ON METHODS OF SUMMABILITY BASED ON
INTEGRAL FUNCTIONS

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1. Introduction. Suppose throughout that

\[ p_n \geq 0, \quad \sum_{r=n}^{\infty} p_r > 0 \quad (n = 0, 1, \ldots) \]

and that

\[ p(x) = \sum_{n=0}^{\infty} p_n x^n \]

is an integral function. Suppose also that \( l, \ s_n (n = 0, 1, \ldots) \) are arbitrary complex numbers and denote by \( \rho(p_s) \) the radius of convergence of the series

\[ \sum_{n=0}^{\infty} p_n s_n x^n. \]

If \( \rho(p_s) = \rho > 0 \) and there is a function \( p^+_s(x) \) such that

- \( p^+_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \quad (0 \leq x < \rho) \),

- \( p^+_s(x) \) is analytic for all positive \( x \),

- \( p^+_s(x)/p(x) \to l \) when \( x \to \infty \) (through real values),

we write

\[ s_n \to l(P^s). \]

If \( \rho(p_s) = \infty \) and \( s_n \to l(P^s) \), we write

\[ s_n \to l(P). \]

This defines the IF (integral function) methods of summability \( P^s, P \). It is known (\S, p. 80) that \( P \) is regular, i.e. \( s_n \to l(P) \) whenever \( s_n \to l \); consequently \( P^s \) is also regular.

Suppose in what follows that

\[ \mu_n > 0, \quad g_n = p_n/\mu_n \quad (n = 0, 1, \ldots). \]

Let \( \rho(g) \) be the radius of convergence of

\[ \sum_{n=0}^{\infty} g_n x^n = g(x), \]

and, whenever \( \rho(g) = \infty \), denote the IF methods associated with the sequence \( \{g_n\} \) by \( Q^s, Q \). Suppose also that \( N \) is an arbitrary non-negative integer.
The following theorem has been proved elsewhere (Borwein (1)).

**Theorem A.** If

\[ p_n = \int_0^\infty e^\tau \, d\chi(t) \geq \frac{\delta}{\alpha} \int_0^\infty e^\tau \, d\chi(t) > 0 \quad (\alpha > 0, \, n \geq N), \]

where \( \chi(t) \) is a real function of bounded variation in \([0, 1]\), then \( \rho(q) = \infty \) and \( s_n \to I(\mathcal{Q}) \) whenever \( s_n \to I(\mathcal{Q}) \).

In this paper two further theorems of the same type are proved and are used in conjunction with Theorem A to obtain inclusion relations between some special I\( \mathcal{F} \) methods of summability. The first of these theorems is

**Theorem 1.** If \( \chi(t) \) is a real function of bounded variation in \((0, \infty)\) such that

\[ \infty > \int_0^\infty e^\tau \, d\chi(t) \geq \frac{\delta}{\alpha} \int_0^\infty e^\tau \, d\chi(t) > 0 \quad (\alpha > 0, \, n \geq N), \]

and if

\[ \rho_n = \int_0^\infty e^\tau \, d\chi(t) \quad (n \geq N), \]

then \( \rho(q) = \infty \).

If, in addition, \( s_n \to I(\mathcal{Q}) \) and \( \rho(p) = \infty \), then \( s_n \to I(\mathcal{Q}) \).

Before formulating the second theorem we define a class \( \Omega \) of functions \( \phi(q) \) as follows:

\( \phi \in \Omega \) if there are positive numbers \( \Delta, \delta \), and a non-negative integer \( N \) such that

\( (\Omega_1) \phi(q) \) is analytic in the region \(|z| > 0, -\Delta < \arg z < \Delta \); and, when \( \tau \to 0 +, T \to \infty \), the integrals

\[ \int_0^\infty \left| \frac{\phi(t + \alpha)}{t} \right| \, dt, \quad \int_0^\infty \left| \frac{\phi(t + \alpha)}{t} \right| \, dt \]

tend to finite limits uniformly in the interval \(-\Delta < \theta < \Delta \);

\( (\Omega_2) \phi(q) \) is real for \( t > 0 \) and

\[ \infty > \int_0^\infty \left| \frac{\phi(t + \alpha)}{t} \right| \, dt \geq \int_0^\infty \left| \frac{\phi(t + \alpha)}{t} \right| \, dt > 0 \quad (n \geq N). \]

**Theorem 2.** If \( \phi \in \Omega \) and

\[ \rho_n = \int_0^\infty e^\tau \, d\chi(t) \quad (n \geq N), \]

then \( \rho(q) = \infty \).

