ON PRODUCTS OF SEQUENCES

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1. Introduction.

It is well known (Hardy [3], 286) that if \( \sigma > -1, \lambda > -1 \) and

\[
\sum_{n=0}^{\infty} w_n = \sigma (C, \sigma, \lambda), \quad \sum_{n=0}^{\infty} v_n = \tau (C, \lambda),
\]

and if the sequence \( \{w_n\} \) is the Cauchy product of the sequences \( \{u_n\}, \{v_n\} \),

i.e. \( w_n = \sum_{k=0}^{n} u_k v_{n-k} \), then

\[
\sum_{n=0}^{\infty} w_n = \sigma \tau (C, \sigma + \lambda + 1).
\]

Now put

\[
\alpha_n = \sum_{n=0}^{\infty} \alpha_n, \quad \beta_n = \sum_{n=0}^{\infty} \beta_n, \quad \gamma_n = \sum_{n=0}^{\infty} \gamma_n,
\]

Then

\[
\frac{1}{n+1} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n = \frac{1}{n+1} \sum_{n=0}^{\infty} \gamma_n,
\]

and, in consequence of this and a well-known property of Cesaro means, we obtain

**Theorem A.** If \( \sigma > -1, \lambda > -1, \sigma + \lambda > -1 \), and \( \sigma_n \rightarrow \sigma (C, \sigma) \),

\( \alpha_n \rightarrow \alpha (C, \lambda) \),

then

\[
\frac{1}{n+1} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n \rightarrow \sigma (C, \sigma + \lambda).
\]

This theorem is concerned with the Cesaro method of summability and the Cauchy product of the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \). The object of this paper is to obtain results of this type involving other methods of summability and products more general than the Cauchy product.

2. Notation, definitions and preliminary results.

Suppose throughout that \( \sigma, \tau \) are arbitrary complex numbers and that

\( \{\alpha_n\}, \{\beta_n\} (n = 0, 1, \ldots) \) are arbitrary sequences of complex numbers.

Given two summability methods \( P \) and \( Q \), \( P \) is said to **include** \( Q \) and we write \( P \supseteq Q \) if \( \alpha_n \rightarrow \sigma (Q) \) whenever \( \alpha_n \rightarrow \sigma (Q) \). If \( P = Q \) or \( Q \subseteq P \), \( P \) and \( Q \) are said to be **equivalent**, and we write \( P \approx Q \). The method \( P \) is said to be **regular** if \( \alpha_n \rightarrow \sigma (P) \) whenever \( \alpha_n \rightarrow \sigma \).

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ON PRODUCTS OF SEQUENCES. 333

Given sequences \( a = (a_n), b = (b_n) (n = 0, 1, \ldots) \), we denote by \( a \ast b \) the sequence \( \{c_n\} \), where

\[
c_n = \sum_{n=0}^{\infty} a_{n+m} b_m,
\]

and we write \( (a \ast b) \rightarrow c \). Further, we write

\[
a(x) = \sum_{n=0}^{\infty} a_n x^n,
\]

and denote the radius of convergence of the power series by \( \rho_a \); we also use this notation with \( b \) and \( c \) in place of \( a \).

We define next two methods of summability of which the second is new.

**The power series method** \( (f, a) \) (see Hardy [3], 79-81). Let \( a = (a_n) \) be a sequence of real non-negative numbers, not all zero, such that \( \rho_a > 0 \). We write \( \alpha_n \rightarrow \sigma (f, a) \) if, as \( x \rightarrow \rho_a \), in the open interval \( (0, \rho_a) \),

\[
\frac{1}{a(x)} \sum_{n=0}^{\infty} \alpha_n x^n \rightarrow \sigma.
\]

The **generalized Nörlund method** \( (N, a, b) \). Let \( a = (a_n), b = (b_n) \) be sequences of real numbers such that \( \alpha_n = (a \ast b) \neq 0 \). Given a sequence \( \{\alpha_n\} \), we define the sequence \( \{\beta_n\} \) of its \( (N, a, b) \) means by the relation

\[
\beta_n = \frac{1}{n+1} \sum_{n=0}^{\infty} \alpha_n \beta_n \gamma_n;
\]

and we write \( \alpha_n \rightarrow \sigma (N, a, b) \) if \( \beta_n \rightarrow \sigma \).

