WEIGHTED CONVOLUTION OPERATORS ON $\ell_p$

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Abstract. The main results deal with conditions for the validity of the weighted convolution inequality
\[ \left| b_n \sum_{k \in \mathbb{Z}} a_{n-k} x_k \right|^{q} \leq C \sum_{k \in \mathbb{Z}} \left| x_k \right|^{q} \]
when $p \geq 1$.

1. Introduction and main result.

We suppose throughout that $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq r \leq \infty$, $\frac{1}{r} + \frac{1}{s} = 1$,
and observe the convention that $q = \infty$ when $p = 1$.

Given a two-sided complex sequence $x = (x_n)_{n \in \mathbb{Z}}$, we define
\[ \|x\|_p := \left( \sum_{k \in \mathbb{Z}} |x_k|^p \right)^{1/p} \]
for $1 \leq p < \infty$, and $\|x\|_\infty := \sup_{n \in \mathbb{Z}} |x_n|$;
and we say that $x \in \ell_p$ if $\|x\|_p < \infty$. Given a two-sided complex sequence $a = (a_n)$ and a two-sided complex sequence $b = (b_n)$ of weights, we define the weighted convolution linear transformation $y = (y_n) = \lambda x$ by
\[ y_n := (\lambda x)_n := b_n \sum_{k \in \mathbb{Z}} a_{n-k} x_k, \]
and aim to obtain sufficient conditions for $\lambda$ to be a bounded operator on $\ell_p$. In other words, our objective is to establish conditions under which there is a positive constant $C$ such that, for all $x \in \ell_p$,
\[ \|y\|_p \leq C \|x\|_p, \]
in which case the operator norm of $\lambda$, defined as $\|\lambda\|_p := \sup_{\|x\|_p \leq 1} \|\lambda x\|_p \leq C$. When $1 \leq p < \infty$, (1) amounts to
\[ \sum_{n \in \mathbb{Z}} \left| b_n \sum_{k \in \mathbb{Z}} a_{n-k} x_k \right|^p \leq C^p \sum_{k \in \mathbb{Z}} |x_k|^p. \]

Our main result is the following

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Theorem. If $1 \leq p \leq \infty$, $1 \leq r \leq q$, $a \in \ell_r$, $b \in \ell_s$, then (1) holds for all $x \in \ell_p$ with $C = \|a\|_r \|b\|_s$.

Note that all the above concerns two-sided sequences. The situation is very different when one-sided sequences are considered. This amounts to having $a_n = b_n = x_n = 0$ for all $n < 0$. In this case (2) reduces to

$$\sum_{n=0}^{\infty} \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right|^p \leq C^p \sum_{k=0}^{\infty} |x_k|^p,$$

and when $a_n \geq 0$, $A_n := a_0 + a_1 + \cdots + a_n > 0$ for $n \geq 0$, and $b_n := \frac{1}{A_n}$ for $n \geq 0$ we get the following known proposition about the Nörlund transform (see [1, Theorem 2] or [2, Theorem 1]).

Proposition. If $1 < p < \infty$ and $n a_n = O(A_n)$ as $n \to \infty$, then there is a positive constant $C$ such that

$$\sum_{n=0}^{\infty} \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right|^p \leq C^p \sum_{k=0}^{\infty} |x_k|^p.$$

2. Lemmas. We prove two lemmas.

Lemma 1. If $1 < p < \infty$ and $\sum_{k \in \mathbb{Z}} c_k x_k$ is convergent whenever $\sum_{k \in \mathbb{Z}} |x_k|^p < \infty$, then

$$\sum_{k \in \mathbb{Z}} |c_k|^q < \infty.$$

Proof. A version of this result with the stronger hypothesis that $\sum_{k \in \mathbb{Z}} c_k x_k$ is absolutely convergent whenever $x \in \ell_p$ appears as a problem in [3, p. 198, Problem 7] where $\ell_q$ is referred to as being the Kőthe-Toeplitz dual of $\ell_p$. It may well be that the result as stated is also known. We offer the following elementary non-functional analytic proof. The hypothesis is equivalent to the pair of statements:

$$\sum_{k=0}^{\infty} c_k x_k \text{ is convergent whenever } \sum_{k=0}^{\infty} |x_k|^p < \infty,$$

and

$$\sum_{k=1}^{\infty} c_k x_k \text{ is convergent whenever } \sum_{k=1}^{\infty} |x_k|^p < \infty.$$

Suppose $\sum_{k=0}^{\infty} |c_k|^q = \infty$. Let $D_n := \sum_{k=0}^{n} |c_k|^q$. Assume without loss in generality that $D_0 > 0$, and take

$$x_k := \begin{cases} \frac{|c_k|^{q-1} |c_k|}{D_k} & \text{when } c_k \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

and when $a_n \geq 0$, $A_n := a_0 + a_1 + \cdots + a_n > 0$ for $n \geq 0$, and $b_n := \frac{1}{A_n}$ for $n \geq 0$ we get the following known proposition about the Nörlund transform (see [1, Theorem 2] or [2, Theorem 1]).