If, in addition, \( s_n \to I(\mathcal{Q}) \) and \( \rho(p) > 0 \), then \( s_n \to I(\mathcal{P}) \).

2. **Proofs of Theorems 1 and 2.** Suppose in what follows that

\[ p_n - \sum_{n=0}^\infty \alpha^n q_n = 0 \quad (0 \leq n < N); \]

it is evident that this leads to no loss in generality in either theorem.

Let \( \chi(t), \rho_n \) satisfy the hypotheses of Theorem 1. Then, since

\[ \infty > \int_0^\infty e^\tau \, d\chi(t) > 0, \]

there is a number \( a > 0 \) such that

\[ \infty > K = \frac{\delta}{\alpha} \int_0^\infty |d\chi(t)| > 0, \]

and so

\[ p_n = \delta \int_0^\infty e^\tau \, d\chi(t) > K a^\alpha q_n > 0. \]

Suppose now that

\[ q_n (x) = \sum_{n=0}^\infty \alpha^n q_n \]

is an integral function and that \( \rho(q) = \infty \). Then

\[ \sum_{n=0}^\infty p_n \alpha^n q_n = \sum_{n=0}^\infty \alpha^n \phi(q_n) \]

is the inversion being legitimate since, for \( 0 < \alpha < \rho \),

\[ \infty > \delta \int_0^\infty \sum_{n=0}^\infty \alpha^n q_n \int_0^\infty e^\tau \, d\chi(t) = \int_0^\infty e^\tau \, d\chi(t) \sum_{n=0}^\infty \alpha^n q_n \alpha^n. \]

Further, taking \( \alpha_n = 1 \), we get

\[ p(x) = \int_0^\infty e^\tau \, d\chi(t) \geq \delta \int_0^\infty e^\tau \, d\chi(t) \quad (x > 0). \]

**Proof of Theorem 1.** Since \( p(x) \) is an integral function, it follows from (1) that \( \rho(q) = \infty \). We now suppose that \( s_n \to I(\mathcal{Q}) \) and that \( \rho(p) = \rho \). In view of (3) we have, for \( x > a > 0 \),

\[ \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) = \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) - \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) \]

\[ \leq \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) \sum_{n=0}^\infty \alpha^n q_n \]

\[ \leq \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) \sum_{n=0}^\infty q_n \]

Since \( p(x) \to \infty \) and \( q_n (x) \to \infty \) when \( x \to \infty \), it follows that

\[ \lim_{x \to \infty} \frac{1}{p(x)} \int_0^\infty e^\tau \, d\chi(t) = 1. \]

and hence, by (3), that \( s_n \to I(\mathcal{P}) \).

**Proof of Theorem 2.** In view of conditions \((\Omega_1) \) on \( \phi \), the hypotheses of Theorem 1 are satisfied by

\( \chi(t) = \int_0^t \phi(s) \, ds \quad (t > 0), \)

and consequently, as above, \( \rho(q) = \infty \). Suppose that \( s_n \to I(\mathcal{Q}) \) and that \( \rho(p) = \rho > 0 \). Then, by (2),

\[ \sum_{n=0}^\infty p_n \alpha^n q_n = \sum_{n=0}^\infty q_n \phi(q_n) \]

Since \( \phi \) satisfies conditions \((\Omega_1) \) and \( q_n (t) \) is continuous and bounded for \( t > 0 \), it is easily seen that, when \( \tau \to 0 +, T \to \infty \), the integrals

\[ \int_0^\infty e^\tau \, d\chi(t) \]

are finite.
tend to finite limits uniformly in the region $-\Delta < \arg z < \Delta$, $C > |z| > 0$, where $C$ is any positive number. Hence, by standard results, the function

$$
\psi_\alpha(z) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty q(t) \phi(t) \frac{dt}{t^\alpha}
$$

is analytic in the region $|z| > 0$, $-\Delta < \arg z < \Delta$. Also, for $x > 0$,

$$
\psi_\alpha(x) = \int_0^x q(t) \phi(t) \frac{dt}{t^\alpha},
$$

and so, as in the proof of Theorem 1,

$$
\lim_{x \to +\infty} \psi_\alpha(x) = I.
$$

It follows that $L \to I(P^+)$.

