Matrix Transformations of Series of Orthogonal Polynomials

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Abstract. For a sequence of polynomials \( (P_n) \) orthonormal on the interval \([-1, 1]\), we consider the sequence of transforms \( (g_n) \) of the series \( \sum_{n=0}^{\infty} a_n P_n(u) \) given by \( g_n(u) = \sum_{n=0}^{\infty} b_n a_n P_n(u) \). We establish necessary and sufficient conditions on the matrix \( (b_{nk}) \) for the sequence \( (g_n) \) to converge uniformly on compact subsets of the interior of an appropriate ellipse to a function holomorphic on that interior.

§1. Introduction. Suppose throughout that \( 1 < p \leq \infty \), \( 1 < R < \infty \), and that all sequences and matrices are complex with indices running through \( 0, 1, 2, \ldots \). We make the following definitions.

\( \gamma_R \) is the ellipse with foci \( \pm 1 \) and half-axes \( a = \frac{1}{2} (R + R^{-1}) \), \( b = \frac{1}{2} (R - R^{-1}) \). Note that an ellipse with foci \( \pm 1 \) having \( R \) as the sum of its two half-axes is necessarily \( \gamma_R \).

\( D'_R \) is the interior of the ellipse \( \gamma_R \), and \( D''_R = \mathbb{C} \).

\( (P_n) \) is an orthonormal sequence of polynomials with respect to a fixed non-negative weight function \( w \) on the interval \([-1, 1]\). That is, \( P_n \) is a polynomial of degree \( n \), and

\[
\int_{-1}^{1} P_n(u) P_m(u) w(u) du = \delta_{nm}.
\]

We assume throughout that

\( w \in L(-1, 1) \) and \( w^{-\varepsilon} \in L(-1, 1) \) for some \( \varepsilon > 0 \).

The first of these integrability conditions is standard, and the second is imposed for the purposes of the present paper. The classical Jacobi polynomials, for which \( w(u) = (u - 1)^\alpha (u + 1)^\beta \) with \( \alpha, \beta > -1 \), satisfy the conditions.

\( \mathcal{E} \) is the set of all sequences \( a = (a_n) \) such that \( \lim |a_n|^{1/(n+1)} = 0 \).

\( \mathcal{E}^\beta \) is the set of all sequences \( a = (a_n) \) such that \( \limsup |a_n|^{1/(n+1)} < \infty \).

\( \mathcal{A}_R \) is the set of all sequences \( a = (a_n) \) such that \( \sum_{n=0}^{\infty} |a_n| R^n < \infty \).

\( A \) is the set of all sequences \( a = (a_n) \) such that \( \limsup |a_n|^{1/(n+1)} = 1/R \).

The following lemma, the proof of which appears in [1], shows that \( \mathcal{E}^\beta \) is the \( \beta \)-dual of \( \mathcal{E} \).

Lemma 1. A sequence \( b \) has the property that \( \sum_{n=0}^{\infty} b_n a_n \) is convergent for each \( a \in \mathcal{E} \) if, and only if, \( b \in \mathcal{E}^\beta \).

[Mathematika, 42 (1995), 427–443]
The following are the first three of eight theorems we shall prove concerning matrix transformations of series of orthogonal polynomials. They are analogs of Theorems 1, 2 and 3 in [1] concerning matrix transformations of power series.

**Theorem 1.** A matrix \(B = (b_{ak})\) has the property that whenever the sequence \(u = (u_n) \in \mathcal{D}_\psi\) the sequence of functions \((g_n)\) given by

\[
g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u), \quad n = 0, 1, \ldots
\]

converges uniformly on every compact subset of \(D_\psi\), each series \(\sum_{k=0}^{\infty} b_{ak} P_k(u)\) of orthogonal polynomials being convergent on \(D_\psi\), if, and only if,

(i) \(\lim_{m \to \infty} b_{ak} = b_k\) for \(k = 0, 1, \ldots\); and

(ii) \(M(p) = \sup_{|p| = 1} |b_k| |p|^k < \infty\) whenever \(1 < p < P\).

And then \(\lim_{m \to \infty} g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u)\) on \(D_\psi\).

