MATRIX OPERATORS ON $l_p$ TO $l_q$

DAVID BORWEIN AND XIAOPENG GAO

ABSTRACT. Workable necessary and sufficient conditions for a non-negative matrix

REPRINTED FROM / RÉIMPRIMÉ DE

Canadian Mathematical Bulletin

canadien de mathématiques
MATRIX OPERATORS ON $l_p$ TO $l_q$

DAVID BORWEIN AND XIAOFENG GAO

ABSTRACT. Workable necessary and sufficient conditions for a non-negative matrix to be a bounded operator from $l_p$ to $l_q$ when $1 < q < p < \infty$ are discussed. Alternative proofs are given for some known results, thereby filling a gap in the proof of the case $p = q$ of a result of Kostka's. The case $1 < q < p < \infty$ of Kostka's result is refined, and a weakened form of the von Neumann conjecture concerning matrix operators on $l_p$ is shown to be false.

1. Introduction. Suppose throughout that $1 \leq p, q < \infty$, and write

$$p' := \frac{p}{p-1} \quad \text{when } p > 1, \quad \text{and } q' := \frac{q}{q-1} \quad \text{when } q > 1.$$ 

Suppose also that, unless otherwise stated, the indices of all sequences and matrices run through $\mathbb{N} := \{1, 2, \ldots \}$. Let $l_p$ be the Banach space of all complex sequences $x := (x_j)$ with norm

$$\|x\|_p := \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < \infty.$$ 

Let $A := (a_{jk})$ be a real matrix. We say that $A \in (l_p, l_q)$ if for every $x := (x_j) \in l_p$, $y := \sum_{j=1}^{\infty} a_{jk} x_k$ is convergent for all $k \in \mathbb{N}$ and $Ax := (y_k) \in l_q$. We define

$$\|A\|_{p \to q} := \sup \{ \|Ax\|_q : \|x\|_p = 1 \}$$

so that $A \in (l_p, l_q)$ if and only if $\|A\|_{p \to q} < \infty$, in which case $\|A\|_{p \to q}$ is the $p,q$-norm of $A$. Following usual practice, we shall write $\|A\|_p$ for $\|A\|_{p,p}$. The matrix $A$ is said to be non-negative (or positive) if $a_{jk} \geq 0$ (or $a_{jk} > 0$) for all $j, k \in \mathbb{N}$, and likewise a sequence $u := (u_j)$ is said to be non-negative (or positive) if $u_j \geq 0$ (or $u_j > 0$) for all $j \in \mathbb{N}$. To avoid trivial cases we assume in all that follows that no matrix $A$ is identically zero.

The problem of obtaining workable necessary and sufficient conditions for $A \in (l_p, l_q)$ has been addressed by a number of authors. Ladyzhenskii [7] proved the first part of the following result, and the complete result is essentially the case $p = q > 1$ of Kostka's Theorem 1 in [8].

This research was supported in part by the National Sciences and Engineering Council of Canada. Received by the editors February 12, 1993.

AMS subject classification: 47B37, 47A30.

Key words and phrases: operators on $l_p$ to $l_q$, infinite matrices.


448
THEOREM A. Let $p > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_p)$ if and only if there exist a positive number $C$ and a positive sequence $u := (u_i)$ such that

$$
\sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} u_i \right)^{p_j} \leq C u_j^{-p_j}, \quad j = 1, 2, \ldots.
$$

and then $\|A\|_{p_j} \leq C^{1/p_j}$. Further, if the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence $u$ for which (1) holds with $C = \|A\|_{p_j}$.

However, as noted in §3 below, Koskela's proof of the "only if" or necessity part of the above result is flawed. In this note we provide an alternative proof for the necessity part of Theorem A which also corrects the gap in Koskela's argument.