3. Lemma. We first show that

$$
\phi(z) = \gamma e^{-\gamma z} (x > 0, \beta > 0, \gamma > 0)
$$

is in the class $\Omega$, it being assumed that for real $\lambda, \beta = |z| e^{i\theta}$ where $\theta$ is the principal value of $\arg z$.

It is evident that $\phi$ satisfies conditions $(\Omega_2)$. Also, $\phi(x)$ is analytic in the region $|z| > 0$, $\arg z < \pi/3 < \Delta$ and, for $t > 0$, $-\Delta < \arg t < \Delta$,

$$
|\phi(xy)| = e^{-\gamma x} \frac{\Gamma(\beta x + 1)}{\Gamma(\beta x + 1)} = \gamma e^{-\gamma x} M(x)
$$

say. Since $\int_0^\infty M(t) dt < \infty$, we see that $\phi$ also satisfies conditions $(\Omega_3)$ and so $\phi \in \Omega$.

Next we prove three lemmas.

**Lemma 1.** If $x > 0$, $y > 0$, $\beta > 0$ (for $\gamma = 1, 2, 3, \ldots$), then

$$
\mu_n = \frac{1}{\Gamma(\alpha + \beta)} \Gamma(\alpha + \beta) (n = 0, 1, \ldots)
$$

satisfies the conditions of Theorem 1.

**Proof.** Let $a > 0$, $\gamma > 0$; then

$$
\lambda = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \gamma)} = \frac{1}{\alpha \Gamma(\gamma - \beta)} \int_0^\infty e^{-\lambda t} \Gamma(\alpha + \beta + 1) dt
$$

is totally monotone. It follows (68), (71), (79) that $\mu_n$ is totally monotone and hence that it satisfies the conditions of Theorem 1.

**Lemma 2.** If $x > 0$, $\beta > 0$, $1 > \gamma > 0$, then

$$
\mu_n = \frac{1}{\Gamma(\alpha + \beta)} \Gamma(\alpha + \beta) (n = 0, 1, \ldots)
$$

satisfies the conditions of Theorem 1.

**Proof.** It has been shown by Good (3) that

$$
\mu_n = \int_0^\infty e^{-\lambda t} \phi(t) \frac{dt}{t^\alpha}
$$

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where

$$
\sum_{n=0}^\infty (-1)^n x^n \frac{\Gamma(\alpha + 1)}{\Gamma(n + 1)} \sin \frac{\pi n x}{\alpha + \beta + \alpha}
$$

the series being convergent for all $t > 0$. When $t \to 0^+$,

$$
\sum_{n=0}^\infty (-1)^n x^n \frac{\Gamma(\alpha + 1)}{\Gamma(n + 1)} \sin \frac{\pi n x}{\alpha + \beta + \alpha}
$$

and Good has proved (33), p. 150) that, when $t \to \infty$,

$$
\phi(t) \sim K e^{-\beta t}
$$

where

$$
K = (2\pi(1 - \gamma)^{\alpha(1 + \gamma - \beta)}), \quad a = \frac{1}{2a(1 - \gamma) + 1},
$$

$$
\mu = \frac{1}{a(1 - \gamma)}, \quad \nu = 1
$$

It follows that

$$
\int_0^\infty t^n |\phi(t)| dt < \infty (n = 0, 1, \ldots),
$$

and that there is a number $T > 0$ such that $\phi(t) > 0$ for $t \geq T$. Consequently

$$
\mu_n = \int_0^\infty t^n |\phi(t)| dt = \frac{1}{\Gamma(\alpha + \beta)} \Gamma(\alpha + \beta)
$$

and hence, since $T^n |\phi(t)| dt < \infty$, we see that $\phi$ also satisfies conditions $(\Omega_3)$ and so $\phi \in \Omega$.

