A high indices Tauberian theorem

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Communicated by F. Móricz

Abstract. We prove a Tauberian theorem concerning the summability method $D_{\lambda, a}$ based on the Dirichlet series $\sum a_ne^{-\lambda_n x}$ with $a_{n+1} \sim a_n > 0$ when the sequence $(\lambda_n)$ satisfies the 'high indices' condition $\lambda_{n+1} > c\lambda_n \geq 0$ with any $c > 1$.

1. Introduction

Suppose throughout that $(\lambda_n)$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geq 0$, and that $(a_n)$ is a sequence of positive numbers. Suppose also that

$$A_n := \sum_{k=1}^{n} a_k \to \infty,$$

and define

$$a(x) := \sum_{n=1}^{\infty} a_ne^{-\lambda_n x}$$

whenever this Dirichlet series converges.

Let $s, s_1, s_2, \ldots$ be complex numbers, and define

$$\sigma_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k\sigma_k \quad \text{and} \quad \sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n\sigma_ne^{-\lambda_n x}.$$

Received June 20, 1997.

AMS Subject Classification (1991): 40E05, 40G99.

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.
The weighted mean summability method \( M_x \) and the Dirichlet series summability method \( D_{\lambda x} \) (see [1]) are defined as follows:

\[
s_n \rightarrow s(M_x) \quad \text{if} \quad s_n \rightarrow s;
\]
\[
s_n \rightarrow s(D_{\lambda x}) \quad \text{if} \quad s(x) \exists \text{for } x > 0 \text{ and } s(x) \rightarrow s \text{ as } x \rightarrow 0+.
\]

It is known (see [1]) that, since \( A_n \rightarrow \infty \), both methods are regular (i.e., \( s_n \rightarrow s \) implies \( s_n \rightarrow s(M_x) \) and \( s_n \rightarrow s(D_{\lambda x}) \)), and that \( s_n \rightarrow s(M_x) \) implies \( s_n \rightarrow s(D_{\lambda x}) \).

The purpose of this paper is to prove the following result:

**Theorem 1.** Suppose that the sequence \( (a_n) \) satisfies the condition

\[
a_{n+1} \sim a_n,
\]
and that the sequence \( (\lambda_n) \) satisfies the high indices condition

\[
\lambda_{n+1} > c \lambda_n \quad \text{with} \quad c > 1.
\]

Suppose also that \( s_n \rightarrow s(D_{\lambda x}) \), and that the Tauberian condition

\[
s_{n+1} - s_n = O \left( \frac{a_n}{A_n} \right)
\]
holds. Then \( s_n \rightarrow s \).

**Remark.** Observe that conditions (1) and (2) imply that, for \( x > 0 \),

\[
a_{n+1}e^{-\lambda_{n+1}x} \leq a_ne^{-\lambda_nx} \leq a_0e^{-\lambda_nx} \rightarrow 0,
\]

from which it follows that the Dirichlet series \( a(x) \) converges for all \( x > 0 \).

It is interesting to compare Theorem 1 with Theorem 114 in [3], namely:

**Theorem H1.** Suppose that (2) holds, and that

\[
\sum_{n=1}^{\infty} (s_{n+1} - s_n)e^{-\lambda_nx} \rightarrow 0 \quad \text{as} \quad x \rightarrow 0+.
\]

Then \( s_n \rightarrow s \).

A known Tauberian result concerning sequences \( (\lambda_n) \) not satisfying the high indices condition (2) is the following theorem due to Borwein [2, Theorem 6] (see also Tietz [6, Satz 3.9] for the case \( \lambda_n = n \)).

**Theorem B1.** Suppose that

\[
\lambda_{n+1} \sim \lambda_n,
\]

\[
\frac{A_n}{A_m} \rightarrow 1 \quad \text{when} \quad \frac{\lambda_n}{\lambda_m} \rightarrow 1, \quad m > n \rightarrow \infty,
\]

\[
\lim_{x \rightarrow 0+} s_n - s(x) \geq 0 \quad \text{when} \quad \frac{A_n}{A_m} \rightarrow 1, \quad m > n \rightarrow \infty,
\]

and that \( s_n \rightarrow s(D_{\lambda x}) \). Then \( s_n \rightarrow s \).

We won't use the above two results, but will use another known Tauberian result which follows immediately from Theorem 67 in [3]:

**Theorem H2.** If (3) holds and \( s_n \rightarrow s(M_x) \), then \( s_n \rightarrow s \).

We will also use the following Tauberian theorem due to Borwein [1, Theorem 2]:

**Theorem B2.** Let \( s_n \rightarrow s(D_{\lambda x}) \), let \( s_n > -H \) where \( H \) is a constant, and let

\[
\lim_{x \rightarrow 0+} \frac{a(mx)}{a(x)} = a_0 > 0 \quad \text{for} \quad m = 2 \quad \text{and} \quad m = 3.
\]

Then \( s_n \rightarrow s(M_x) \).

