUCTION

Suppose throughout that $1 < p < \infty$, and that $A := (a_{nk})$ is a triangular matrix of complex numbers, that is $a_{nk} = 0$ for $n > k$. Let $l_p$ be the Banach space of all complex sequences $x = \{x_n\}$ with norm

$$
\|x\|_p := \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} < \infty,
$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on $l_p$. Thus $A \in B(l_p)$ if and only if $Ax \in l_p$ whenever $x \in l_p$, $Ax$ being the sequence with $n$th term $(Ax)_n := \sum_{k=0}^{\infty} a_{nk} x_k$. Let

$$
\|A\|_p := \sup_{1 \leq x \leq 1} \|Ax\|_p,
$$

so that $A \in B(l_p)$ if and only if $\|A\|_p < \infty$, in which case $\|A\|_p$ is the norm of $A$.

* This research was supported in part by the Natural Sciences and Engineering Council of Canada.

517

0022-247X/93 $5.00

Copyright © 1993 by Academic Press, Inc. 
All rights of reproduction in any form reserved.
Generalized Hausdorff Matrices

Suppose that \( \lambda = (\lambda_n) \) is a sequence of real numbers with \( \lambda_0 > 0, \lambda_n > 0 \) for \( n \geq 1 \), and that \( x \) is a function of bounded variation on \([0, 1]\). For \( 0 \leq k \leq n \), let

\[
\lambda_{ak}(t) := -\lambda_{a+1} \cdots \lambda_{a+k+1} \int_0^t \frac{dx}{(\lambda_{a+k} - x) \cdots (\lambda_{a+k-1} - x)} \quad 0 \leq t \leq 1,
\]

\[
\lambda_{ak}(0) := \lambda_{ak}(0+),
\]

\( C = C_{\lambda} \) being a positively sensed Jordan contour enclosing \( \lambda_0, \lambda_{a+1}, \ldots, \lambda_n \).

Here and elsewhere we observe the convention that empty products, like \( \lambda_{a+1} \cdots \lambda_n \) when \( k = n \), have the value 1. Let

\[
\lambda_n := \int_0^1 \lambda_{ak}(t) \, dt \quad 0 \leq k \leq n,
\]

\( \lambda_n := 0 \) for \( k > n \),

and denote the triangular matrix \( (\lambda_{ak}) \) by \( H(\lambda, a) \). This is called a generalized Hausdorff matrix.

Weighted Mean Matrices

Let \( a = (a_n) \) be a sequence of positive numbers and let \( A_n := \sum_{k=0}^{n} a_k \).

The weighted mean matrix \( \mathcal{M}_a := (a_{ak}) \) is defined by setting

\[
a_{ak} := \frac{a_k}{a_a} \quad 0 \leq k \leq n, \quad a_{ak} := 0 \quad 0 \leq k > n.
\]

Let

\[
D_0 := (1 + \lambda_0) \, d_0 = 1,
\]

\[
D_n := (1 + \lambda_n) \cdots \frac{1}{1 + \lambda_0} = (1 + \lambda_0) \, d_n \quad \text{for } n \geq 1.
\]

Then

\[
\lambda_n := \frac{D_n}{d_0} = \frac{\sum_{k=0}^{n} d_k}{d_0} \quad \text{for } n \geq 0.
\]

Also, it was proved in [5] that when all the \( \lambda_n \)'s are distinct

\[
\int_0^1 \lambda_{ak}(t) \, dt = \frac{d_k}{D_n} \quad 0 \leq k \leq n.
\]

Although a continuity argument shows that this is also true for more general \( \lambda_n \), Lemma 1 (below) affords a shorter and more direct proof. When \( a(t) := t \) and \( \lambda_0 := 0 \), \( H(\lambda, a) \) reduces to the weighted mean matrix \( M_a \) with \( d := (d_n) \) given by (2). Conversely if \( d := (d_n) \) is a sequence of positive numbers with \( d_0 := 1 \), then (2) yields a sequence \( \lambda := (\lambda_n) \) such that \( H(\lambda, a) \) becomes \( M_a \) when \( a(t) := t \).