Koskela [8] (see also [1] and [10] for the sufficiency part) showed that the first part of Theorem A could be expressed in the more usable form:

THEOREM B. Let $p > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_p)$ if and only if there exist positive numbers $C_1$ and $C_2$ and a positive sequence $u := (u_i)$ such that

$$
\begin{align*}
\sum_{i=1}^{\infty} a_{i1} u_i^{p_1} & \leq C_1 u_1^{-p_1}, \quad i = 1, 2, \ldots, \\
\sum_{i=1}^{\infty} a_{i2} u_i^{p_2} & \leq C_2 u_2^{-p_2}, \quad i = 1, 2, \ldots,
\end{align*}
$$

and then $\|A\|_{p} \leq C^{1/p} C_2^{1/p_2}$.

The sufficiency part of this result has proved particularly effective in establishing conditions for standard summability matrices to be in $(l_p, l_p)$. (For Nörlund matrices see [3, 4, 6], and for Hausdorff matrices see [1, 2, 3].)

It was conjectured by Vere-Jones [10] that if $p > 1$ and the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence $u$ for which (2) holds with $C_1 = C_2 = \|A\|_{p_j}$, Koskela [8] showed this conjecture to be false. A more compelling version of the conjecture would seem to be:

CONJECTURE V-J. If $p > 1$ and the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence $u$ for which (2) holds with $C^{1/p} C_2^{1/p_2} = \|A\|_{p}$. We show in §5 that this conjecture is also false.

A useful adjunct to Theorem B is the following theorem which can sometimes be used (as in [2] and [3], for example) to show that $A \notin (l_p, l_p)$. It is essentially Lemma 2 of [2], and can be proved likewise. Its genesis is Theorem 4 of [1].

THEOREM C. Let $A := (a_{ij})$ be a non-negative matrix, let $u$ a bounded sequence of positive numbers such that $\sum_{i=1}^{\infty} u_i = \infty$, and let $\varepsilon_j := \sum_{i=1}^{\infty} a_{ij} u_i^{1/j}$ where $p > 1$. If $p = \liminf_{j \to \infty} \varepsilon_j$ and $\|A\|_{p} \geq u$ in particular, if $\varepsilon_j = 0$, then $A \notin (l_p, l_p)$. In [8] Koskela gave essentially the following theorem concerning conditions for non-negative matrices to be in $(l_p, l_p)$ with $p > q > 1$:

THEOREM D. Let $p > q > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_p)$ if and only if there exist a positive constant $C$ and a positive sequence $u := (u_i)$ with the following properties:

(a) $\|u\|_{p} \leq 1$;

(b) $\sum_{i=1}^{\infty} a_{ij} u_i^{p_j} \leq C u_j^{-p_j}, \quad j = 1, 2, \ldots$;

and then $C^{1/q} \geq \|A\|_{p}$, and the necessity part of Theorem D is true with any $C > \|A\|_{p}$, and that the necessity part of Theorem D is true with any $C > \|A\|_{p}$ even when $q > p$. The proof that we exhibit in §2 is an adaptation of that of Theorem 7.16 in [9]. We are indebted to Gordon Simmon for elaborating the details.

THEOREM 1. Suppose that $p, q > 1$, that the non-negative matrix $A := (a_{ij}) \in (l_p, l_q)$, and that $C \geq \|A\|_{p}$. Then there exists a positive sequence $u := (u_i)$ such that $\|u\|_{p} \leq 1$ and (3) is true.

2. Preliminary results and proof of Theorem 1. In order to prove Theorem 1 we first adopt some notations and prove two lemmas. We define $B^*_{p, q}$ to be the set of non-negative sequences $u = (u_i)$ with $\|u\|_{p} \leq 1$; and $E_p := (a_{ij})$ for $u := (u_i) \geq 0$.

LEMMA 1. Let $p \geq 1$. Suppose that $S$ is a continuous, order preserving map from $B^*_{p, q}$ to $B^*_{p, q}$ and that $0 < t < 1$. Then there exists a positive $u \in B^*_{p, q}$ such that $s(u) < u$.