The method \( (N, a, b) \) reduces to the ordinary Nörlund method \( (N, a) \) (Hardy [3], 64) when \( b_n = 1 \), and to the method \( (N, b) \) (Hardy [3], 57) when \( a_n = 1 \). Further, when \( a_n = a^n \) and \( b_n = b^n \) \((a > 0, \beta > 0)\) we find that \( (N, a, b) \) is equivalent to the Euler-Knopp method \( (E, a, b) \) (E2, 180).

We say that the method \( (N, a, b) \) is **regular** if both \((N, a, b)\) and \((N, b, a)\) are regular.

The following two results are consequences respectively of Toeplitz' theorem (Hardy [3], 43) and a simple extension of part of this theorem (Knopp [4], 73).

1. Necessary and sufficient conditions for the method \( (N, a, b) \) to be regular are

(i) \( \sum_{n=0}^{\infty} |a_n| |b_n| < H |c_n| \), where \( H \) is a positive number independent of \( n \);

(ii) for each integer \( x \geq 0 \), \( a_{n+m} |b_m| \rightarrow 0 \) as \( n \rightarrow \infty \).
II. If \( a_n \to a, \ l_n \to \sigma \) and the method \( (N, a, b) \) is bi-regular, then, as \( n \to \infty \),

\[
\frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} \to \sigma r.
\]

3. The main theorems.

**Theorem 1.** If

(i) \( a, b, c, k, l, m \) are sequences of real numbers such that \( c = a * b \), \( m = k * l \),

(ii) \( a_n \neq 0, (k * a)_n \neq 0, (l * b)_n \neq 0, (m * c)_n \neq 0 \),

(iii) the method \( (N, k a, l b) \) is bi-regular,

(iv) \( a_n \to c (N, k a, l b) \), then

\[
u_n = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} \to \sigma (N, m, c).
\]

Proof. Let

\[
f = k * a, \ g = l * b, \ h = f * g,
\]

\[
e_{n} = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n}, \ l'_{n} = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n}, \ v_{n} = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} e_{n}.
\]

We now have the formal identities

\[
\sum_{n=0}^{m} a_{k_n} b_{l_n} = \sum_{n=0}^{m} e_{n} e_{n}', \ \sum_{n=0}^{m} a_{k_n} b_{l_n} = \sum_{n=0}^{m} l_{n} k_{n} a_{n} b_{n}
\]

\[
= \sum_{n=0}^{m} l_{n} k_{n} a_{n} b_{n}, \ \sum_{n=0}^{m} a_{k_n} b_{l_n} = \sum_{n=0}^{m} l_{n} k_{n} a_{n} b_{n}, \ \sum_{n=0}^{m} l_{n} k_{n} a_{n} b_{n}
\]

from which we deduce that

\[
u_n = \frac{1}{c_n} \sum_{n=0}^{m} v_{n} e_{n}.
\]

Similarly we find that

\[b_n = (m * c)_n \neq 0.
\]

Now, by hypothesis (iv), \( e_{n} \to c, l'_{n} \to \sigma \) and consequently, in virtue of II and hypothesis (iii),

\[
u_n b_n \to \sigma r.
\]

Hence, \( u_n \to \sigma (N, m, c) \), and the proof is complete.

\footnote{Cf. Messrs \([5], \text{Theorem 2}\).}

**Theorem 2.** If \( a, b, c \) are sequences of real non-negative numbers such that \( e_{n} = (a * b)_n \to 0 \) and \( p_n = p \to 0 \). \( i \), \( a_n \to a (J, a), l_n \to \tau (J, b) \), then

\[
u_n = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} \to \sigma (J, c).
\]

Proof. Note that, for \( |x| < \rho \),

\[
c(x) = c(a(x) \ b(x)),
\]

so that, by a familiar result concerning the singularities of a power series on its circle of convergence, \( p_n \to 0 \).