The following theorem is due to Cass and Kratz [4].

**Theorem A.** Suppose that \( a_t = f(t) \) where \( f(x) \) is a logarithmico-exponential function for \( x > x_0 \), and that \( A_{n+a} / a_n \to e \). Suppose also that \( p < 1 \) and \( (1/p) + (1/q) = 1 \). Then \( M_a \in B((L_p \cap L_q)) \) if and only if \( e < p \), in which case

\[
\frac{p}{p-e} \leq \|M_a\|_p \leq e^{\gamma (1/p)} < \infty,
\]

where

\[
\gamma := \sup_{x > x_0} \left( \frac{a_x}{a_{x+1}} \right)^{1/2}, \quad \sigma := \sup_{x > x_0} \left( \frac{a_x}{a_{x+1}} \right)^{1/2}
\]

Cass and Kratz showed that \( A_{n+a} / a_n \) necessarily tends to a finite or infinite limit when \( a_n \) is generated by a logarithmico-exponential function. Borwein [1] proved the following theorem.

**Theorem B.** If \( p \geq 1, e > 0 \) and

\[
\mu := \sup_{0 \leq t < 1} \left( \frac{\lambda_{a+1} \cdots \lambda_n}{(1 + \lambda_0) \cdots (1 + \lambda_{n-1})} \right) < \infty,
\]

and if \( \int_0^1 t^{-\alpha} \, |dx(t)| < \infty \), then \( H(\lambda, a) \in B(L_p) \), and

\[
\|H(\lambda, a)\|_p \mu t^{-\alpha} \int_0^1 t^{-\alpha} |dx(t)|.
\]

**Note.** Although the proof of Theorem 1 in [1] in fact establishes Theorem B, the statement of Theorem 1 has the simpler condition

\[
\lambda_a \leq e + \tilde{\lambda}_a \quad \text{for } n \geq n_0
\]

in place of (4). Evidently (5) implies (4), but as illustrated in Example 1 (below) it is possible for (4) to hold for some \( e \) and (5) to fail to hold for any \( e \).

One of the objectives of this paper is to show that, subject to the existence of \( e := \lim A_{n+a} / a_n \), the condition in Theorem A that \( a_n \) be generated
by a logarithmic-exponential function is redundant when \( c > 0 \), and can be replaced by the far less restrictive condition that \( \{a_n\} \) be eventually monotonic when \( c = 0 \). This objective is achieved by means of Theorem 2 (below) which is largely a specialization of our main result, Theorem 1 (below). In view of (2) and (3), the existence of \( \lim a_n/a_n \) in Theorem 2 corresponds to the existence of \( \lim a_n/a_n \) in Theorem 1.

We shall prove the following two theorems:

**Theorem 1.** Suppose \( p \geq 1 \). Let \( c_1 := \lim \inf \lambda_n/n \) and \( c_2 := \lim \sup \lambda_n/n \).

(i) If \( \infty > c_1 > 0 \), \( \sum_{n=1}^{\infty} 1/\lambda_n = \infty \), and \( \alpha \) is a non-decreasing function on \( [0,1] \) such that \( \alpha(0+) = \alpha(0) \), then

\[
\left\{ H(\lambda, \alpha) \right\}_p = p^{\frac{1}{p} - 1/p} \int_0^1 \alpha(t) \frac{dt}{t}.
\]

In particular, if \( \int_0^1 \alpha(t) \frac{dt}{t} = \infty \), then \( H(\lambda, \alpha) \notin B(1) \).

(ii) If \( \lim \lambda_n/n = \infty \) (i.e., \( c_1 = \infty \)), \( \sum_{n=1}^{\infty} 1/\lambda_n = \infty \), and \( \alpha \) is a non-decreasing function on \( [0,1] \) such that \( \alpha(0+) = \alpha(0) < \infty \) for some \( r \in (0,1) \), then \( H(\lambda, \alpha) \notin B(1) \).

