BINARY AND TERNARY TRANSFORMATIONS OF SEQUENCES

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1. Introduction

Agnew (1) has defined a binary transformation \( T(x) \), with \( x \) real, as one which takes the sequence \( \{s_i\} \), \( i=0, 1, \ldots \), into the sequence \( \{s_i(1, x)\} \) where

\[
s_i(1, x) = \begin{cases} \alpha s_0 & \text{for } i=0, \\ \alpha s_i + (1-\alpha)s_{i-1} & \text{for } i=1, 2, \ldots . \end{cases}
\]

An \( r \)-fold application of \( T(x) \) yields the transformation \( T^r(x) \) which takes \( \{s_i\} \) into \( \{s_i(r, x)\} \) where, in general, if \( s_n(0, x) = s_n \) and \( s_n(r, x) = 0 \) for negative integral \( n \) then, for all \( n \) and \( r \geq 0 \),

\[
s_n(l+1, x) = (1-\alpha)s_{n-l}(1, x) + (1-\alpha)s_{n-l}(l, x).
\]

It easily follows by induction that

\[
s_n(l+r, x) = \sum_{k=0}^{r} \binom{r}{k} (1-\alpha)^{r-k}\alpha^{k}s_{n-r+k}(l, x), \tag{i}
\]

with the convention that \( 0^0 = 1 \).

Putting \( l=0, n = 1/\alpha - 1, \alpha \neq 0 \), we obtain

\[
s_n(r, x) = (q+1)^{-r} \sum_{k=n-r}^{n} \binom{r}{k} q^{n-k}\alpha^k, \tag{ii}
\]

and

\[
s_n(n, x) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k}\alpha^k. \tag{iii}
\]

If \( s_n(r, x) \) tends to a finite limit \( s \) as \( n \) tends to infinity then \( \{s_i\} \) is said to be summable \( T^r(x) \) to \( s \). If \( s_n(n, x) \) tends to a finite limit \( s \) as \( n \) tends to infinity then \( \{s_i\} \) may be said to be summable \( T^\infty(x) \) to \( s \). From (iii) and Hardy (2), equation (8.3.4), it follows that summability \( T^r(x) \) is equivalent to Euler summability \((E, q) \). It should also be noted that summability \( T^\infty(x) \) is equivalent to convergence.

We shall use the notation \( P \Rightarrow Q \) to mean that any sequence summable \((P) \) to \( s \) is necessarily summable \((Q) \) to \( s \), and \( P \Leftrightarrow Q \) to mean that both \( P \Rightarrow Q \) and \( Q \Rightarrow P \).

2. Relations between \( T^r(x) \) and \( T^\infty(x) \)

Knopp (5) has shown that for \( 0 < \alpha \leq 1 \) convergence to \( s \) implies summability \((E, 1/\alpha - 1) \) to \( s \), i.e. that \( T^\infty(x) \Rightarrow T^r(x) \); and from a general result on compounded matrices Agnew (1) has deduced that \( T^r(x) \Rightarrow T^\infty(x) \) for \( r > 0, 0 < \alpha < 1 \).
The case \( x=1 \) of this result was familiar to Hutton \( (3) \) who first considered the \( T(1) \) process early in the nineteenth century without giving rigorous proofs. The following proof is more direct than Agnew’s.

**Theorem.** For \( r \geq 0 \),

(i) \( T^{(r)}(a) = T^{-r}(a) \) for any \( a \); \( T^{(r)}(a) = T^{-r}(a) \) if and only if \( 0 < a \leq 1 \).

**Proof.** (i) Sufficiency. Let \( p > 1 \geq 1 \geq 1 \geq 0 \) and suppose that \( \{a_n\} \) is summable \( T^{(r)}(a) \) to \( a \). Applying the \( (S, q) \) process, which is known (see e.g. \( (2) \), p. 179) to be regular for \( q > 0 \), to the sequence \( a_i \) for \( r \geq 1 \), \( a_{r+1} \), \( a_{r+2} \), \( a_{r+3} \), \( \ldots \) which converges to \( a \), we get that

\[
(g+1)^{r} \sum_{k=0}^{g} \left( \frac{1}{g+1} \right)^{r} T^{-r}(a) k = a \quad \text{as} \quad g \to \infty.
\]

In virtue of identity \( (i) \) with \( r, i \) and \( n \) replaced by \( a, r \) and \( n + r \) respectively, it follows that \( \{a_n\} \) is summable \( T^{(r)}(a) \) to \( a \). Necessity. If \( \{a_n\} \) is summable \( T^{(r)}(a) \) to \( a \), and for \( n > 1 \), \( a_n \) is summable \( T^{(r)}(a) \) and so summable \( T^{(r)}(a) \), but is not summable \( T^{(r)}(a) \).