**Theorem 2.** A matrix \(B = (b_{ak})\) has the property that whenever the sequence \(u = (u_n) \in \mathcal{A}_\psi\) the sequence of functions \((g_n)\) given by

\[
g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u), \quad n = 0, 1, \ldots
\]

converges uniformly on every compact subset of \(D_\psi\), each series \(\sum_{k=0}^{\infty} b_{ak} P_k(u)\) of orthogonal polynomials being convergent on \(D_\psi\), if, and only if,

(i) \(\lim_{m \to \infty} b_{ak} = b_k\) for \(k = 0, 1, \ldots\); and

(ii) \(M(p) = \sup_{|p| = 1} |b_k| |p|^k < \infty\) whenever \(1 < p < P\).

And then \(\lim_{m \to \infty} g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u)\) on \(D_\psi\).

**Theorem 3.** A matrix \(B = (b_{ak})\) has the property that whenever the sequence \(u = (u_n) \in \mathcal{C}\) the sequence of functions \((g_n)\) given by

\[
g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u), \quad n = 0, 1, \ldots
\]

converges uniformly on every compact subset of \(C\), each series \(\sum_{k=0}^{\infty} b_{ak} P_k(u)\) of orthogonal polynomials being convergent on \(C\), if, and only if,

(i) \(\lim_{m \to \infty} b_{ak} = b_k\) for \(k = 0, 1, \ldots\); and

(ii) \(M = \sup_{0 < |z| < 1} |b_k|^{|z|^{k+1}} < \infty\).

And then \(\lim_{m \to \infty} g_n(u) = \sum_{k=0}^{\infty} b_{ak} P_k(u)\) on \(C\).

These theorems show that if the series-to-sequence transform given by \(B\) is regular, then it is necessary in each case that \(\lim_{m \to \infty} b_{ak} = b_k\) for \(k = 0, 1, \ldots\), and this in turn implies that \(P \in \mathcal{B}\) in Theorems 1 and 2 (i.e., the sequence \((g_n)\) cannot converge uniformly in the interior of any ellipse \(\gamma\) with \(P > R\)). Regular sequence-to-sequence transforms of power series have been considered by Petermuhlb [8] and Luh [7] among others. One of the novel features of our approach is that with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let \((B_k)\) be a sequence of non-zero complex numbers. The associated Nörlund series-to-sequence matrix \(N_\psi\) is the triangular matrix \((b_{nk})\) with

\[
b_{nk} = \begin{cases} \frac{B_{n-k}}{B_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}
\]

The following theorem is an immediate consequence of Theorem 1.

**Theorem N.** The Nörlund matrix \(N_\psi\) has the property that whenever the sequence \(u = (u_n) \in \mathcal{D}_\psi\) the sequence of functions \((g_n)\) given by

\[
g_n(u) = \frac{1}{B_k} \sum_{k=0}^{n} B_{n-k} u_{n-k} P_k(u), \quad n = 0, 1, \ldots
\]

converges uniformly on every compact subset of \(D_\psi\), if, and only if,

\[
\lim_{n \to \infty} \frac{B_{n-k}}{B_n} = b_k \quad \text{with } |b_k| = \frac{R}{P},
\]

And then \(\lim_{n \to \infty} g_n(u) = \sum_{k=0}^{n} b_{nk} P_k(u)\) on \(D_\psi\).

Note. In view of Theorem 2, Theorem N remains true if \(\mathcal{D}_\psi\) is replaced by \(\mathcal{A}_\psi\).

§2. Orthogonal polynomials. In this section we set out some of the properties of orthogonal polynomials required in our proofs. Note that the function \(u = \frac{1}{2}(1 + z)\) maps the region \([z: |z| > 1]\) bijectionally onto the region \([u: |u| \in [1, 1]]\), and that each circle \(|z| = R\) is mapped onto \(\gamma_R\). The inverse of this function is \(z = u + \sqrt{u^2 - 1}\). Here and elsewhere in the paper the sign of the square root is chosen so that \(|u + \sqrt{u^2 - 1}| > 1\) when \(|u| > 1\). We then have, for \(z = u + \sqrt{u^2 - 1}\), that \(|z| = R\) when \(u = \gamma_R\), and \(|z| < R\) when \(u < \gamma_R\). The function \(u = \frac{1}{2}(1 + z)\) maps both the top half and the bottom half of the unit circle \([z: |z| = 1]\) onto \([-1, 1]\).