(iii) If \( \sum_{n=1}^{\infty} |\alpha(t)| < \infty \) for some \( c > c_1 \), and \( c_1 > 0 \), then \( H(\lambda, \alpha) \in B(1) \), and

\[
\left\{ H(\lambda, \alpha) \right\}_p \leq p^{1/p} \int_0^1 \alpha(t) \frac{dt}{t} < \infty
\]

where \( \mu \) is given by (4).

(iv) If \( \lim \lambda_n/n = 0 \) and \( \lim (\lambda_{n+1}/\lambda_n) \) exists, and if \( \int_0^1 |\alpha(t)| \frac{dt}{t} < \infty \) for some \( c > 0 \), then \( H(\lambda, \alpha) \in B(1) \) for all \( p \geq 1 \).

(v) If the sequence \( \{\alpha_n\} \) given by (2) is eventually non-decreasing, and \( \sum_{n=1}^{\infty} |\alpha(t)| < \infty \) for some \( p \geq 1 \), then \( H(\lambda, \alpha) \in B(1) \).

**Theorem 2.** Suppose \( p \geq 1 \). Let \( c_1 := \lim \inf a_n/a_n \) and \( c_2 := \lim \sup a_n/a_n \).

(i) If either \( \sum a_n \) is convergent, or \( \infty > c_1 \geq p \), then \( M \in B(1) \).

(ii) If \( 0 < c_1 < c_2 < p \), then \( M \in B(1) \) and

\[
\frac{p}{p-c_1} \leq \left\| M \right\|_p \leq \frac{p}{p-c_2} < \infty
\]

where \( \mu \) is given by (4) with \( \lambda_{p+1} := a_n/a_n - 1 \) and any \( c \in (c_1, p) \). Furthermore, if \( 0 < \lim a_n/a_n = c < p \) and

\[
\frac{A_n}{a_{n+1}} \leq c + \frac{A_n}{a_n}
\]

for \( n \geq 0 \), then \( \left\{ M_n \right\}_p = p(p-c) \).

(iii) If \( \lim a_n/a_n = 0 \), and

\[
\frac{A_n}{a_{n+1}} \leq \frac{A_n}{a_n}
\]

tends to a limit, then \( M \in B(1) \) for all \( p \geq 1 \).

(iv) If \( \lim a_n/a_n = 0 \) and \( \{a_n\} \) is eventually monotonic, then \( M \in B(1) \) for all \( p \geq 1 \).

2. Preliminary Results

**Lemma 1.** If \( D_1 \) and \( d_n \) satisfy (2), then (3) holds.

**Proof.** Let \( \Gamma \) be a circle enclosing \( \lambda_1, ..., \lambda_n \) and lying in the half-plane Re \( z > -i \) where \( 0 < \delta < 1 \). For \( 0 < \delta < 1 \) and \( z \in \Gamma \), we have

\[
\left| \frac{\gamma}{(\lambda_1 - z) \cdots (\lambda_n - z)} \right| \leq M e^{-\delta}
\]

for some positive \( M \) independent of \( t \) and \( z \). Hence, by Fubini's theorem,

\[
\int_0^1 \lambda_n(t) dt = \frac{\lambda_n}{2\pi i} \int \frac{1}{(\lambda_1 - z) \cdots (\lambda_n - z)} dz
\]

Let

\[
f(z) := \frac{1}{(z+1)(\lambda_1 - z) \cdots (\lambda_n - z)}
\]

Then \( \int_{|z|=\delta} f(z) dz \to 0 \) as \( \delta \to \infty \), and so

\[
\frac{1}{2\pi i} \int f(z) dz = -\text{Res}(f(z), -1) = \frac{1}{(\lambda_1 + 1) \cdots (\lambda_n + 1)}
\]
Consequently
\[ \int_0^1 \lambda_n(t) \, dt = \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k+1) \cdots (\lambda_n+1)} \cdot \frac{d_k}{D_n}. \]

The following lemma is included here for convenience. Its proof is given in [2].