If \( \{a_n\} \) is the sequence 1, 0, 0, \( \ldots \) then its \( T^{(r)}(a) \) transform is \( (1-1)^{r} \).

For every \( n \), \( \{a_n\} \) is summable \( T^{(r)}(a) \), and so summable \( T^{(r)}(a) \), to 0; but the sequence is summable \( T^{(r)}(a) \) to 1 and is not summable \( T^{(r)}(a) \) for any \( r < 0 \).

The condition \( 0 < a \leq 1 \) is therefore necessary.

3. Norlund means, etc.

The following results will be used later:

**Kubota’s theorem.** (6). If \( a_n, a_{n+1}, \ldots, a_{n+k} \) are fixed real or complex numbers then, in order that \( a_k \) should tend to \( 0 \) as \( k \) tends to \( \infty \), it is necessary and sufficient that all roots of the equation \( \sum_{n=0}^{\infty} a_n x^n = 0 \) should lie within the unit circle.

**Norlund means.** Suppose that \( p_n > 0 \), \( P_n = p_0 + p_1 + \ldots + p_n \), where \( p_n \) is real, and that \( P_n \geq 0 \) for \( n \in M \).

For \( 0 < M \) let \( t_k = \sum_{k=0}^{M} p_k / P_k \).

And for \( M < 0 \) let \( t_k = \sum_{k=0}^{M} p_k / P_k \).

We shall say that sequence \( \{a_n\} \) is summable by the Norlund method \( (N, p_n) \) to \( s \) if \( t_k \) tends to \( s \) as \( k \) tends to \( \infty \). In (2), Hardy imposes the further condition \( p_n \to 0 \) (and takes \( M = 0 \)), but this is too restrictive for our purposes.

**Binary and Ternary Transformations of Sequences.**

It follows from formula (ii) that for \( s \neq 0 \) the \( T^{(r)}(a) \) transformation is a Norlund transformation with

\[
M = r, \quad p_n = \begin{cases} p_n, & p_n = 1 | x - 1, \text{for} \ 0 < x < r, \\ 0 & \text{for} \ x = r, \\ (1 + p_n)^{r} & \text{if} \ n \geq r, \end{cases}
\]

and

\[
\sum_{n=0}^{\infty} p_n x^n = (1 - (1 + p_n)^{r}).
\]

It is also known (2, p. 109) that the Cesaro mean \( (C, r) \) with \( r \geq 0 \) can be expressed as a Norlund mean \( (N, p_n) \) with

\[
M = 0, \quad p_n = \frac{n + r - 1}{r - 1} \quad \text{if} \quad r > 0,
\]

and

\[
p_n = 1, \quad p_n = 0 \quad (n = 1, 2, \ldots).
\]

For \( r \geq 0 \),

\[
\sum_{n=0}^{\infty} p_n x^n = (1 - r)^{-r} \quad \text{and} \quad p_n = \frac{n^r}{r(r+1)}. \tag{7}
\]

The following simple extensions of Hardy’s theorems 16, 17, 19 and 21 can be established by using the methods of his proofs and (in the case of theorem 17) a result due to Zink and Peayemoff (4, lemma 1).

**Theorem 16.** The Norlund method \( (N, p_n) \) is regular, i.e., the convergence of a sequence to a finite limit implies its summability \( (N, p_n) \) to the same limit, if and only if there is a constant \( H \) independent of \( n \) such that

\[
\sum_{k=0}^{\infty} |p_k| |H| < |P_k| \quad \text{for} \quad n \geq M
\]

and \( p_n / P_n \to 0 \) as \( n \to \infty \).

**Theorem 17.** Any two regular Norlund methods \( (N, p_n) \), \( (N, q_n) \) are consistent; i.e., if a sequence is summable \( (N, p_n) \) to \( s \) and \( (N, q_n) \) to \( t \) then \( s = t \).

**Theorem 19.** If \( (N, p_n) \) and \( (N, q_n) \) are regular and \( p(x) = \sum p_k x^k \), \( q(x) = \sum q_k x^k \), then in order that \( (N, p_n) \) of a sequence should imply its summability \( (N, q_n) \) it is necessary and sufficient that

\[
\sum_{k=0}^{\infty} k p_k q_k x^k < H \quad \text{for} \quad n \geq M,
\]

where \( H \) is independent of \( n \), and that \( k p_k q_k \to 0 \).