For other Tauberian theorems of the type \( D_{\lambda x} \Rightarrow M \) see [1, Theorem 3], [2, Theorem 7], and (for the case \( \lambda_n = n \)) [5, Korollar 4.2].

2. Auxiliary results

**Lemma 1.** Suppose \( (x_n) \) is a non-increasing sequence of positive numbers such that \( \sigma(x_n) \) exists. Suppose also that (1) and (2) hold, and that

\[
|x_n+1 - x_n| \leq c_1 \frac{a_n}{\lambda_n} \quad \text{with} \quad \lambda_n := a(x_n)e^\lambda x_n,
\]

where \( c_1 \) is a positive constant. Then \( |\sigma(x_n) - s_n| \leq c_1 \).
Proof. (Cf. the proof of [4, Theorem].) We have

$$|r(z_n) - s_n| = \left| \sum_{k=1}^{n-1} a_k (s_k - s_{k-1}) e^{-\lambda_k z_n} \right| \leq |\Sigma_1 + \Sigma_2|,$$

where

$$\Sigma_1 := \frac{1}{a(z_n)} \sum_{k=1}^{n-1} a_k |s_k - s_{k-1}| e^{-\lambda_k z_n}, \quad \Sigma_2 := \frac{1}{a(z_n)} \sum_{k=m+1}^{\infty} a_k |s_k - s_{k-1}| e^{-\lambda_k z_n}.$$ 

We have

$$\Sigma_1 \leq \frac{1}{a(z_n)} \sum_{k=1}^{n-1} a_k e^{-\lambda_k z_n} \sum_{j=k}^{n-1} g_j \sum_{k=1}^{j-1} e^{-\lambda_k z_n} \leq c_1 \frac{1}{a(z_n)} \sum_{k=1}^{n-1} a_k e^{-\lambda_k z_n},$$

and

$$\Sigma_2 \leq \frac{1}{a(z_n)} \sum_{k=m+1}^{\infty} a_k e^{-\lambda_k z_n} \sum_{j=m}^{\infty} g_j \sum_{k=1}^{j-1} e^{-\lambda_k z_n} \leq c_1 \frac{1}{a(z_n)} \sum_{k=m+1}^{\infty} a_k e^{-\lambda_k z_n},$$

since $\lambda_k \leq \lambda_j$ and $z_j \geq z_n$ when $k \leq j < n$. Next we have

$$\Sigma_1 \leq \frac{1}{a(z_n)} \sum_{k=m+1}^{\infty} a_k e^{-\lambda_k z_n} \sum_{j=m}^{\infty} \sum_{k=1}^{j-1} g_j \sum_{k=1}^{j-1} e^{-\lambda_k z_n} \leq c_1 \frac{1}{a(z_n)} \sum_{k=m+1}^{\infty} a_k e^{-\lambda_k z_n},$$

since $\lambda_k \geq \lambda_j$ and $z_j \leq z_n$ when $n \leq j < k$. Hence

$$\Sigma_1 + \Sigma_2 \leq -\frac{c_1}{a(z_n)} \sum_{k=1}^{n-1} a_k e^{-\lambda_k z_n} = c_1,$$

which yields the desired conclusion.

Lemma 2. Suppose that (1) and (2) hold. Then

(i) $A_{n+1} = A_n$;
(ii) for all sufficiently large $n$, $(1/e)A_n \leq a(1/\lambda_n) \leq c_1 A_n$,
where $c_2 := 1 + \sum_{k=1}^{\infty} e^{-\lambda_k} < \infty$ with $c > 1$ from (2), and (iii) for all $t > 0$, $(a(t)/a(x)) \to 1$ as $x \to 0+$.

Proof. (i) It follows from (1) and the regularity of $M_n$ that

$$\frac{A_{n+1}}{A_n} = \frac{1}{A_n} \left( a_1 + \sum_{k=1}^{n+1} a_k \right) \to 1.$$ 

Hence, for sufficiently large $n$ and all $k \geq 0$,

$$\frac{A_{k+1}}{A_k} \leq e^k.$$ 

(ii) Next we have, for sufficiently large $n$,

$$\frac{1}{A_n} \leq \sum_{k=1}^{n} a_k e^{-\lambda_k z_n} \leq a n \leq \frac{1}{\lambda_n} \leq A_n \left( 1 + \sum_{k=m+1}^{\infty} a_k e^{-\lambda_k z_n} \right)$$

$$\leq A_n \left( 1 + \sum_{k=m+1}^{\infty} \frac{a_k}{A_k} e^{-\lambda_k z_n} \right) \leq A_n \left( 1 + \sum_{k=m+1}^{\infty} e^{\lambda_n} e^{-\lambda_k} \right) = c_2 A_n,$$

since $(A_{k+1}/A_k) \leq e^k$ and $(\lambda_{k+1}/\lambda_m) \geq e^k$.