Further, in view of hypothesis (ii), we have for \( |x| < \rho \),

\[
\sum_{n=0}^{m} a_{k_n} b_{l_n} x^n = \sum_{n=0}^{m} a_{k_n} b_{l_n} x^n, \ \sum_{n=0}^{m} b_{l_n} x^n.
\]

The theorem follows.

3. Special cases.

We proceed now to obtain corollaries of the main theorems by considering special cases of the methods \( (N, a, b) \) and \( (J, a) \).

For convenience we denote the binomial coefficient \( \binom{\alpha + \beta}{\alpha} \) by \( e_{\alpha}^\beta \).

Note that, if \( d_n - e_{\alpha}^\beta, d'_n - e_{\alpha}^\beta \), then \( (d * d')_n = e_{\alpha+eta}^{\alpha+eta} \).

**The Centre method \((C, \epsilon)\).** The definition of this method is standard for the range \( \epsilon > -1 \) and various equivalent definitions have been given for the range \( \epsilon < -1 \) (see Borwein \([2]\) for references).

For \( k_n = e_{\epsilon}^{\epsilon}, a_n = e_{\epsilon}^{\epsilon}, \) we denote the method \( (N, k_n, a_n) \) by \( (C, \epsilon, a) \).

We then have the following result (proved in \([3]\)).

**Lemma.** If \( \alpha > -1, \ k + \alpha > -1, \) then \( (C, \epsilon, a) \Rightarrow (C, \epsilon, a) \).

The next theorem generalizes Theorem A.

**Theorem 3.** If \( (i) \alpha > -1, \beta > -1, \ k + \alpha > -1, \lambda + \beta > -1, \)

(ii) \( a_n \to a \ (J, a), \ l_n \to \tau \ (J, b), \) then

\[
u_n = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} \to \sigma (C, \epsilon, \lambda).
\]

Proof. Let \( a_n = e_{\lambda}^{\lambda}, \ b_n = e_{\beta}^{\beta}, \ c_n = e_{\alpha}^{\alpha}, \ k_n = e_{\lambda}^{k}, \ l_n = e_{\beta}^{l}, \ m_n = e_{\alpha}^{m} \), so that

\[
c = a * b, \ m = k * l,
\]

and

\[
u_n = \frac{1}{c_n} \sum_{n=0}^{m} a_{k_n} b_{l_n} \to \sigma.
\]

Then, in virtue of hypothesis (i),

\[
\epsilon_n > 0, \ (k * a)_n = e_{\lambda}^{k+\alpha} > 0, \ (l * b)_n = e_{\beta}^{l+\beta} > 0, \ (m * c)_n = e_{\alpha+eta}^{\alpha+eta} > 0.
\]
and, by the Lemma,
\[(N, k, a) \simeq (C, \kappa), \quad (N, l, b) \simeq (C, \lambda), \quad (N, m, c) \simeq (C, \kappa + \lambda),\]
\[(N, k \ast a, l \ast b) \simeq (C, \kappa + \lambda + 1), \quad (N, l \ast b, k \ast a) \simeq (C, \kappa + \lambda + 1 + \frac{1}{2}).\]
Since \(\kappa + \kappa + 1 > 0, \lambda + \lambda + 1 > 0\), it follows from the final two equivalences that \((N, k \ast a, l \ast b)\) is bi-regular.

The theorem is now an immediate consequence of Theorem 1.