PROOF: Choose $u^{(1)} \in B^*_{p, q}$ such that $\|u^{(1)}\|_{p} > 0$ and $\|u^{(1)}\|_{q} = 1 - t$. For $n \in N$, define $u^{(n+1)} := u^{(n)} + t u^{(n-1)}$. Note that if $u^{(n)} \in B^*_{p, q}$ for any $n \in N$, then $S u^{(n)} \in B^*_{p, q}$, so $\|u^{(n)}\|_{p} \leq \|u^{(n)}\|_{p} + t \|u^{(n)}\|_{q} \leq 1 - t - 1 = -t$, and hence $u^{(n+1)} = S u^{(n)}$. Also $u^{(1)} - u^{(0)} \geq 0$, and if $u^{(n)} - u^{(n-1)} \geq 0$ for any $n \in N$, then $u^{(n)} - u^{(n-1)} = t u^{(n-1)} - u^{(n-1)} \geq 0$. Since $S$ is order preserving, it follows that the sequence of sequences $(u^{(n)})$ is term-wise...
non-decreasing in $B^+_p$. By the monotonic convergence theorem, $u$, the term-wise limit of $u^{(n)}$, is also the $l_p$-limit of $(u^{(n)})$. Hence $u \in B^+_p$, and since $S$ is continuous, we have that $u = u^{(1)} + Su > Su \geq 0$ as required.

**Lemma 2.** Let $p \geq 1$ and $r > 0$. Then $E_r$ is a continuous, order preserving map from $B^+_p$ to $B^+_p$.

**Proof.** Only the continuity of $E_r$ is not immediately evident. Let $x := (x_i)$ and $y := (y_i)$ be sequences in $B^+_p$. If $r \leq 1$, then $|x_i - y_i| \leq |x_j - y_j|$ by basic calculus, and so

$$
\|E_r x - E_r y\|_p = \left( \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{\infty} |x_j - y_j|^p \right)^{1/p} = \|x - y\|_p.
$$

If $r > 1$, then $|x_i - y_i| \leq r|x_j - y_j|/(r-1)$ by the mean value theorem, and so, using Hölder’s and Minkowski’s inequalities, we get the estimate

$$
\|E_r x - E_r y\|_p \leq \left( \sum_{i=1}^{\infty} \frac{r |x_j - y_j|^p}{(r-1)^p} \right)^{1/p} \leq \frac{r}{(r-1)^{1/p}} \left( \sum_{i=1}^{\infty} |x_j - y_j|^p \right)^{1/p} \leq \|x - y\|_p.
$$

It follows that $E_r$ is a continuous map from $B^+_p$ to $B^+_p$.

**Lemma 1.** Since $|x|_{B^+_p} > 0$, we can divide $A$ by $|x|_{B^+_p}$ and thereby reduce the problem to the case $|x|_{B^+_p} = 1$. Note that the transpose matrix $A^*$ satisfies $\|A^*\|_{W^{p',r'}} = 1$. Further, $A$ is a continuous, order preserving map from $B^+_p$ to $B^+_p$ and $A^*$ is an order preserving map from $B^+_p$ to $B^+_p$. Therefore

$$
S := E_r f A^* E_r \bar{g} A
$$

is a continuous, order preserving map from $B^+_p$ to $B^+_p$. Let $0 < i < 1$. Then, by Lemma 1, there is a positive $u := (u_i) \in B^+_p$ such that $Su \leq u$, that is

$$
(C + \sum_{i=1}^{\infty} |x|^p A^*)^{-1/p} u_j \leq u_j, \quad j = 1, 2, \ldots.
$$

and therefore

$$
\sum_{i=1}^{\infty} d_i \left( \sum_{i=1}^{\infty} e_i^p a_i u_i \right)^{-1/p} \leq \sum_{i=1}^{\infty} d_i^{i/p} \sum_{i=1}^{\infty} e_i a_i u_i = \sum_{i=1}^{\infty} d_i^{i/p} \sum_{i=1}^{\infty} e_i a_i u_i.
$$