(iii) Since $a(x)$ decreases as $x$ increases and $c > 1$, we have, by (1), (2) and the regularity of $D_{\lambda_n}$, that, for $x > 0$,

$$\frac{1}{\lambda_n} \leq \frac{a(x)}{a(z_n)} \leq \frac{1}{a(z_n)} \sum_{k=1}^{\infty} a_k e^{-\lambda_k z_n} \geq \frac{1}{a(z_n)} \sum_{k=1}^{\infty} a_k e^{-\lambda_k z_n}$$

$$= \frac{1}{a(z_n)} \sum_{k=1}^{\infty} a_k e^{-\lambda_k} \to 1 \text{ as } x \to 0+,$$

which implies (iii).
3. Proof of Theorem 1

Put \( x_n := 1/n \) whence, in the notation of Lemma 1, \( \delta_n = o(1/n) \). Hence, by parts (i) and (ii) of Lemma 2,

\[
s_{n+1} - s_n = O \left( \frac{a_n}{A_n} \right) = O \left( \frac{a_n}{\delta_n} \right),
\]

and \( s_n \to s \) because \( s_n \to s(D_{n+1}) \). Therefore, by Lemma 1, the sequence \( (s_n) \) is bounded, and so by Lemma 2(iii) and Theorem B2, \( s_n \to s(M_n) \). It follows, by Theorem B2, that \( s_n \to s \).

Remark. The order of magnitude in our Tauberian condition (3) is best possible in the following sense: if \( (\gamma_n) \) is any sequence of positive numbers with \( \gamma_n \to \infty \), then there is a divergent sequence \( (\alpha_n) \) such that

\[
s_{n+1} - s_n = O \left( \frac{\alpha_n}{\gamma_n} \right) \quad \text{and} \quad s_n \to O(M_n),
\]

and hence \( s_n \to O(D_{n+1}) \).

This is certainly well-known and can be established by elementary means, but we have not seen a published proof.

4. Examples

In the following two examples we consider the case \( s = 0 \) of Theorem 1, and assume that each sequence \( (\alpha_n) \) satisfies (2). The order of magnitude of \( a(x) \) as \( x \to 0^{+} \) is easily determined from the observation that, by parts (i) and (ii) of Lemma 2,

\[
\frac{1}{c} A_n \leq a(x) \leq c_2 A_{n+1} \sim c_2 A_n \quad \text{when} \quad \frac{1}{\log x} \leq x \leq \frac{1}{\delta_n} \quad \text{and} \quad n \to \infty.
\]

(I) Suppose that \( \lambda_n \geq \lambda^* \) for some \( \lambda > 1 \) and all sufficiently large \( n \), and that \( a_n \sim kn^\alpha \) for some \( \alpha > -1 \). Then

\[
A_n \sim \frac{n^{1+\alpha}}{1+\alpha}
\]

and, for \( x > 0 \) and some positive constant \( M \),

\[
a(x) \leq M \left( \log \left( \frac{1}{x} \right) \right)^{-1-\alpha}.
\]

Thus, if

\[
\left( \log \left( \frac{1}{x} \right) \right)^{1-\alpha} \sum_{k=1}^{\infty} \frac{k^\alpha e^{-\lambda k}}{k} \to 0 \quad \text{as} \quad x \to 0^{+}
\]

and \( s_{n+1} - s_n = O(1/n) \), then \( s_n \to 0 \). By putting \( t = e^{-r} \), we can transform this into the following result involving power series with 'Hadamard gaps': if

\[
\left( \log \left( \frac{1}{1-t} \right) \right)^{1-\alpha} \sum_{k=1}^{\infty} \frac{k^\alpha t^k}{1-t} \to 0 \quad \text{as} \quad t \to 1^{-}
\]

and \( s_{n+1} - s_n = O(1/n) \), then \( s_n \to 0 \).

(ii) Suppose that \( \lambda_n \geq \exp(2^\alpha) \) for all sufficiently large \( n \). As above, we now have that if

\[
\frac{1}{\log \log \left( \frac{1}{t} \right)} \sum_{k=1}^{\infty} \frac{k^\alpha t^k}{1-t} \to 0 \quad \text{as} \quad t \to 1^{-}
\]

and \( s_{n+1} - s_n = O(1/n) \), then \( s_n \to 0 \).

References


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