**Lemma 2.** For \(\varepsilon > 0\) let the non-negative weight function \(w \in L(-1, 1)\) associated with the orthonormal sequence of polynomials \((P_n)\) be such that \(w \in L(-1, 1)\), and let \(|z| > 1\) and \(w = \frac{1}{2}(1 + z)\). Then

\[
|P_n(u)| \leq K(u)(1 + n)^{\alpha/2 - \alpha/2} |u|^n
\]

for \(n = 0, 1, \ldots\), where \(K(u)\) is a positive number independent of \(n\).
Proof. By Bernstein's inequality (see [5, Theorem 7])

$$|P_n(u)| \leq \max_{|t| < 1} |P_n(t)||z|^4,$$

and by a result due to Erdélyi [2, Theorem 5]

$$\max_{|t| < 1} |P_n(t)||K(t)(1 + 11^2)|^{1/2} \leq |P_n(t)||w(t)|dt.$$

Finally, by the Cauchy–Schwarz inequality,

$$\int_{-1}^{1} |P_n(t)||w(t)|dt \leq \left( \int_{-1}^{1} P_n(t)^2|w(t)|dt \right)^{1/2} \left( \int_{-1}^{1} |w(t)|dt \right)^{1/2}.$$

Combining the above inequalities we get the required result.

**Lemma 3.** (Expansion of a holomorphic function in terms of orthogonal polynomials). Let the non-negative weight function \( w \in L(-1,1) \) be associated with the orthonormal sequence of polynomials \( (P_n) \) such that \( w \in L(-1,1) \) for some \( c > 0 \). Let \( f(u) \) be holomorphic on the closed segment \([-1,1] \), and let \( \gamma_n \) denote the largest ellipse with foci \( \pm 1 \) on the interior of which \( f(u) \) is holomorphic. The Fourier series expansion of \( f(u) \) on \( D_k \), the interior of \( \gamma_n \), is given by

$$f(u) = \sum_{k=0}^{\infty} a_k P_k(u),$$

where

$$a_k = \int_{-1}^{1} f(t) P_k(t) dt$$

The Fourier series is absolutely convergent on \( D_k \), and is also uniformly convergent on compact subsets of \( D_k \). It is divergent on the exterior of \( \gamma_n \). Further, the sum \( R \) of the semi-axes of the ellipse of convergence is given by

$$\frac{1}{R} = \lim_{k \to \infty} |a_k|^{1/k}.$$

Proof. All but the statement about absolute convergence follows from Theorems 12.73 and 12.74 in [11], since the conditions on the weight \( w \) are more stringent than those in the said theorems. To prove the absolute convergence part, let

$$\frac{1}{R} = \lim_{k \to \infty} |a_k|^{1/k}.$$
Proof of Theorems 1 and 2. We prove these two theorems together.

Sufficiency. We assume that

\[ b_k = b_k \quad \text{for} \quad k = 0, 1, \ldots ; \]
\[ M(p) = \sup_{n \geq k} \left| a_n \right| (\frac{p}{R})^k < \infty \quad \text{for} \quad 1 < p < P. \]

Let \( a = a_{\alpha A} \) or \( a = \alpha A \). For 1 < p < P choose \( s \) so that 1 < s < R and \( p/r < P/R \). Now choose \( p_1 \) so that \( p/p_1 < P \) and \( p_1/R < p/R \). Suppose \( u \in D_k^s \). Then \( u = \frac{1}{|z + z|^k} \) with 1 < |z| < \( \rho \), and therefore, by Lemma 2,

\[ |b_k u_n|P_k|w| \leq K(c) ||u_n||_a \frac{(1 + k)^{1+c_1+k}p^k}{k!} \]
\[ = K(c)|b_k| \left( \frac{p}{R} \right)^k ||u_n||_a \frac{(1 + k)^{1+c_1+k}p^k}{k!} \]
\[ = K(c)|b_k| \left( \frac{p}{R} \right)^k ||u_n||_a \frac{(1 + k)^{1+c_1+k}p^k}{k!} \leq K(c)M(p)|P_k||u_n||_a \frac{(1 + k)^{1+c_1+k}p^k}{k!}. \]