**Theorem 3.** Let $p > 1$. Suppose that $A := (a_{ij})$ is a non-negative matrix, that $C > 0$, and that $u := (u_i)$ is a positive sequence satisfying

$$
\sum_{i=1}^{\infty} d_i \left( \sum_{i=1}^{\infty} e_i a_i u_i \right)^{-1/p} \leq C a_i^{-1/p}, \quad j = 1, 2, \ldots
$$

Then

(i) for any fixed $m \in \mathbb{N}$, and $a > 0$, (4) continues to hold if $u$ is replaced by $v := (v_i)$ where

$$
v_i := \begin{cases} u_i & \text{if } i \neq m, \\ a u_i & \text{if } i = m; \end{cases}
$$

(ii) for fixed $i, j \in \mathbb{N}$, with $i > m$ and $i \neq k$, there is a positive integer $r$ and positive constants $K_1, K_2$ such that

$$
K_1 C^{i/m-1} u_i \leq v_j \leq K_2 C^{j/m-1} u_i.
$$

**Proof.** (i) Let $N_m := \{ k \in \mathbb{N} \mid k = m \}$. Then

$$
\sum_{i=1}^{\infty} d_i \left( \sum_{i=1}^{\infty} e_i a_i u_i \right)^{-1/p} \leq \sum_{i=1}^{\infty} d_i^{i/p} \sum_{i=1}^{\infty} e_i a_i u_i = \sum_{i=1}^{\infty} d_i^{i/p} \sum_{i=1}^{\infty} e_i a_i u_i.
$$
from which the desired result follows since

\[ \sum_{k=n}^{m} a_k u_k = 0 \quad \text{when} \quad j = m, \quad \text{and} \quad \sum_{k=n}^{m} a_k u_k u_m = 0 \quad \text{when} \quad j \neq m. \]

(ii) Let \( j_1, j_2, \ldots, j_r = k \) be the chain of integers and \( i_1, i_2, \ldots, i_r \) be the corresponding indices of the definition of \( j \sim k \). It follows from (4) that, for \( \nu = 1, 2, 3, \ldots, r - 1, \)

\[ u_k^{(\nu)} \geq \sum_{i=1}^{\infty} a_i \sum_{i=1}^{\nu} a_i u_i u_k \geq 0. \]

Combining these inequalities we see that \( u_j \geq K_j C_{(\nu+1)}^{-1} u_0 \) for some positive constant \( K_j \). Likewise there is a positive constant \( K \) such that \( u_j \geq K C_{(\nu+1)}^{-1} u_0 \).

PROOFS OF THE NECESSITY PARTS OF THEOREMS A AND D. Suppose that \( p > q > 1 \) and that the non-negative matrix \( A := (a_{ij}) \in (l_p, l_q) \). Let \( C := \|A\|_{l_p 
abla l_q} \) for \( n \in N \).

Then, by Theorem 1, there is a positive sequence \( u_0 := (u_0) \) such that \( \|u_0\|_p \leq 1 \) and

\[ \sum_{j=1}^{\infty} a_j \left( \sum_{k=1}^{n} a_k u_k \right)^{\nu} \leq C \left( \sum_{k=1}^{n} a_k u_k \right)^{\nu}, \quad j = 1, 2, \ldots. \]

CASE 1. Let \( p > q > 1 \). Define \( u := (u) \) where \( u_j := \lim inf_{n \to \infty} u_j^{(n)} \). Then \( \|u\|_p \leq 1 \) and

\[ \sum_{j=1}^{\infty} a_j \left( \sum_{k=1}^{n} a_k u_k \right)^{\nu} \leq (\|A\|_{l_p 
abla l_q} p^{-q})^{-\nu}, \quad j = 1, 2, \ldots. \]

Hence, for every \( j \in N \), \( a_j \leq (\|A\|_{l_p 
abla l_q} p^{-q})^{-\nu} \). It follows that \( u_j \geq 0 \) whenever \( j \in N \).