**Theorem 21.** A necessary condition that two regular Norlund methods \( (N, p_n) \) and \( (N, q_n) \) cannot be equivalent is that \( \sum k p_k \) and \( \sum k q_k \) be finite, where \( \sum k p_k \neq 0 \) or \( q(x) \).

**Corollary.** Regular Norlund methods \( (N, p_n) \) and \( (N, q_n) \) cannot be equivalent if \( p(x) \) and \( q(x) \) are rational and one of them has a zero, inside or on the unit circle, which is not a zero of the other.
4. Relation of $T'(n)$ to the Cesàro and Abel processes

If $(N, p_n)$ is taken as the $(C, c)$ process with $c > 0$, and $(N, q_n)$ as $T'(n)$, then

$$k(n) = (1 - p_n)^r (1 - q_n)^r$$

and $q_n \to 0$ for $r > 1$. By Theorem 19 it follows that, for $s > 0$, summability $(C, c)$ cannot imply summability $T'(n)$.

In the reverse direction we have the following results:

$$s \leq 1$$

By Kubota's theorem a sequence which is summable by Cesàro to $s$ converges to $x$ if and only if

$$|s - 1|/|x| < 1$$

and if only if $s > 1$. Since the $T'(n)$ transform is the $T(n)$ transform of the $T^{(r)}(n)$ transform it follows that

$$0 \leq q_n$$

and $q_n \to 0$ for $s > 1$.

By the above results, for $s > 1$, summability $(C, c)$ implies summability $T'(n)$. In the reverse direction we have the following results:

$$s < 1$$

By the above results, for $s > 1$, summability $(C, c)$ implies summability $T'(n)$. In the reverse direction we have the following results:

$$s < 1$$

The result is "best possible" in the sense that, for any integer $r$ there is a sequence which is summable by Cesàro to $s$ but which is not summable by Cesàro to $s + 1$ for any $s > 1$. This is shown by considering the example $s_n = (1 - 1/n)^r, n > 1$, the case $r = 1$ of which is due to Slivovitz and Straus (7). Since $s_n \neq O(n^{-a})$, $s > 0$, the sequence $(s_n)$ is not summable by Cesàro, $s > 1$.

If, however, $s_n = (1 - 1/n)^r$ where $f(n)$ is a polynomial of degree $s$ then

$$s_n(1, s, (1 - f(n))/n) = (1 - f(n))/n = \log O(n^{s-1}),$$

Putting $f(n) = n^{s-1}$ is a non-negative integer gives

$$\sum_{k=1}^{n} (1 - f(k))/n = O(n^{s-1})$$

from which it easily follows, on using the identity $k - n < (n - k)$, that

$$\sum_{k=1}^{n} (1 - f(k))/n = O(n^{s-1})$$

6. Relation of $T(n)$ to the $(C, c)$ and $T(n)$ processes

Let $f(x) = x^{s-1}$ be the ternary transformation which takes $(s_n)$ to the sequence $(s'_n)$ where

$$s'_n = 2s_n - 2s_n + 2s_n$$

It follows immediately that $T(n, 1 - s)$ is equivalent to $T(n, s)$, and that the $T(n, s)$ transformation is a Nörlund transformation $(N, p_n)$ with $p_n = -s, 1 - s = -s, p_n = -s, p_n = -s, p_n = -s, p_n = -s$.

Further, for $s > 2p > r, s \geq r > 0$, we have

$$\log n \log \frac{s}{\log n} = \left( \sum_{k=1}^{n} \log \left( 1 - \frac{k}{n} \right) \right) - \frac{1}{2} \log \left( \frac{s}{\log n} \right)$$

$$= \frac{1}{2} \log \left( \frac{s}{\log n} \right) + O(n(\log n)^{-r})$$

$$= 1 = A_n + A_{n+1} + \cdots + A_{n+r} + O(n^{-r})$$

where the $A_n$'s are bounded functions of $n$ independent of $k$. It follows that if $I_n = (1 - n)^{2 - 1/r} \sum_{k=1}^{n} \left( \frac{k}{n} \right)^{2 - 1/r} (n - k)^{2 - 1/r}$

then $I_n \log n = O(1)$, so that $I_n \to 0$. But $I_n = s_n(r, s)$ for the sequence $n(1 - n)^{r-1}$, so that this sequence is summable by Cesàro to $0$.