Further, by (i) of either Theorem 1 or Theorem 2,

\[ \lim_{n \to \infty} b_k u_n P_k(u) = b_k u_n P_k(u). \]

Since \( \sum_{n=0}^{\infty} ||u_n||_a (1 + k)^{1+c_1+k}p^k \leq \infty \), and since \( p \) can be chosen arbitrarily close to \( P \) in \( (1, P) \), it follows, by the Weierstrass M-test, that \( g_n(u) \) exists for \( n = 0, 1, \ldots \), and

\[ g_n(u) = \lim_{n \to \infty} \sum_{k=0}^{n} b_k u_n P_k(u) = \sum_{k=0}^{n} b_k u_n P_k(u) \]

on \( D_k^s \), and the sequence \( (g_n) \) is uniformly convergent on compact subsets of \( D_k^s \). This completes the proof of the sufficiency of conditions (i) and (ii) both for Theorem 1 and Theorem 2.

Necessity. Let \( n = 1/R^k (k + 1)^k \). Then \( a = a_{\alpha A} \) and \( a = \alpha A \). Under the hypotheses of either Theorem 1 or Theorem 2 the series

\[ g_n(u) = \sum_{k=0}^{n} b_k u_n P_k(u) \]

is convergent on \( D_k^s \) and the sequence \( (g_n) \) is uniformly convergent on compact subsets of \( D_k^s \). Therefore, by the Weierstrass double-series theorem, \( (g_n) \)

converges to a holomorphic function on \( D_k^s \). By Lemma 3, we get, for the above sequence \( a \), that

\[ b_k u_n = \int_{-1}^{1} g_n(t)P_k(t)dt \quad \text{for} \quad n = 0, 1, \ldots. \]

Since \( g_n(t) \) converges uniformly on \( [-1, 1] \) to \( g(t) \), we say, that we get

\[ \lim_{n \to \infty} b_k u_n = \int_{-1}^{1} g(t)P_k(t)dt = d_k. \]

Hence, for \( k = 0, 1, \ldots \),

\[ \lim_{n \to \infty} b_k u_n = d_k, \]

where \( d_k = \frac{1}{R} (k + 1)^k \). This proves the necessity of condition (i) in both Theorem 1 and Theorem 2.

Suppose now that \( p \) and \( b \) are fixed with \( 1 < p < b < P \). Since \( a \) satisfies the hypotheses of both Theorem 1 and Theorem 2, the sequence \( (g_n) \) is uniformly convergent on \( D_k^s \). Hence we have, for \( a = a_{\alpha A} \) and \( n = 0, 1, \ldots \), that

\[ |g_n(u)| \leq M(b) \quad \text{for} \quad M(b) < \infty, \]

\( M(b) \) being independent of \( n \). By Lemma 4 we get that

\[ |b_k u_n P_k(u)| \leq c(b)M(b) \quad \text{for} \quad n, k = 0, 1, \ldots. \]

Since \( a_k = 1/R^k (k + 1)^k \), it follows that

\[ b_k \left( \frac{p}{R} \right)^k \leq c(b)M(b) \quad \text{for} \quad n, k = 0, 1, \ldots, \]

and hence that

\[ \sup_{\alpha \in A} \left( \frac{p}{R} \right)^k \leq c(b)M(b) \quad \text{for} \quad n, k = 0, 1, \ldots. \]

Therefore the condition

\[ \sup_{\alpha \in A} \left( \frac{p}{R} \right)^k < \infty \quad \text{whenever} \quad 1 < p < P, \]

is necessary, i.e., condition (iii) is necessary in both Theorem 1 and Theorem 2.