The Dueter-Knopp method \((E, \lambda)\). Suppose that \(\lambda > 0, \delta > 0\), and recall that the sequence \(\{a_n^+\}\) of \((E, \lambda)\) means a sequence \(\{a_n\}\) is given by
\[a_n^+ = (\lambda + \beta)^{n-1}\sum_{\omega=0}^{n} \frac{\beta^\omega}{\omega!} a_n;\]
so that, if \(i_n = (\delta + \beta)^{n-1}\), then \((N, i, b) \simeq (E, \lambda),\)

**Theorem 4.** If \((i) \kappa > 0, \lambda > 0, \delta > 0, (ii) a_n \to c, k \to \tau (E, \lambda),\)
then \(u_n = (\beta + \delta)^{n-1}\sum_{\omega=0}^{n} \frac{\beta^\omega}{\omega!} a_n \to \tau (E, \lambda);\)

Proof. Let \(a_n = \lambda^{|n|}, \quad b_n = \beta^{|n|}, \quad c_n = (\delta + \beta)^{|n|}, \quad k_n = \kappa^{|n|},\)
\(l_n = (\delta + \beta)^{|n|}, \quad m_n = (\delta + \kappa + \beta)^{|n|}, \quad \text{so that}\)
\(a = a^+ \ast b, \quad m = k \ast l, \quad \text{and}\)
\(u_n = \lambda^{|n|} \sum_{\omega=0}^{n} \frac{\beta^\omega}{\omega!} a_n \to \tau (E, \lambda);\)
Then
\(c_n > 0, \quad (k \ast b)_n = (\kappa + \kappa + 1)^{|n|} > 0, \quad (l \ast c)_n = (\beta + \kappa + \beta + 1)^{|n|} > 0,\)
Further,
\[(N, k, a) \simeq (E, \kappa), \quad (N, l, b) \simeq (E, \lambda), \quad (N, m, c) \simeq (E, \kappa + \lambda),\]
and \((N, k \ast a, l \ast b) \simeq (E, \kappa + \lambda + 1),\)
\[(N, l \ast b, k \ast a) \simeq (E, \kappa + \lambda + 1).\]
Since \((E, \lambda)\) is regular for \(\lambda > 0\), it follows from the final two equivalences that \((N, k \ast a, l \ast b)\) is bi-regular.

We now complete the proof by appealing to Theorem 1.

The method \(A\). Suppose that \(a > -1\) and \(a_n = a_n^+\), so that \(a_n \to 1\) and \(a(x) = (1 - x)^{-1}\); and denote the method \((J, a)\) by \(A\). Then \(A\)
is the Abel method. It has been proved elsewhere (Borwein [1]) that if \(\beta > (1 - a \gamma) \geq 0\), then \(A_\beta \simeq A_\gamma \simeq (C, \gamma).

The next theorem is a simple corollary of Theorem 2.

**Theorem 5.** If \(a > -1, \beta > -1\) and \(a_n \to \tau (A_\beta), \quad t_n \to \tau (A_\gamma),\)
then
\[1 < \frac{1}{\lambda + \lambda + 1} \sum_{\omega=0}^{n} \frac{\beta^\omega}{\omega!} a_n \to \tau (A_{\lambda + \lambda + 1}).\]

The Borel exponential method \(B\). Suppose that \(\delta > 0, \quad a_n = 1/|n|, \quad b_n = \beta^{|n|}, \quad c_n = (\delta + \beta)^{|n|}, \quad \text{so that} \quad c = a \ast b, \quad \gamma = \lambda = \infty \quad \text{and} \quad u(x) = e^x, \quad \beta(x) = e^x \to \delta^{|x|}.\)
We recall that the Borel method \(B\) is in fact the method \((J, a)\). Clearly, \(B \simeq (J, \beta) \simeq (J, \lambda).\)

Hence, as a special case of Theorem 2, we obtain

**Theorem 6.** If \(a_n \to \tau (B), \quad t_n \to \tau (B),\)
then, for any \(\delta > 0, \quad (\delta + 1)^{-1} \sum_{\omega=0}^{n} \frac{\beta^\omega}{\omega!} a_n \to \tau (B).\)

References.

4. K. Knopp, "Infinite series" (Glasgow, 1945).

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