Proposition. If $1 < p < \infty$ and $n a_n = O(A_n)$ as $n \to \infty$, then there is a positive constant $C$ such that

$$\sum_{n=0}^{\infty} \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right|^p \leq C^p \sum_{k=0}^{\infty} |x_k|^p.$$
Then, by the Abel-Dini theorem,
\[ \sum_{k=0}^{\infty} c_k x_k = \sum_{k=0}^{\infty} \frac{|c_k|^q}{d_k} = \infty \quad \text{while} \quad \sum_{k=0}^{\infty} |x_k|^p = \sum_{k=0}^{\infty} \frac{|c_k|^q}{d_k^p} < \infty, \]
contrary to hypothesis. Thus we must have \( \sum_{k=1}^{\infty} |c_k|^q < \infty \), and likewise \( \sum_{k=1}^{\infty} |c_{-k}|^q < \infty \).

**Lemma 2.** If \( 1 \leq p < \infty, 1 < r \leq q \), and some finite \( t \geq 1 \) is such that
\[
\sum_{n \in \mathbb{Z}} \left( \left| \sum_{k \in \mathbb{Z}} a_{n-k} x_k \right|^p \right) < \infty
\]
whenever \( a \in \ell_r, x \in \ell_p, b \in \ell_t \), then \( t \leq s \).

**Proof.** Suppose, to the contrary, that \( t > s \), and let \( 3 \varepsilon := \frac{1}{s} - \frac{1}{t} \). Let
\[
a_n := \begin{cases} (n+1)^{-\frac{1}{t} - \varepsilon} & \text{for } n \geq 0 \\ 0 & \text{otherwise}, \end{cases}
\]
\[
x_n := \begin{cases} (n+1)^{-\frac{1}{p} - \varepsilon} & \text{for } n \geq 0 \\ 0 & \text{otherwise}, \end{cases}
\]
\[
b_n := \begin{cases} (n+1)^{-\frac{1}{r} - \varepsilon} & \text{for } n \geq 0 \\ 0 & \text{otherwise}. \end{cases}
\]
Then \( a \in \ell_r, x \in \ell_p, b \in \ell_t \), but
\[
\sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} a_{n-k} x_k \right)^p = \sum_{n=0}^{\infty} \left( (n+1)^{-\frac{1}{r} - \varepsilon} \sum_{k=0}^{n} (n+1-k)^{-\frac{1}{r} - \varepsilon} (k+1)^{-\frac{1}{r} - \varepsilon} \right)^p \geq \sum_{n=0}^{\infty} \left( (n+1)^{-\frac{1}{r} - \varepsilon} (n+1)^{-\frac{1}{r} - \varepsilon} (n+1)^{-\frac{1}{r} - \varepsilon} \right)^p = \sum_{n=0}^{\infty} (n+1)^{-1} = \infty.
\]

**3 Proof of the Theorem.**

**Case 1.** \( 1 < p < \infty \). For inequality (2) to be meaningful and non-trivial, observe that, for any \( n \) for which \( b_n \neq 0 \), \( \sum_{k \in \mathbb{Z}} a_{n-k} x_k \) has to be convergent whenever \( \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \). It thus follows from Lemma 1 that we must have \( \sum_{k \in \mathbb{Z}} |a_{n-k}|^q = \)}
\[ \sum_{k \in \mathbb{Z}} |a_k|^q < \infty. \] This explains why we make the restriction \( 1 \leq r \leq q \) in the hypothesis, and Lemma 2 shows why it is not sufficient to require \( b \in \ell_t \) for any \( t > s \).