Proof of Theorem 3. Sufficiency. We assume that

\[ \left\{ \begin{array}{l}
\lim_{n \to \infty} b_k = b_k \quad \text{for} \quad k = 0, 1, \ldots ;
M = \sup_{n \geq k} \left| a_n \right|^{1/(1+k)} < \infty.
\end{array} \right. \]
Let $a \in \mathcal{E}$, and let $w \in B_{\mathcal{D}}$. Then $u = (z + z^{-1})$ with $1 \leq |z| < R < \infty$, and so, by Lemma 2,
\[
|b_n a_n P_n(u)| \leq C|\epsilon| b_n \|a_n\| (1 + k)^{2\gamma + \gamma'} R^k \leq C|\epsilon| b_n \|a_n\| (1 + k)^{2\gamma + \gamma'} R^k \leq C|\epsilon| M a_n (1 + k)^{2\gamma + \gamma'} (MR)^k < \infty.
\]
From (i) we get
\[
\lim_{k \to \infty} b_n a_n P_n(u) = b_n a_n P_n(u).
\]
Since $\sum_{n=1}^{\infty} |a_n| (1 + k)^{2\gamma + \gamma'} (MR)^k < \infty$, and since $R$ can be arbitrarily large, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n = 0, 1, \ldots$, and
\[
\lim_{n \to \infty} g_n(u) = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_n a_n P_k(u) = \sum_{k=0}^{\infty} b_n a_n P_k(u)
\]
on $C$, and that the sequence $(g_n)$ is uniformly convergent on compact subsets of $C$.

**Necessity.** Let $a_n = k^{-n}$, so that $a \in \mathcal{E}$. Then, by hypothesis, the series
\[
g_n(u) = \sum_{k=0}^{\infty} b_n a_n P_k(u)
\]
is convergent on $C$, and the sequence $(g_n)$ is uniformly convergent on compact subsets of $C$. By the Weierstrass double-series theorem, $(g_n)$ converges to an entire function on $C$. By Lemma 3 we have
\[
b_n a_n P_k(u) = \int_{-1}^{1} g_n(t) P_k(t) dt \quad \text{for} \quad n = 0, 1, \ldots.
\]
Since $g_n(t)$ is uniformly convergent on $[-1, 1]$ to $g(t)$ say, we get, for $k = 0, 1, \ldots$, that
\[
\lim_{n \to \infty} b_n a_n = \int_{-1}^{1} g(t) P_k(t) dt = d_k,
\]
and hence that
\[
\lim_{n \to \infty} b_n a_n = d_k,
\]
where $d_k = d_k k^k$ for $k = 0, 1, 2, \ldots$. Thus condition (i) is necessary.

Suppose now that $a$ is an arbitrary sequence in $\mathcal{E}$, and that $R > 1$. Since the sequence $(g_n)$ is uniformly convergent on $B_{R_{\mathcal{D}}}$, we have, for $w \in B_{R_{\mathcal{D}}}$ and $n = 0, 1, \ldots$, that $|g_n(w)| \leq M(R, a) < \infty$. From Lemma 4 we get that
\[
|b_n a_n| \leq C(R) M(R, a) R^{-n} \quad \text{for} \quad n = 0, 1, \ldots.
\]
Then, for each \( a \in A_k \) and each \( R > P \),
\[
\lim_{n \to \infty} \sup \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \leq R_{n} / P.
\]

Proof. Choose \( R > P \) and suppose \( a \in A_k \). Let \( 1 < k \leq 1 \) and take \( p = \lambda > P \). Then \( 1 < p < P \). Since \( \lim_{n \to \infty} |a_{n}|^{(1/p) \cdot 1/n} = 1 / R \), there is a positive constant \( c(\lambda) \) such that
\[
|a_{n}| < c(\lambda) \left( \lambda R \right)^{1/n}.
\]

By Lemma 2, for \( u \in \gamma_{n} \), we have \( |P_{a}(u)| \leq K(\varepsilon) \left( 1 + k \right)^{\varepsilon + (1/2 \cdot P)} \) and hence
\[
\sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \leq K(\varepsilon) \sum_{x=0}^{n} |a_{x}| P_{R} \left( 1 + k \right)^{\varepsilon + (1/2 \cdot P)} \leq K(\varepsilon) M(p) \left( \lambda R \right)^{1/n} \sum_{x=0}^{n} \left( R_{n} / P \right)^{1/n} (1 + k)^{\varepsilon + (1/2 \cdot P)}.
\]

Since \( R_{n} / \lambda^{2} > R_{n} / P > 1 \) it follows that
\[
\limsup_{n \to \infty} \left( \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \right)^{1/n} \leq \limsup_{n \to \infty} \left( \sum_{x=0}^{n} R_{n} / \lambda^{2} P \right)^{1/n} = R_{n} / \lambda^{2} P.
\]