Thus $(C, c)$ summability $T(n, s)$ is trivially equivalent to convergence. Otherwise consider, so does Argaez (1), the sequence $(s_n)$ where $s_n = (1 - 1/n)^s$. It is summable to 0 and so is also summable $(C, c)$ to 0, but $(C, c)$ summability would imply convergence $(1/n - 1) < 1$ so that $(s_n)$ is not Abel summable. Hence for $s < 1$, summability $T(n, s)$ does not imply Abel summability.
Before investigating the behaviour of \( T(x, \beta) \) for \((x, \beta) \in \mathcal{S}_4\), we delimit the sets \( S_1, S_2 \) and \( S_3 \). Since \( S_3 \) is the complement of \( S_1 \cup S_2 \), it is sufficient to consider only \( S_1 \) and \( S_2 \).

The set \( S_1 \). We show that \( S_1 \) consists of the points \((0, 0), (\beta, 1), (2\beta, \beta) \leq 1\), \( x > 1\), \( x > 1\), \( \beta > 1\).

It is easily seen that \((0, \beta) \in S_1\), if and only if \( \beta = 0 \) or \( \beta > 1\). It remains to prove that when \( x > 1\), \((x, \beta) \in S_1\), if and only if \( 2x + \beta > 1, \beta \in S_1\).

(i) Suppose that both zeros \( x_1, x_2 \) of \( f(x) \) lie in \(|x| < 1\). Since \( f(-1) \neq 0 \) and \( f(1) = 1, f(-1) = 1 - 2\beta \) must be positive, for otherwise \( f(x) \) would have one real zero in the range \(-1 < x < 1\) and another outside this range. Hence \( \beta > 1\).

Further, \(-1 < x_1 x_2 = -1 - (1 - \beta) < 1\), so that \( 0 < (1 - \beta) < 2\). Since \( \beta > 1\), \( x_2 \) must be positive and so \( 2x + \beta > 1\).

(ii) Suppose \( 2x + \beta > 1, x < 1\), and, as above, \(-1 < x_2 < 1\).

Hence, if the zeros of \( f(x) \) are not real, both must lie in \(|x| < 1\). If both zeros are real, one must lie in the range \(-1 < x < 1\) and, since \( f(-1) > 0, f(1) > 0\), so must the other.

The set \( S_2 \).

(i) \((x, \beta) \in S_2\) if and only if \( x > 1\), \( x > 1\), \( \beta > 1\), which is equivalent to \( 2x + \beta > 1\).

(ii) Since \( f(1) = 1\), \((x, \beta) \in S_2\) if and only if \( f(-1) = 1 - \beta > 0\), \((-1 < \beta < 1\).

Hence \( S_2 \) consists of the part \( x > 1\) of the line \( 2x + \beta = 1\) and the part \( x > 1\) of the line \( \beta = 1\).

7. Relation of \( T(x, \beta) \) to the Cesàro process in \( S_3 \).

(i) The segment \( x > 1\) of the line \( \beta = 1\).

Here \( \beta > 1\) and \( x > 1\), \( x > 1\), \( \beta > 1\).

In theorem 19 take \((x, \beta) \in \mathcal{S}_4\) the \( T(x, \beta) \) process and \((x, \beta) \) the Cesàro \((C, \beta) \) process. Then

\[
k(n) = \begin{cases} a(1-x)^{-1}(1-x)^{2} \left( 1 - \frac{2x-1}{2x} \right) \\ \frac{1}{a} \left( 1 + (s-1)x + \frac{(s-1)x^2}{2} + \cdots \right) + \frac{2x-1}{2x} x + \frac{2x-1}{2x} \left( 1 + \frac{(s-1)x^2}{2} + \cdots \right) \end{cases}
\]

If \( s = 1\), then

\[
k(n) = \frac{1}{a} \left( 1 - \frac{2x-1}{2x} \right) \left( 1 - \frac{2x-1}{2x} \right) + \frac{2x-1}{2x} x + \left( 1 + \frac{(s-1)x^2}{2} + \cdots \right) \left( 1 + \frac{(s-1)x^2}{2} + \cdots \right) \]

where \( \gamma = 2x - 1 + i \sqrt{4(2x - 1)} \).

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Hence \( k(n) = 0 \) if \( x > 1\) and \( k(n) = 2n \) if \( x = 1\). Also, \( q(n) = x = 1\). All the conditions of theorem 19 are satisfied if \( x > 1\). Hence for \( x > 1, T(x, 1) \in \mathcal{S}_1\).

If \( x = 1\) and \( x > 1\), then \( k(n) = 4(1 - x)^2 \) and the conditions of theorem 19 are easily seen to hold, so that \( T(x, 1) \in \mathcal{S}_1\).