**Lemma 3.** If $x, \beta, \gamma, x$ are positive, then

$$
\mu_n = \frac{1}{\alpha \Gamma(\gamma - \beta)} \int_0^\infty e^{-\lambda t} \Gamma(\alpha + \beta + 1) dt
$$

satisfies the conditions of Theorem 2.

**Proof.** In order to establish this lemma we have only to observe that

$$
\psi_\alpha = \frac{1}{\alpha \Gamma(\gamma - \beta)} \int_0^\infty e^{-\lambda t} \Gamma(\alpha + \beta + 1) dt
$$

and that

$$
\phi(t) \sim K e^{-\beta t}
$$

is in the class $\Omega$.

4. Special Inclusions. (i) Let

$$
\psi_\alpha = 1/\Gamma(\alpha + 1) (x > 0, n = 0, 1, \ldots),
$$

and denote the IF methods of summability associated with the sequences $\{\psi_\alpha\}$ by $P^+_{\alpha}$, $P^-_{\alpha}$. Then the Borel exponential method.
In a recent paper (7) Wlodarczyk stated the following result:

If \( \alpha = 2^{-\delta}, \beta = 2^{-\delta + \epsilon} \) \((k = 0, 1, \ldots; \delta = 1, 2, \ldots,)\), and if the series

\[
\sum_{n=0}^{\infty} a_n \Gamma(n+\alpha + 1) = w(\alpha), \sum_{n=0}^{\infty} a_n \Gamma(n+\beta + 1) = w(\beta)
\]

are convergent for all \( t > 0 \) and

\[
\lim_{t \to \infty} a_t e^{-\alpha u(t)} = 1, \quad \lim_{t \to \infty} b_t e^{-\beta u(t)} = 1,
\]

then

\[
\sum_{n=0}^{\infty} a_n \Gamma(n+\alpha + 1) = 1.
\]

Now it is known (88, pp. 197–8) that, for \( \delta > 0, \)

\[
\lim_{n \to \infty} \delta e^{-\alpha n} \sum_{n=0}^{\infty} a_n \Gamma(n+\alpha + 1) = 1.
\]

and so Wlodarczyk's result is equivalent to:

If \( \alpha = 2^{-\delta}, \beta = 2^{-\delta + \epsilon} \) \((k = 0, 1, \ldots; \delta = 1, 2, \ldots,)\), and if \( s_n \to l(P^2) \) and \( \Sigma s_n \Gamma(n+\alpha + 1) \)

is an integral function, then \( s_n \to l(P^2) \).

We shall prove the more general result:

1. If \( \alpha > \beta > 0 \), and if \( s_n \to l(P^2) \) and the radius of convergence of \( \Sigma s_n \Gamma(n+\alpha + 1) \)

is greater than zero, then \( s_n \to l(P^2) \).

Proof. Let \( a, b \) be integers such that

\[
a > b > 2, \quad b > \beta > \frac{\alpha}{\beta} > 0,
\]

and let

\[
\gamma = \frac{\alpha}{\beta}, \quad c = a - b, \quad m = \frac{n-1}{b} \quad (n = 1, 2, \ldots).
\]

Then, using Gauss's multiplication theorem (22, p. 225) for gamma functions, we get

\[
P_a^P = \Gamma(m + \alpha + 1) \Gamma(\gamma + 1) \Gamma(m + \beta + 1) \Gamma(\gamma + \alpha) \Gamma(m + \gamma + 1) \Gamma(m + \alpha + \beta)
\]

where

\[
\Gamma(m + \gamma + 1) \Gamma(m + \alpha + \beta) = \prod_{\gamma = 0}^{m-1} \prod_{\alpha = 0}^{m-1} \prod_{\beta = 0}^{m-1} \Gamma(m + \gamma + 1) \Gamma(m + \alpha + \beta)
\]

and, by Lemma 1, \( \rho_\alpha \) satisfies the conditions of Theorem A.