Letting \( \lambda \to 1 \) we get
\[
\limsup_{n \to \infty} \left( \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \right)^{1/n} \leq \frac{R_{n}}{P}.
\]

Remark. Assume that a triangular matrix \( B \) satisfies
\[
M(p) = \sup_{x=0}^{n} |b_{a_{x}}| \left( \frac{p}{R} \right)^{1/n} \leq 1 \quad \text{for} \quad 1 < p < P.
\]

Then
\[
|b_{a_{x}}|^{1/n} \leq M(p)^{1/n} \leq 1 \quad \text{as} \quad n \to \infty,
\]
and hence
\[
\limsup_{n \to \infty} |a_{n}|^{1/n} \leq \frac{R_{n}}{P} \quad \text{for each} \quad p \in (1, P).
\]

Letting \( p \to P \) we get
\[
\limsup_{n \to \infty} |a_{n}|^{1/n} \leq \frac{R_{n}}{P}.
\]

This suggests that it is not inappropriate to impose the condition
\[
|b_{a_{x}}|^{1/n} \leq \frac{R_{n}}{P},
\]
as we do in the following theorem.

Theorem 5. Let \( B \) be a triangular matrix. Suppose that
\[
\lim_{n \to \infty} |b_{a_{x}}|^{1/n} \leq \frac{R_{n}}{P},
\]
where \( P \) and \( R \) are finite numbers greater than 1. Then for each \( a \in A_k \) and each \( R > P \) we have
\[
\limsup_{n \to \infty} \left( \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \right)^{1/n} \geq \frac{R_{n}}{P}.
\]

Proof. Assume that the conclusion of the theorem is not true. Then there is an \( x \in A_k \) and an \( R > P > 1 \) such that
\[
\limsup_{n \to \infty} \left( \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \right)^{1/n} < \frac{R_{n}}{P}.
\]

Therefore there exists a number \( \bar{R} \) such that \( 1 < \bar{R} < R \) and, for all \( n \) sufficiently large,
\[
\max_{x=0}^{n} |b_{a_{x}}|^{1/n} \leq \frac{R}{P},
\]
and hence
\[
\max_{x=0}^{n} b_{a_{x}} P_{a}(u) \leq \left( \frac{R}{P} \right)^{1/n}.
\]

Applying Lemma 4 to the function \( g(u) = \sum_{x=0}^{n} b_{a_{x}} P_{a}(u) \) we get in particular that, for all large \( n \),
\[
|b_{a_{x}}| \leq c(R) \left( \frac{R}{P} \right)^{1/n},
\]
and therefore
\[
|b_{a_{x}}|^{1/n} \leq c(R) \frac{R}{P}.
\]

From the last inequality we get that
\[
\bar{R} \geq \limsup_{n \to \infty} (|b_{a_{x}}|^{1/n} R_{n}) = R_{n} \lim_{n \to \infty} |a_{n}|^{1/n}, \quad \text{and sup} \quad |a_{n}|^{1/n} = \frac{R_{n}}{P}.
\]

But this is a contradiction since \( 1 < \bar{R} < R \). Hence the conclusion of the theorem must hold.

The next two theorems are analogues of Theorems 6 and 7 (concerning matrix transformations of power series) in [1], which in turn generalize results about regular and non-regular Nörlund matrices due respectively to Luh [6].
and K. Stadtmüller [9, Theorems 6 and 7]. The first of these new theorems, which follows immediately from Theorems 4 and 5, shows, inter alia, that the sequence \( (g_r) \) specified in Theorem 2 cannot converge uniformly in the interior of any ellipse \( \gamma R \) with \( R > P \) when \( B \) is a triangular matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

**Theorem 6.** Suppose that \( P \) and \( R \) are finite numbers greater than 1, and that \( B \) is a triangular matrix satisfying

\[
M(P) = \sup_{a \in A} |a| \left( \frac{P}{R} \right)^{\alpha} < \infty \quad \text{for} \quad 1 < P < R,
\]

and

\[
\lim_{n \to \infty} |b_n|^{1/n} = \frac{R}{P}.
\]

Then, for each \( a \in A \), each \( R \), and each \( P \),

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

The next theorem shows that the ellipse \( \gamma R \), in the conclusion of Theorem 6 can be replaced by any arc of that ellipse (provided condition (i) of Theorem 2 is also satisfied when \( R_1 = P \)).