An application of Hölder's inequality yields

\[
\left| \sum_{k \in \mathbb{Z}} a_{n-k}x_k \right|^p \leq \|a\|_r^{(p-1)} \sum_{k \in \mathbb{Z}} |a_{n-k}|^{(q-r)(p-1)} |x_k|^p,
\]

and hence that

\[
\sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} a_{n-k}x_k \right|^p \leq \|a\|_r^{(p-1)} \|x\|_p \sum_{n \in \mathbb{Z}} |b_n| |a_{n-k}|^{(q-r)(p-1)} \|b\|_s^p.
\]

since \( \|x\|_p = \sum_{k \in \mathbb{Z}} |x_k|^p < \infty \) and \( \|b\|_s = \sum_{n \in \mathbb{Z}} |b_n|^s < \infty \), and this establishes (1) with

\[ C = \|a\|_r \|b\|_s. \]

Note that Hölder’s inequality with \( \tilde{r} = \frac{r}{q-r}(p-1), \tilde{s} = \frac{s}{p} \) is used in the penultimate step above.

Case 2. \( p = 1 \), \( q = \infty \) or \( p = \infty \), \( q = 1 \). When \( p = 1 \) the result follows by changing the order of summation in (2) and then applying Hölder’s inequality, and when \( p = \infty \) the desired conclusion is even more immediate.

We have shown that if \( 1 \leq p < \infty \), \( 1 < r \leq q \), \( a \in \ell_r \), then (2) holds for all \( x \in \ell_p \) provided \( b \in \ell_s \), but may fail to hold if \( b \in \ell_t \) with a finite \( t > s \). In the following section we show by means of an example that, if \( 1 < p < \infty \), then (2) may hold for all \( x \in \ell_p \) when \( b \not\in \ell_t \) for any finite \( t > 1 \).

4. Example.

Suppose \( 1 < p < \infty \). Let \( A_n := a_0 + a_1 + \cdots + a_n \) for \( n \geq 0 \), where

\[ a_n := \begin{cases} 
\frac{1}{n+1} & \text{for } n \geq 0 \\
0 & \text{otherwise,} \end{cases} \]

let

\[ b_n := \begin{cases} 
\frac{1}{A_n} & \text{for } n \geq 0 \\
0 & \text{otherwise,} \end{cases} \]

and let

\[ y_n := \left| \sum_{k \in \mathbb{Z}} a_{n-k}x_k \right| = \left| \sum_{k=0}^{\infty} a_k x_{n-k} \right| \leq y_{1,n} + y_{2,n}, \]
where
\[ y_{1,n} := \left| \frac{1}{A_n} \sum_{k=0}^{n} a_k x_{n-k} \right| \quad \text{and} \quad y_{2,n} := \left| \frac{1}{A_n} \sum_{k=n+1}^{\infty} a_k x_{n-k} \right|. \]

Note that \( \sum_{k \in \mathbb{Z}} |a_k| = \infty \) and \( \|a\|_r = \sum_{k \in \mathbb{Z}} |a_k|^r < \infty \) for all \( r > 1 \). Suppose that the sequence \( x = (x_n) \in \ell_p \). Since \( A_n \sim \log n \) and \( na_n A_n \sim 1 \log n = O(1) \) as \( n \to \infty \), it follows from the Proposition that
\[ \sum_{n=0}^{\infty} y_{1,n}^p \leq C_1 \sum_{k=0}^{\infty} |x_k|^p \leq C_1 \|x\|_p^p. \]

Further, by Hölder’s inequality,
\[ \sum_{n=0}^{\infty} y_{2,n}^p \leq \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p} \left( \sum_{k=n+1}^{\infty} a_k^p \right)^{p-1} \leq \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p} \left( \int_{n+1}^{\infty} \frac{dt}{t^q} \right)^{p-1} = (q-1)^{-p} \|x\|_p^p \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} = C_2 \|x\|_p^p, \]

where \( C_2 = (q-1)^{-p} \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} < \infty. \) Hence
\[ \sum_{n \in \mathbb{Z}} y_n^p = \sum_{n=0}^{\infty} y_n^p \leq 2^p \sum_{n=0}^{\infty} (y_{1,n}^p + y_{2,n}^p) \leq 2^p (C_1 + C_2) \|x\|_p^p. \]

Thus (2) is satisfied but \( b \not\in \ell_t \) for any finite \( t > 1 \), since \( \|b\|_t^t = \sum_{n=0}^{\infty} \frac{1}{A_n^t} = \infty. \)

A similar but slightly more complicated argument can be used to show that we could get the same result by taking, for any real \( \alpha \),
\[ a_n := \begin{cases} \frac{\log^\alpha (n+1)}{n+1} & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
in the example.

References