**Lemma 2.** Under the hypotheses in the definition of a generalized Hausdorff matrix, we always have

\[ 0 \leq \lambda_n(t) \leq \sum_{k=0}^n \lambda_n(t) \leq 1 \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 \leq j \leq n. \]

If, in addition, \( \sum_{k=0}^n \lambda_n(t) = \infty \), then

\[ \lim_{n \to \infty} \sum_{k=0}^n \lambda_n(t) = \begin{cases} 0 & \text{if } t = 0 \text{ and } \lambda_n > 0 \\ 1 & \text{otherwise.} \end{cases} \]

The next lemma is essentially Theorem 4 of [3] with superficial changes to make it a little more general and also easier to apply. We supply a proof here for completeness.

**Lemma 3.** Suppose that \( a_{k,n} \geq 0 \) for \( 0 \leq k \leq n, a_{k,n} = 0 \) for \( k > n \), and that \( \{b_n\} \) is a bounded sequence of positive numbers such that \( \sum b_n = \infty \). Let

\[ \sigma_n := \sum_{k=0}^n \frac{a_{k,n}}{b_n} \frac{1}{\lambda_k}. \]

If \( \sigma := \lim \inf \sigma_n \) and \( A := (a_{k,n}) \), then \( \|A\| \geq \sigma \). In particular, if \( \sigma = \infty \), then \( A \in BV(\lambda) \).

Proof. Suppose without loss in generality that \( \sigma > 0 \), and let \( 0 < \mu < \lambda < \sigma \). Let

\[ T_n := \left( \prod_{k=0}^n \left( 1 - \frac{b_k}{b} \right) \right)^{-1} \quad \text{where} \quad b > \sup b_k. \]

Then \( T_n := T_{n-1} - T_n b_n / b \) for \( n \geq 1 \), and \( T_n = T_0 + t_1 + \cdots + t_n \to \infty \).

Let

\[ y_n := \sum_{k=0}^n a_{k,n} x_k \quad \text{where} \quad x_k := \left( \frac{b_k}{T_k} \right)^{1/\lambda_k}, \quad n > 0. \]

By Dini's theorem, \( \{x_k\} \in L_1 \). Further, there is a positive integer \( N \) independent of \( c \) such that for \( n \geq N \)

\[ y_n = x_n \sum_{k=0}^n a_{k,n} \left( \frac{b_k}{T_k} \right)^{1/\lambda_k} = \frac{x_n}{T_n} \sum_{k=0}^n a_{k,n} \left( \frac{b_k}{T_k} \right)^{1/\lambda_k} \geq \lambda x_n. \]

Now choose \( e > 0 \) so small that

\[ \sum_{n=N}^\infty x_n^e \leq \frac{1}{e} \sum_{n=N}^\infty x_n^e. \]

Then

\[ \sum_{n=N}^\infty x_n^e \leq \frac{1}{e} \sum_{n=N}^\infty x_n^e. \]

Therefore \( \|A\| \geq \sigma \) and, since \( \mu \) is an arbitrary number in the interval \( (0, \sigma) \), it follows that \( \|A\| \geq \sigma \).

3. Proofs of the Main Results

To prove part (ii) of Theorem 1 we need the following lemma.

**Lemma 4.** If \( \lim \inf \lambda_n / a > 0 \), and \( c := \lim \sup \lambda_n / n < c < \infty \), then

\[ \mu := \sup_{a+k=c} \left( \lambda_{k+1} \cdots \lambda_k \right)^{1/\lambda_k} < \infty. \]

Proof. Let \( n_0 \) be a positive integer such that \( \lambda_n \leq n \) when \( n \geq n_0 \). For \( n \geq k > n_0 \) we have

\[ \frac{\lambda_{k+1} + \cdots + \lambda_k}{\lambda_k} \leq \frac{\lambda_{k+1} + \cdots + \lambda_n}{\lambda_n} \leq \frac{1}{n-1} \left( 1 + \frac{1}{n-1} \right) \leq \frac{n}{k}. \]

so that

\[ \frac{\lambda_{k+1} + \cdots + \lambda_k}{\lambda_k} \leq \frac{\lambda_{k+1} + \cdots + \lambda_n}{\lambda_n} \leq \frac{1}{n-1} \left( 1 + \frac{1}{n-1} \right) = \frac{n}{k}. \]