Now let

\[
P_a = \rho_\alpha P_a, \quad q_n = \frac{\Gamma(n+\alpha + 1)}{\Gamma(n+\gamma + 1)} P_a,
\]

so that

\[
P_a^P = \Sigma q_n \Gamma(n+\alpha + 1) \Gamma(n+\gamma + 1)
\]

Suppose that

\[
s_n \to l(P^2),
\]

and that the radius of convergence of \( \Sigma q_n \Gamma(n+\alpha + 1) \)

is \( \rho > 0 \). Then, since \( \lim \frac{p_n}{q_n} \Gamma(n+\alpha + 1) = 0 \),

\( \Sigma q_n \Gamma(n+\alpha + 1) \)

is an integral function. Also, by Theorem A,

\[
s_n \to l(P^2).
\]

Consequently, by Lemma 2 and Theorem 1,

\[
s_n \to l(Q^2),
\]

and hence, by Lemma 3 and Theorem 2,

\[
s_n \to l(P^2).
\]

Remark. The above proof can be basically simplified if we impose the additional condition that \( \alpha/\beta \) be rational. For then we can avoid the use of Lemma 2, and consequently of Good's asymptotic approximation, by putting \( \alpha/\beta = \sigma/\beta \) and \( \gamma = 1 \).

(ii) For \( \alpha > 0 \), denote by \( Q_\alpha \) the IF method of summability associated with the function \((n!^{1/n})\). It is known that, when \( x \to \infty, \)

\[
\sum_{n=0}^{\infty} a_n x^n = (2\pi)^{1/2} e^{-\pi x} \Gamma(1/2) \Gamma(n+\alpha + 1), \Gamma(n+\alpha + 1) \Gamma(n+\alpha + 1)
\]

(see Hardy (6), p. 55).

We prove:

\[
\Pi, (\pi - \beta) > 0 \quad \text{and} \quad \alpha - \beta \text{ is an integer, and if } s_n \to l(Q_\alpha) \text{ and the radius of convergence of } \Sigma s_n \Gamma(n+\alpha + 1)
\]

is greater than zero, then \( s_n \to l(Q^2) \).

Proof. Let \( k = \alpha - \beta \). Then

\[
\Gamma(\pi - \beta) = \Gamma(x + \beta) \Gamma(n+\alpha + 1) \Gamma(n+\alpha + 1) \Gamma(n+\alpha + 1)
\]

and

\[
\rho_\alpha = \frac{\Gamma(n+\alpha + 1)}{\Gamma(n+\alpha + 1)} \Gamma(n+\alpha + 1) \Gamma(n+\alpha + 1)
\]

The proof can now be completed (as above in (i)) by first appealing to Lemma 1 and Theorem A and then to Lemma 3 and Theorem 2.

We conclude by considering, in relation to result II, the sequences \( s_n \), \( t_n \), where

\[
s_n = (-1)^n a_n \Gamma(n+\alpha + 1), \quad t_n = (-1)^n a_n \Gamma(n+\alpha + 1) \quad (a > 0).
\]

Following Hardy (6), p. 80) we find that \( s_n \to 0 (Q_\alpha) \), but, since \( \Sigma a_n \) is divergent when \( x > 1/a \), the Bessel method \( Q_\alpha \) cannot be applied to the sequence \( s_n \). However,

\[
\Gamma(\pi - \beta) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha + 1)}{\Gamma(n+\alpha + 1)} \Gamma(n+\alpha + 1)
\]

and so, as expected in view of II, \( s_n \to 0 (Q^2) \).

Next, it is easily verified that

\[
t_n \to 0 (Q_\alpha), \quad t_n \to 0 (Q^2).
\]

On the other hand, \( \Sigma a_n \) is zero radius of convergence and so the method \( Q^2 \) cannot be applied to \( t_n \).

I am indebted to the referee for suggesting the present form of the conditions (\( Q_\alpha \)), which are less restrictive than those of my original manuscript.
REFERENCES


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