**Theorem 7.** Suppose that \( P \) and \( R \) are finite numbers greater than 1, and that \( B \) is a triangular matrix such that

\[
M(P) = \sup_{a \in A} |a| \left( \frac{P}{R} \right)^{\alpha} < \infty \quad \text{for} \quad 1 < P < R,
\]

and

\[
\lim_{n \to \infty} |b_n|^{1/n} = \frac{R}{P}.
\]

(i) Then, for each \( a \in A \) and each \( R > P \),

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

where \( \Gamma \) is any closed non-trivial arc of \( \gamma R \).

(ii) If, in addition,

\[
\lim_{n \to \infty} b_n = b_0 \quad \text{for} \quad k = 0, 1, \ldots, \quad \text{where} \quad b_0 \neq 0 \quad \text{for} \quad k > k^*,
\]

then, for each \( a \in A \),

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq 1.
\]

**Proof of (i).** By Theorem 6 we know that

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

Hence it is enough to prove that, for every \( a \in A \),

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P},
\]

which we now proceed to do. Assume that (3) is not true. Then there exists a sequence \( a_n \in A \) and a number \( R \) such that \( P < R < R \) and

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

Hence given \( \epsilon > 0 \) we have, for \( z = u + \sqrt{R^2 - 1} \) and all sufficiently large \( n, \)

\[
\max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

Further, from Theorem 6 we get that, for all large \( n, \)

\[
\max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

and

\[
\max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

Let \( P < r < R \). Since the function \( z = u + \sqrt{R^2 - 1} \) is holomorphic and different from zero on \( \{1, 1, 1, 1, \} \), we have, by Novanlinna's N-constants theorem (see [9, Theorem 18.3]), that there exist positive constants \( \theta_1, \theta_2, \theta_3 \) (depending on \( r \) but not on \( x \)) such that \( \theta_1 + \theta_2 + \theta_3 = 1 \) and

\[
\max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

For all sufficiently large \( n \), Hence, choosing \( \epsilon > 0 \) so small that \( (R/R)^{\theta_2} < 1 \), we get

\[
\limsup_{n \to \infty} \max_{|a| \leq 1} \sum_{k=0}^{n} b_k a_k P_k(u) \geq \frac{R_1}{P}.
\]

Since \( r > P \), the last inequality contradicts the conclusion of Theorem 5. Hence (3) must hold when \( R_1 > P \).
Proof of (ii). By Theorem 6 we know in this case that

$$\lim_{n \to \infty} \sup_{x \in [0,1]} \left| \sum_{k=0}^{n} h_{n} a_{k} P_{k}(x) \right| = 1.$$

Hence it is enough to prove that, for every $a \in A_{p}$,

$$\lim_{n \to \infty} \sup_{x \in [0,1]} \left| \sum_{k=0}^{n} h_{n} a_{k} P_{k}(x) \right| \geq 1,$$

(4)

Suppose (4) is not true. Then for some $a^* \in A_{p}$ we have

$$\lim_{n \to \infty} \sup_{x \in [0,1]} \left| \sum_{k=0}^{n} h_{n} a_{k}^{2} P_{k}(x) \right| < 1.$$

Write

$$g_{n}(u, a^*) = \sum_{k=0}^{n} h_{n} a_{k}^{2} P_{k}(u).$$

It follows that there exists a positive number $q < R_{l/P} l,P = 1$, such that, for all $n$ sufficiently large,

$$\sup_{x \in [0,1]} \left| g_{n}(u, a^*) \right| < q^{n}.$$

Given $a > 0$ we get from Theorem 6 that, for all $n$ sufficiently large,

$$\max_{x \in [0,1]} \left| g_{n}(u, a^*) \right| < q^{n}.$$

By Nevanlinna’s $N$-constants theorem, there exists a positive number $\theta < 1$ (independent of $a$) such that, for all large $n$,

$$\max_{-1 < x < 1} \left| g_{n}(u, a^*) \right| < \left( \frac{q^{n}}{\theta^{n}} \right)^{\frac{n}{\theta}}.$$

Since we can choose $u > 0$ so small that $\theta^{2^{n-\theta}} < 1$, it follows that

$$\max_{-1 < x < 1} \left| g_{n}(u, a^*) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