The above process could yield \( u_j = 0 \), but only when \( j \in N \setminus N \), that is, when the \( j \)-th column of \( A \) is identically 0. This establishes the necessity part of Theorem D.

CASE 2. Let \( p > q > 1 \). Let \( N^1 \) be the set of first elements in the equivalence classes associated with the equivalence relation \( \sim \) on \( N \). For each \( k \in N^1 \) and \( j \sim k \), we can do without affecting the validity of (5). Thus we now have \( u_j^{(n)} = 0 \) for all \( k \in N^1 \). Also, by Lemma 3(i), we have, for fixed distinct \( j, k \in N \), with \( k \in N^1 \) and \( j \sim k \), that there is a positive integer \( g \) and positive constants \( K_1, K_2 \) such that

\[ K_1 (1 - q^{-g}) \leq u_j^{(n)} \leq K_2 (1 - q^{-g}). \]

Define

\[ u_j := \begin{cases} \lim inf_{n \to \infty} u_j^{(n)} & \text{for} \quad j \in N \setminus N^1, \\ 1 & \text{for} \quad j \in N^1 \setminus N. \end{cases} \]

Then \( \infty > u_j > 0 \) for all \( j \in N \), and \( u := (u) \) is a positive sequence satisfying (1) with \( C := (\|A\|_{l_p 
abla l_q}) \). Note that, for \( j \in N \setminus N^1 \), we could have defined \( u_j \) to be any positive number. This completes the proof of the necessity part of Theorem A.
all $i, j > n$, then

$$\|A + E\|_p = \|A\|_p + 1.$$ 

PROOF. Let $C := \|A + E\|_p$. Applying Theorem B to $A + E$, we see that there is a positive sequence $\omega := (\omega_i)$ such that

$$\sum_{i=1}^n \omega_i |a_{ii}|^{1/p} \leq (C_1 - 1)|a_{ii}|^{1/p}, \quad i = 1, 2, \ldots,$$

$$\sum_{j=1}^n \omega_j |a_{jj}|^{1/p} \leq (C_1 - 1)|a_{jj}|^{1/p}, \quad j = 1, 2, \ldots,$$

with $C = C_1^{1/p} C_2^{1/p}$. By Theorem B and Hölder’s inequality, we get

$$\|A\|_p \leq (C_1 - 1)^{1/p} (C_2 - 1)^{1/p} \leq C - 1 = \|A + E\|_p - 1 \leq \|A\|_p.$$ 

To show that Conjecture V-J is not true in general, we require, in addition to Theorem 3, the following proposition concerning $n \times n$ matrices which is due to Koskela [8]. It should be noted that the $p$-norm of an $n \times n$ matrix $A := (a_{ij})$ with respect to the $l_p$ space of $n$-tuples is the same as the $p$-norm of the infinite form of that matrix obtained by setting $a_{ij} := 0$ for all $i, j > n$.

**PROPOSITION.** Let $p > 1$, let $I$ denote the unit $n \times n$ matrix, and let $A$ be a non-negative $n \times n$ matrix. Then

$$\|A + I\|_p = \|A\|_p + 1$$

if and only if $\|A\|_p = \lambda_0$, the greatest non-negative eigenvalue of $A$.

Since there are non-negative $n \times n$ matrices $A$ with greatest non-negative eigenvalues $\lambda_0 < \|A\|_p$, the failure, in general, of Conjecture V-J follows from Theorem 3(b) and the proposition. A simple example of such a matrix is given by

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for which $\lambda_0 = 0 < \|A\|_p = 1$.

**REFERENCES**