Therefore

\[ M_{n-1} < \infty. \]

By Dini's theorem, \( \{x_k\} \in L_1 \). Further, there is a positive integer \( N \) independent of \( c \) such that for \( n \geq N \)

\[ y_n = x_n \sum_{k=0}^n a_{k,n} \left( \frac{b_k}{T_k} \right)^{1/\lambda_k} = \frac{x_n}{T_n} \sum_{k=0}^n a_{k,n} \left( \frac{b_k}{T_k} \right)^{1/\lambda_k} \geq \lambda x_n. \]

Now choose \( e > 0 \) so small that

\[ \sum_{n=N}^\infty x_n^e \leq \frac{1}{e} \sum_{n=N}^\infty x_n^e. \]

Then

\[ \sum_{n=N}^\infty x_n^e \leq \frac{1}{e} \sum_{n=N}^\infty x_n^e. \]

Therefore \( \|A\| \geq \sigma \) and, since \( \mu \) is an arbitrary number in the interval \( (0, \sigma) \), it follows that \( \|A\| \geq \sigma \).

3. Proofs of the Main Results

To prove part (ii) of Theorem 1 we need the following lemma.

**Lemma 4.** If \( \lim \inf \lambda_n / a > 0 \), and \( c := \lim \sup \lambda_n / n < c < \infty \), then

\[ \mu := \sup_{a+k=c} \left( \lambda_{k+1} \cdots \lambda_k \right)^{1/\lambda_k} < \infty. \]

Proof. Let \( n_0 \) be a positive integer such that \( \lambda_n \leq n \) when \( n \geq n_0 \). For \( n \geq k > n_0 \) we have

\[ \frac{\lambda_{k+1} + \cdots + \lambda_k}{\lambda_k} \leq \frac{\lambda_{k+1} + \cdots + \lambda_n}{\lambda_n} \leq \frac{1}{n-1} \left( 1 + \frac{1}{n-1} \right) \leq \frac{n}{k}. \]

so that

\[ \frac{\lambda_{k+1} + \cdots + \lambda_k}{\lambda_k} \leq \frac{\lambda_{k+1} + \cdots + \lambda_n}{\lambda_n} \leq \frac{1}{n-1} \left( 1 + \frac{1}{n-1} \right) = \frac{n}{k}. \]

Therefore

\[ M_{n-1} < \infty. \]
Let
\[ M_2 := \sup_{0 < k < n} \frac{\lambda_{k+1} \cdots \lambda_n}{\lambda_k + c} \]

Then for \(0 < k < n\) we have
\[ \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} \leq M_2 \]

and so \(M_3 := \max(M_1, M_2, M_3) < \infty\).

Proof of Theorem 1. (i) Let \(0 < w < c_1/p\), let
\[ b_n = \left(\frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + w) \cdots (\lambda_n + w)}\right)^w, \quad \lambda_n := \lambda_n + w, \]
and define \(T_n(t)\) by (1) with \(\lambda_n\) in place of \(\lambda_n\). Since \(w < c_1\), there is a positive integer \(n_0\) such that \(\lambda_n \geq n_0p\) for all \(n \geq n_0\). Hence, for \(n \geq n_0\),
\[ \frac{b_n}{b_{n-1}} \geq \left(1 + \frac{w}{\lambda_n}\right)^w \geq 1 - \frac{nw}{\lambda_n} \geq 1 - \frac{n-1}{n}, \]

and so \(b_n \geq b_{n_0}/n\). It follows that \(\sum b_n = \infty\). Further, for \(0 < k < n\), \(0 < t \leq 1\),
\[ \lambda_{k+1}^{(t)} / \lambda_k^{(t)} = -\left(\lambda_{k+1} + w\right) \cdots (\lambda_n + w) \left(\frac{1}{\lambda_{k+1} - z} \cdots \frac{1}{\lambda_n - z}\right) \]
\[ = -\lambda_{n-k} \left(\frac{1}{\lambda_{k+1} - z} \cdots \frac{1}{\lambda_n - z}\right) \int_0^t \frac{r^* \, dr}{(\lambda_n - z) \cdots (\lambda_k - z)} \]