By Lemma 3 we have

$$h_{n} a_{k}^{2} = \int_{-1}^{1} g_{n}(t, a^*) P_{k}(t) dt \quad \text{for} \quad n = 0, 1, \ldots.$$ 

Since $g_{n}(t, a^*)$ tends uniformly to 0 on $[-1, 1]$ as $n \to \infty$, it follows that

$$0 = \lim_{n \to \infty} h_{n} a_{k}^{2} = h_{k} a_{k}^{2} \quad \text{for} \quad k = 0, 1, \ldots.$$

Since $a^* \in A_{p}$ we have that $a_{k} \neq 0$ for some $k > k^{*}$. Hence $h_{k} = 0$ for such a $k$. But this contradicts the assumption that $h_{k} \neq 0$ for $k > k^{*}$. Therefore (4) must hold.

**Theorem 8.** Suppose that $P$ and $R$ are finite numbers greater than 1, and that $B$ is a triangular matrix such that

(i) $\lim_{n \to \infty} h_{n} = b_{n}$, for $k = 0, 1, \ldots$, where $b_{k} \neq 0$ for $k > k^{*}$;

(ii) $M(p) = \sup_{0 < x < 1} |h_{n}(x/P)^{k}| < \infty$ for $1 < p < P$, $\lim_{n \to \infty} |h_{n}|^{1/n} = R/P$, and

(iii) $|h_{n}| < 1$ for $k = 0, 1, \ldots$.

Suppose that $a \in A_{p}$ and that $\lim \sup_{n \to \infty} |a_{n} R^{n}| > 0$. Let

$$g_{n}(u, a) = \sum_{k=0}^{n} h_{n} a_{k}^{2} P_{k}(u),$$

where $(P_{k})$ is the orthonormal sequence on $[-1, 1]$ of Chebyshev polynomials of the first or second kind, and let $P_{1} > P$. Then $\lim \sup_{n \to \infty} |g_{n}(u, a)|^{1/n} < 1$ for at most a finite number of points $u$ outside the ellipse $\gamma_{p}$, and hence, in particular, the sequence $(g_{n}(u))$ can converge at most at a finite number of points $u$ outside the ellipse $\gamma_{p}$.

**Proof.** Assume that $u$ is a point outside the ellipse $\gamma_{p}$, for which

$$\lim_{n \to \infty} g_{n}(u, a) = 1.$$

Let $z = u + \sqrt{u^2 - 1}$, so that $|z| > P_{1}$, and let

$$g_{n}(z) = \sum_{k=0}^{n} h_{n} a_{k} z^{k}.$$

Then, by (5),

$$2g_{n}(u) - 2 \sum_{k=0}^{n} h_{n} a_{k} P_{k}(u) = g_{n}(z) + g_{n}(z^{-1})$$
when the Chebychev polynomials $P_n$ are of the first kind; and, by (6),

$$z - z_n^*\gamma_n(z) - 2\gamma_n(z) - z_n^*\gamma_n(z)^*$$

when the Chebychev polynomials $P_n$ are of the second kind.

Since $|z|^2 < P_n^2 < P$ it follows from Theorem 2 in [1] that $\gamma_n(z)^*$ tends to a finite limit as $n \to \infty$, and therefore from (7) that, in either case,

$$\lim_{n \to \infty} |\gamma_n(z)|^{1/n} \leq 1.$$  \hspace{1cm} (8)

Theorem 8 in [1] tells us that inequality (8) can hold for at most a finite number of points $z$ satisfying $|z| > P_1$, and thus (7) can hold for at most finitely many points $z$ outside the ellipse $\gamma_n$. 

Remarks. A Nörlund matrix $N_b$ for which

$$\lim_{n \to \infty} \frac{R_n}{b} = b \quad \text{with} \quad |b| = \frac{R}{P}$$

satisfies all the conditions on the matrix in Theorem 8. In this case, however, the condition $\lim sup |a_n| R_n^* > 0$ can be omitted since the corresponding version of the theorem for power series has recently been proved by K. Stadtmüller and Grosse-Erdmann [10, Remark 3.7].

An open and challenging question is whether Theorem 8 holds for other orthogonal polynomials.

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References