By Lemma 2 and Fatou's theorem,
\[ \liminf_{n \to \infty} \sum_{k=0}^n \lambda_{k+1}^{(t)} / b_k^{(t)} = \liminf_{n \to \infty} \int_0^t T_n(t) r^* \, dt(t) \geq \int_0^t r^* \, dt(t), \]

and hence, by Lemma 3,
\[ \|H(\lambda, g)\|_{L^p} \geq \int_0^t r^* \, dt(t). \]
(iv) First we prove that the hypothesis \( \lim a_n/n a_n = 0 \) implies that 
\[
\lim a_n = \infty. \tag{5}
\]
(In fact the hypothesis implies that \( n^{-a_n} \to \infty \) for every real constant \( c \).) Let \( a_2 := \lim a_n/a_n \). Then
\[
\log a_2 = \lim_{n \to \infty} \left( \frac{a_n}{n} \right) = \frac{a_2}{n},
\]
so that \( \log a_2 > \log a_2 + \sum_{n=2}^{\infty} a_n/k \). Since \( a_2 \to \infty \), it follows that \( \log a_n > (c+1) \log n \), and hence that, for any given \( c \) and sufficiently large \( n \), \( \log a_n > (c+1) \log n \) or \( A_n > n^{c+1} \). Therefore \( n^{-a_n} = n^{-a_n/A_n} = a_n \to \infty \) as \( n \to \infty \).

Hence, since \( \{a_n\} \) is eventually monotonic it must be eventually non-decreasing, and so \( M_e \in B_{\infty} \) for every \( p > 1 \) by Theorem 1(v).

4. EXAMPLES

In this section we deal with two examples. For the first we construct a generalized Hausdorff matrix \( H(\lambda, \alpha) \in B_{\infty} \), with \( \alpha \) increasing, for which \( \int_0^{\infty} t^{-c} \lambda(t) = \infty \) for every \( c > c_1 \), where \( p_0 > 1 \) and \( c_1 \) is as in Theorem 1(iii). This will show that the conclusion of Theorem 1(iii) can hold when its main condition is not satisfied. The example will also show that (4) can hold with \( c = 1 \) while (5) cannot fail to hold for any \( c \).

Example 1. Suppose that \( p_0 > 1 \). Let
\[
\lambda_n := n^{\alpha} \quad \text{for} \quad m^2 \leq n < (m+1)^2, \quad m = 0, 1, 2, \ldots,
\]
\[
\alpha(t) := t^{p_0 - 1} \quad \text{for} \quad t \in (0, 1).
\]
Observe that \( c_1 \geq \lambda_n/n \), and \( \int_0^{\infty} t^{-c} \lambda(t) = \infty \) for all \( c > c_1 \). Thus Theorem 1(iii) cannot be used to prove that \( H(\lambda, \alpha) \in B_{\infty} \). Instead we shall appeal to Theorem B. For \( m \geq 1 \), we have that
\[
\beta_n := \frac{\lambda_{m+1} \cdots \lambda_{m+1}}{(\lambda_{m+1} \cdots \lambda_{m+1})^{1/m}} = \frac{(m+1)^{m+1} (m+1)^{m+1}}{(m+1)^{m+1}}.
\]
Since
\[
\frac{(m+1)^{m+1}}{(m+1)^{m+1}} = \frac{1}{m+1} \geq 1 + \frac{1}{m+1},
\]
and hence
\[
\frac{(m+1)^{m+1} (m+1)}{m+1} > 1 + \frac{1}{m+1},
\]

it follows that \( \beta_n \leq 1 \) for \( m \geq 1 \). Also because
\[
\lim_{m \to \infty} \frac{\lambda_{m+1}}{\lambda_{m+1}} = 1,
\]
and \( \lambda_{m+1}/(\lambda_{m+1} + 1) < 1 \) when \( k \) is not a perfect square, we get
\[
\sup_{0 < \lambda_n < \lambda_m} \frac{\lambda_{m+1}}{\lambda_{m+1} + 1} < \infty.
\]
Since
\[
\int_0^{1} t^{-\lambda_n(n)} \lambda(n) = \int_0^{1} \frac{dt}{t^{\log n}} < \infty,
\]

it follows that \( H(\lambda, \alpha) \in B_{\infty} \) by Theorem B with \( c = 1 \). In fact this theorem shows that \( H(\lambda, \alpha) \in B_{\infty} \) for all \( p > p_0 \). On the other hand, a simple consequence of Theorem 1(i) is that \( H(\lambda, \alpha) \in B_{\infty} \) for \( 1 < p < p_0 \).

Finally, we see that (4) holds with \( c = 1 \), but (5) cannot hold for any \( c \) since \( \lim \sup (\lambda_{m+1} - 1) = \infty \).

The second example will show us that it is possible for \( M_e \notin B(\lambda) \) when \( A_n/n a_n \to 0 \), although Theorem 2 tells us that \( M_e \in B(\lambda) \) when \( A_n/n a_n \to c \in (0, p) \). Thus the condition \( A_n/n a_n \to 0 \) needs to be augmented, as in part (iii) and (iv) of Theorem 2, in order to ensure that \( M_e \in B_{\infty} \).

Correspondingly the condition \( \lambda_n/n \to 0 \) needs to be augmented, as in part (iv) of Theorem 1, in order to ensure that \( H(\lambda, \alpha) \in B_{\infty} \).

Example 2. Define the weighted mean matrix \( M_n \) with \( a_n := (\alpha_n) \) as follows:
\[
a_n := \begin{cases}
\frac{1}{2m} & \text{for} \quad n^2 \leq m < (m+1)^2, \\
\frac{1}{2m} & \text{for} \quad m^2 \leq m < (m+1)^2, \\
\frac{1}{2m} & \text{for} \quad m \geq 1.
\end{cases}
\]
Then
\[
a_n := \sum_{k=1}^{m+1} k^{2m+2} = (m+1)^{2m+1} + 9.
\]
Hence, for \( m^2 \leq n < (m + 1)^2 \),
\[
\frac{na_n}{A_n} \geq \frac{m^2 2^m}{(m - 1) 2^{m+3} + 9} \to \infty,
\]
and
\[
\frac{a_m^2}{A_n} \geq \delta_m := \frac{m 2^{m+1}}{(m - 1) 2^{m+3} + 9} \to \frac{1}{4}.
\]
Now let \( p > 1 \), and define \( x := \{ x_k \} \in l_p \), by setting
\[
x_k := \begin{cases} \frac{1}{m^{1/p} \log m} & \text{if } k = m^2, \ m = 2, 3, \ldots \\ 0 & \text{otherwise}, \end{cases}
\]
and let
\[
y_n := \frac{1}{A_n} \sum_{k=0}^{n} a_k x_k.
\]
Then, for \( 4 \leq m^2 \leq n < (m + 1)^2 \),
\[
y_n \geq \frac{a_m^2}{A_n} x_m \geq \frac{\delta_m}{m^{1/p} \log m}.
\]
Thus
\[
\sum_{n = m^2}^{(m+1)^2 - 1} y_n \geq \frac{2m \delta_m^p}{m \log^p m}, \quad \text{and so} \quad \sum_{n=0}^{\infty} y_n^p \geq \sum_{m=2}^{\infty} \frac{2 \delta_m^p}{\log^p m} = \infty.
\]
Consequently \( M_o \notin B(l_p) \), even though \( \lim A_n/na_n = 0 \).

References


Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium