ON STRONG AND ABSOLUTE SUMMABILITY

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1. Introduction. Suppose throughout that \( \lambda > 0, \kappa > 0, -1 < \gamma < 1, \) \( \gamma \) is real and that

\[ s_n = \binom{n + \gamma}{\alpha}, \quad s_n = \sum_{r=0}^{n} r^n, \quad s_n = \frac{1}{\kappa} \sum_{r=0}^{n} r^n, \quad (n = 0, 1, \ldots). \]

The series \( \sum s_n \) is said to be

(i) summable \( (C, \kappa) \) to \( s \) if \( s_n \to s \),

(ii) strongly summable \( (C, \kappa + 1) \) with index \( \lambda \), or summable \( (C, \kappa + 1) \) to \( s \) if

\[ \lim_{n \to \infty} \left( \sum_{r=0}^{n} \left| s_n - s \right|^\kappa \right)^{1/\kappa} = \alpha(1), \]

(iii) absolutely summable \( (C, \kappa) \) with indices \( \gamma, \lambda, \gamma \), or summable \( (C, \kappa, \gamma) \), if

\[ \sum_{n=1}^{\infty} n^n \left| s_n - s_{n-1} \right|^\lambda < \infty. \]

Definitions (ii) and (iii), for general \( \kappa, \lambda, \gamma \), are due respectively to Hyslop [11] and Flett [4].

Their papers contain references to special cases considered earlier.

Let \( Q = (a_{n,m}) \), \( n, r = 0, 1, \ldots \) be a \( (C, \kappa) \) matrix, and let

\[ s_n = Q(s_n) = \sum_{m=0}^{n} a_{n,m}s_m. \]

It is to be supposed that all matrices referred to in this paper are of the above type. The symbol \( P \) will be reserved for matrices \( (p_{n,m}) \) with \( p_{n,m} > 0 \) \( (n, r = 0, 1, \ldots) \). The series \( \sum s_n \) is said to be

(iv) summable \( Q \) to \( s \), and we write \( s_n \to s(Q) \), if \( s_n \) is defined for all \( n \) and tends to \( s \) as \( n \to \infty \).

We now generalize the above definitions of strong and absolute summability in a natural way as follows. We say that \( \sum s_n \) is

(v) summable \( (P, Q) \) to \( s \), and we write \( s_n \to s(P, Q) \), if

\[ P(s_n - s) = \sum_{m=0}^{n} p_{n,m}s_m - s \to 0 \]

is defined for each \( n \) and tends to 0 as \( n \to \infty \).

(vi) summable \( (C, \kappa, \gamma) \), if

\[ \sum_{n=1}^{\infty} n^n \left| s_n - s_{n-1} \right| \gamma < \infty. \]

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We also define "prolate" processes of the form \( Q R, [P, Q], \{ Q R \gamma \} \), where \( R \) is any matrix, by replacing \( Q \) in (iv), (v), (vi) by \( Q R \) and taking \( s_n \) to be \( Q R(s_n) \) ; i.e. \( s_n = Q R(s_n) \)

\[ s_n = R(s_n). \]

Denoting by \( C \), the matrix of the transformation which changes \( (a_{n,m}) \) into \( (p_{n,m}) \), we observe that the summability processes \( (C, \kappa + 1) \), and \( (C, \kappa, \gamma) \), are respectively the same as \( (C, \kappa) \), and \( (C, \kappa, \gamma) \).

The unit matrix will be denoted by \( I \), so that \( I(s_n) = s_n \).

Let \( P \) and \( Q \) be summability processes (or matrices). We shall use the notation

\[ P \equiv Q \]

to mean that any series summable \( V \) to \( s \) is necessarily summable \( W \) to \( s \) provided that neither \( V \) nor \( W \) is an absolute summability process ; otherwise we shall understand the notation to mean simply that every series summable \( V \) is also summable \( W \). In either case we say that \( V \) is included in \( W \). We say that \( V \) and \( W \) are equivalent and write

\[ V \cong W \]

if each is included in the other, and we write \( V = W \) if \( V \) and \( W \) denote the same process (or matrix).

If \( f = V \) and \( f \) is not an absolute summability process, then \( V \) is said to be regular.

In this paper some of the properties of the strong and absolute summability defined above are investigated.

2. Simple inclusions.

Theorem 1. If \( Q \) is any matrix and \( P = (p_{n,m}) \), where

\[ \sum_{m=0}^{n} p_{n,m} < M \quad (n = 0, 1, \ldots) \]

and if \( \lambda > \mu > 0 \), then \( [P, Q], (P, Q) \), \( (P, q) \).

In particular, the conclusion holds if \( \lambda > \mu > 0 \) and \( P \) is regular.

This generalizes a result proved by Hyslop [11, Theorem 1].

Proof. By Hölder's inequality,

\[ \sum_{m=0}^{n} p_{n,m} |v_m|^\kappa \leq \left( \sum_{m=0}^{n} p_{n,m} |v_m|^\lambda \right)^{1/\lambda} M^{1-\lambda} \]

for any sequence \( (v_m) \). The required inclusion follows.

To complete the proof we have only to note that (1) is a necessary condition for the regularity of \( P \) [7, Theorem 2].

Note. Here and elsewhere an inclusion involving an arbitrary matrix \( C \) is essentially no more general than the same inclusion with \( P \) in place of \( Q \), the former being an immediate consequence of the latter.

Theorem 2. If \( Q \) is any matrix and \( \lambda > \mu > 0, \beta > \alpha > 0 \), then \( \{ Q R \gamma \} \leq \{ C \} \).

Proof. Let \( p = \lambda/\mu, q = \mu/(p-1) \) and let \( (v_m) \) be any sequence. Then, by Hölder's inequality (cf. Hyslop [11, Theorem 2]),

\[ C \{ Q R \gamma \} \leq \left( \sum_{m=0}^{n} p_{n,m} |v_m|^\lambda \right)^{1/\lambda} \leq \left( \sum_{m=0}^{n} p_{n,m} |v_m|^\mu \right)^{1/\mu} \]
Theorem 3. If $P, Q$ are matrices and $P$ is regular, then

(i) $Q = (P, Q)_\lambda$ for $\lambda > 0$,

(ii) $(P, Q)_\lambda > PQ$ for $\lambda \geq 1$.

Proof. (i) If $s_\lambda \to s$, then, since $P$ is regular, $P(s_\lambda - s) \to 0$, i.e. $I = (P, I)$, and inclusion (i) follows.

(ii) Suppose that $s_\lambda \to s(P, I)$. Then, by theorem 1, $s_\lambda \to s(P, I)$ and so

$P(s_\lambda - s) \to 0$.

Since $P$ is regular, it follows that $P(s_\lambda) \to s$. Hence $(P, I) = P$ and inclusion (ii) is an immediate consequence.

As a corollary of part (i) of Theorem 3 we have

(iii) If $P, Q$ are regular matrices and $\lambda > 0$, then $(P, Q)_\lambda$ is regular.

Theorem 4. If $\lambda > \mu > 0$, $\gamma > \delta$, then

(i) $\sum_{n=1}^{\infty} \gamma_{n+1} \to \left(\sum_{n=1}^{\infty} \gamma_{n+1} \right)^{1/\gamma}$,

where $\gamma$ is independent of the sequence $(\gamma_n)$.

Proof of (i). The case $\lambda = \mu$ is evident. Suppose therefore that $\lambda > \mu$.

Then, by Holder's inequality,

$\left(\sum_{n=1}^{\infty} \gamma_{n+1} \right)^{1/\gamma} \leq \left(\sum_{n=1}^{\infty} \gamma_{n+1} \right)^{1/\gamma}$,

where $s(1-\mu) = s - s(\lambda + 1 - (\gamma + \lambda - 1)h) = -\mu(y - 3) - (1 - 3/8)h$, so that $\gamma < -1$. The required inequality follows.

Result (ii) is an immediate consequence of (i).
Consequently, if $\sum a_n$ is summable $|Q^\ast|$, $\gamma |\lambda|$, then
$$\|x_n\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$
and so
$$\|x_n\|_{\ell^p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p},$$
from which it follows that the series is summable $|Q^\ast|$, $\gamma |\lambda|$, provided $\lambda > \mu > 0$, i.e., $|Q^\ast|$, $\gamma |\lambda|$, for $\lambda > \mu > 0$.

3. Hausdorff matrices. Given a real sequence $(\xi_n)$, let
$$x_{n,r} = \left\{ \begin{array}{ll}
\xi_n & \text{for } 0 \leq r < n, \\
0 & \text{otherwise},
\end{array} \right. $$
denote the matrix $(x_{n,r})$ by $(h, \xi_n)$. Matrices of this type are said to be real Hausdorff matrices. We shall assume hereafter that all Hausdorff matrices considered are real.

Let $X = (h, \xi_n)$, $Y = (h, \xi_{n,r})$. Then it is known that $XY = YX = (h, \xi_{n,r})$. Consequently $X^{-1} = (h, 1/\xi_n)$ provided $\xi_n \neq 0$, and it is familiar and easily verified that in this case $X = Y$ if and only if $XY^{-1}$ is regular.

Further, it is known that $X$ is regular if and only if
$$\xi_n = \int \frac{dx}{x},$$
where $\chi$ is a real function of bounded variation in $[0, 1]$ such that
$$\chi(0^+) = \chi(0^-) = \chi(1^-) = 1, \quad \chi(0^-) = 0, \quad \chi(0^+) = 1,$$
and
$$\chi(n \cdot) = \chi(n \cdot) = \chi((n+1)^-) = 1, \quad \chi(n \cdot) = 0, \quad \chi((n+1)^-) = 1.$$
Theroll 6. If \( X = (\xi, \xi) \), where
\[
\xi = \int_0^\infty d^2(t) \quad (n = 0, 1, \ldots),
\]
\( \chi \) being a real function of bounded variation in \([0, 1] \), and if
\[
\int_0^\infty \gamma \cdot d^2(t) < \infty \quad \text{ ........................................ (7)}
\]
and \( \lambda > 1 \), then
\[
(1) \quad \sum_{n=1}^{\infty} n^{\alpha-1} |X(na)|^\lambda \leq M \sum_{n=1}^{\infty} n^{\alpha-1} |na|^\beta,
\]
where \( M \) is independent of the sequence \( \{a_n\} \).

(iii) \( \sum_{n=1}^{\infty} n^{\gamma-1} |X(na)|^\lambda \leq M \sum_{n=1}^{\infty} n^{\gamma-1} |na|^\beta \),

where \( M \) is independent of the sequence \( \{a_n\} \).

When \( \gamma > 0 \) the integral in condition (7) should be interpreted in the Lebesgue-Stieltjes sense; when \( \gamma < 0 \) the condition is redundant.

Proof of (i). Suppose first that \( \gamma < 0 \). Then, by Lemma 1, since \( n^{\alpha} \leq n^\beta \) for \( n \geq 1 \),
\[
\sum_{n=1}^{\infty} n^{\alpha-1} |X(na)|^\lambda \leq \left( \sum_{n=1}^{\infty} n^{\alpha-1} |na|^\beta \right) \left( \sum_{n=1}^{\infty} n^{\gamma-1} |na|^\beta \right),
\]
and \( \lambda > 1 \).

The proof of part (ii) is thus complete.

It follows from (i) that \( |X| \gamma = |X|_\gamma \), and inclusion (ii) is an immediate consequence.

The next theorem generalises a result given by Hyslop [11, Theorem 3].

Theorem 7. If \( P \) is a regular matrix, \( Q \) is a matrix and \( \lambda > 1 \), then necessary and sufficient conditions for a series to be summable \( \{P \cdot Q \} \), to \( s \) are that it be summable \( \{P \cdot Q \} \) to \( s \) and summable \( \{P \cdot (P \cdot Q) \} \) to \( s \).

Proof. Let \( s_n = Q(a_n) \), \( s_n = P(a_n) \). We have to prove that
\[
P \left( \{ s_n - \gamma \} \right) = o(1) \quad .............................(8)
\]
and if only if \( \gamma \rightarrow s \).

Suppose now that \( \gamma > 0 \), and let
\[
f_\gamma(t) = \sum_{n=1}^{\infty} \left( \frac{t}{n} \right)^{\alpha-1} |na|^\beta
\]
where \( 0 < t < 1 \). Then, by Holder’s inequality,
\[
|f_\gamma(t)| \leq \sum_{n=1}^{\infty} n^{\alpha-1} |na|^\beta \left[ \sum_{n=1}^{\infty} \left( \frac{t}{n} \right)^{\gamma-1} \right]^{\lambda-1}
\]
and so
\[
\sum_{n=1}^{\infty} n^{\alpha-1} |f_\gamma(t)| \leq M \sum_{n=1}^{\infty} n^{\alpha-1} \left( \frac{t}{n} \right)^{\gamma-1} |na|^\beta
\]
where \( M \) and \( M_1 \) are independent of \( \{a_n\} \).

Now
\[
X(na) = \int_0^\infty f_\gamma(t) \cdot d^2(t)
\]
and so, by a form of Minkowski’s inequality,
\[
\left( \sum_{n=1}^{\infty} n^{\alpha-1} |X(na)|^\lambda \right)^{\lambda/\lambda} \leq \sum_{n=1}^{\infty} |d^2(t)| \left( \sum_{n=1}^{\infty} n^{\alpha-1} |f_\gamma(t)|^\lambda \right)^{\lambda/\lambda}
\]
and
\[
\sum_{n=1}^{\infty} n^{\alpha-1} |X(na)|^\lambda \leq M \sum_{n=1}^{\infty} n^{\alpha-1} |na|^\beta
\]

The proof of the theorem is thus complete.
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Now by (11), \( C_{\alpha, n} \approx C_{\alpha} \) \((\alpha > 0)\), and so, by result (II), \([ C_{\alpha} C_{\alpha + 1}] \approx [ C_{\alpha}, C_{\alpha} ]\) \((\alpha > 0, \lambda \geq 1)\).

Consequently, by (III), we have

(IV). If \( \lambda \gg 1, \alpha > 0, 0, \) then necessary and sufficient conditions for a series \( \sum b_n \) to be summable \([ C, s] \) to \( s \) are that it be summable \([ C, a] \) to \( a \) and that \( \sum b_n (a_n m)^{1/\alpha} = o(m) \).

This result has been proved directly by Hyslop [14] and it is suggested that the following definition of summability \([ C, 0] \), to \( 1 \) \( m \) if it is convergent with sum \( a \) and

\[ \sum_{n \geq 0} |a_n|^{1/\alpha} = o(m) \]

4. Equivalence of Cesàro and Hölder summability processes. For any real \( \lambda \) \( \leq 0 \), let \( \lambda \) \((\alpha \gg 1)\). Then \( C_{\lambda, n} \approx H_{\lambda}, H_{\lambda} \approx H_{\lambda+1}\), and it is known [7, Theorem 211] that

\[ C_{\lambda} \approx H_{\lambda} \approx H_{\lambda+1} \quad (\alpha \gg 1) \]

In conformity with the notation described in 1, we denote the Hölder type summability processes \( H_{\lambda}, (H_{\lambda}, H_{\lambda+1}) \), and \( H_{\lambda}, y \), \((y, y)\), \((H, y)\), \((H, y), (H, y), y\), \( (y, y), y\), respectively.

We now prove two theorems.

Theorem 8. If \( \alpha > 0, \lambda \gg 1, \) then \([ C, \alpha] \approx [ H, \alpha]\).

For \( \alpha > 0 \) this follows from (13) by result (II), and for \( \alpha = 0 \) it is a consequence of (III) with \( X = H_{\alpha} \approx C_{\alpha}^{-1}\).

The next theorem is a generalisation of the known results (see Knopp and Lorentz [12] and Morley [14]) that

\[ [ C, \alpha, 0] \approx [ H, \alpha, 0] \approx [ C, \alpha, 0] \quad (\alpha > -1) \]

Theorem 9. (i) If \( \alpha > -1, \lambda \gg 1, y > \min 1, 1 + \alpha, \alpha > 0 \), then

\[ [ C, \alpha, y] \approx [ H, \alpha, y] \approx [ H, \alpha, y] \]

(ii) If either \( \alpha > -1, \lambda \gg 1, y \leq 1 + \alpha \), then \( \lambda \gg 1, y > 2, \) then

\[ [ H, \alpha, y] \approx [ C, \alpha, y] \approx [ H, \alpha, y] \]

In connection with the second part of (ii) it should be noted that

\[ H_{\lambda}, y \approx C_{\lambda}, y \approx H_{\lambda+1}, y \]

The cases \( y < 0 \) of the propositions contained in Theorem 9 follow directly from (13) by Theorem 6(ii). To deal with the remaining cases we shall use

Lemma 2. If \( \alpha > \beta > 0 \) and \( g(\alpha) \) is an analytic function of \( s = \sigma + it \) in the region \( \sigma > \sigma_c \), and if, for \( \sigma > \sigma_c \) and large \( |t| \),

\[ g(\sigma) = K + O(|t|^{-\beta}) \]

where \( K, \beta \) are constants and \( \beta > 0 \), then

\[ g(\sigma) = \int_0^\sigma \phi(t) \, dt \quad (\sigma > 0) \]

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where \( \chi \) is a function of bounded variation in \([ 0, 1] \) such that

\[ \int_0^1 |dx(t)| < \infty \]

for every \( c > \sigma_c \).

Proof. Let \( f(\sigma) = g(\sigma) - K_\sigma \). Then, for \( c > \sigma_c + c > \sigma_c \)

\[ \int_0^c |f(c + it)|^2 \, dt \leq \mathcal{M}_c \]

where \( \mathcal{M}_c \) is a finite number independent of \( c \). Hence, by a result due to Rogosinski [14, 185-6],

\[ f(\sigma) = \int_0^c \phi(t) \, dt \quad (\sigma > 0) \]

where \( \phi(t) \in L(0, 1) \) for every \( c > \sigma_c + c \) and so for every \( c > \sigma_c \).

Consequently

\[ g(\sigma) = \int_0^c \phi(t) \, dt \quad (\sigma > 0) \]

It is evident that \( \int_0^c f(\sigma) \, d\sigma | < \infty \) for every \( c > \sigma_c \).

The lemma is thus proved.

Completion of the proof of Theorem 9. Let

\[ w(\sigma) = (\sigma + 1)^{-1} \int_0^{\sigma + 1} \frac{1}{(\sigma + 1)^{1/\beta} + 1} \]

and let \( W \) be the Hausdorff matrix \( (h, w) \), where \( w(\sigma) = w(\sigma) \).

(i) By Stirling's theorem, \( w(\sigma) \) satisfies the hypotheses of \( g(\sigma) \) in Lemma 2 with \( \delta = 1, \sigma_c = \max (-1, -1 - \sigma) \). Hence by Theorem 6(ii), with \( X = W, \)

\[ C_{\lambda}, y \approx |W C_{\lambda}, y \]

for \( y > \sigma_c \) i.e. for \( y < \min 1, 1 + \alpha \). Since \( C_{\lambda}, y \approx H_{\lambda}, y \), the proof of part (i) is complete.

(ii) The function \( 1(\phi(\sigma)) \) satisfies the hypotheses of \( g(\sigma) \) in Lemma 2 with \( \delta = 1, \sigma_c = -1 \) when \( \sigma > -1 \) and with \( \delta = 1, \sigma_c = -2 \) when \( \sigma = -2, \ldots \). Hence by Theorem 6(ii), with \( X = W^{-1} \)

\[ W C_{\lambda}, y \approx |W^{-1} H_{\lambda}, y \]

for \( -y > -1 \) when \( \sigma > -1 \), and for \( -y > -2 \) when \( \sigma = -2, \ldots \). Since \( W C_{\lambda}, y \approx C_{\lambda}, y \), this completes the proof of part (ii).

5. Hausdorff matrices associated with functions of class \( L^p \). In this section we deal with Hausdorff matrices \( (h, \xi) \) such that \( \xi = \int_0^1 \phi(t) \, dt \) where \( \phi(t) \in L(0, 1) \) and \( \phi(t) \in L^p(0, 1) \) for some real \( p \) and some \( p > 1 \). It is known [7, Theorem 215] that a Hausdorff matrix \((a_{n, m})\) satisfies these conditions with \( c < 0 \) if and only if
\[
\sum_{n=0}^{\infty} |x_n| \sigma < M (n+1)^{-p} \quad (n = 0, 1, \ldots),
\]
where \(M\) is independent of \(n\). Note that if \(\sigma(\theta)\) is in \(L^p(0,1)\) then it is necessarily in \(L^q(0,1)\).

We establish two theorems which augment Theorems 6 and 6. In the proof of the first of these we use

**Lemma 3.** Let \(\psi(t)\) be a real function in the class \(L^p(0,1)\), where \(p > 1\), and let

\[
\check{x}_{n} = \int_{0}^{1} \psi(t) \, dt, \quad \hat{x}_{n} = \int_{0}^{1} \tau \psi(t) \, dt \quad (n = 0, 1, \ldots, X = (\check{x}_n, \hat{x}_n), \quad \Xi^{(p)} = (0, \hat{x}_2),
\]

If \(\mu > \lambda + 1\) and \(1 + \mu - 1/\lambda = 1/p\), then, for any sequence \((w_n)\),

\[
|X(w_n)| \leq (p^{\mu-1})(\lambda^{\mu-1}) \sum_{n=0}^{\infty} (|w_n|^{\mu} + |w_n|^{\mu-1} X^{(p)}(w_n)).
\]

**Proof.** Let

\[
\check{x}_{n} = \sum_{r=1}^{n} \lambda^{r-1} \check{x}_r,
\]

where \(0 < r < 1\). Then, as in the proof of Theorem 6,

\[
|X(w_n)| = \sum_{n=0}^{\infty} (|w_n|^{\mu} + |w_n|^{\mu-1} X^{(p)}(w_n)),
\]

so that

\[
\int_{0}^{1} \check{x}_{n} \, dt = \frac{1}{n+1} \sum_{r=1}^{n} \check{x}_r \quad (n = 0, 1, \ldots, X = (\check{x}_n, \hat{x}_n), \quad \Xi^{(p)} = (0, \hat{x}_2),
\]

and

\[
\int_{0}^{1} \hat{x}_{n} \, dt = X^{(p)}(w_n).
\]

Further, using Hölder's inequality twice, we have

\[
|X(w_n)| \leq \left( \int_{0}^{1} \check{x}_{n} \, dt \right)^{\mu} \left( \int_{0}^{1} \check{x}_{n} \, dt \right)^{\mu-1} \left( \int_{0}^{1} \hat{x}_{n} \, dt \right)^{\mu} \left( \int_{0}^{1} \hat{x}_{n} \, dt \right)^{\mu-1}
\]

\[
\leq \left( \int_{0}^{1} \check{x}_{n} \, dt \right)^{\mu} \left( \int_{0}^{1} \hat{x}_{n} \, dt \right)^{\mu-1} \left( \int_{0}^{1} \check{x}_{n} \, dt \right)^{\mu-1} \left( \int_{0}^{1} \hat{x}_{n} \, dt \right)^{\mu-1}
\]

The required result follows from (14), (16), and (18).

**Theorem 10.** Let \(\mu > \lambda > 1\), \(1/p = 1 + \mu - 1/\lambda\), and let \(X = (\check{x}_n, \hat{x}_n)\), where

\[
\check{x}_{n} = \int_{0}^{1} \check{x}_{n} \, dt \quad \text{with} \quad \psi(t) \in L^p(0,1) \quad \text{and} \quad \hat{x}_{n} = 1.
\]

Then \(\mathcal{C}_p \Xi^{(p)} = \mathcal{C}_p \Xi^{(p)}\), for any matrix \(Q\).

**Proof.** Observe that \(X\) is a regular Hausdorff matrix and (in the notation of Lemma 3) that \(X^{(p)}\) is a Hausdorff matrix such that \(X^{(p)}(w_n) = 0\) whenever \(w_n \to 0\).
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(VI). If \( Q \) is any matrix and either (i) \( \mu \geq \lambda > 1 \), \( p > 1/2 - 1/\mu \), \( \alpha + 1 > \gamma > 0 \) or (ii) \( \mu > \lambda > 1 \), \( p = 1/2 - 1/\mu \), \( \alpha > \gamma > 0 \), then

\[
|C_V Q|_{\gamma} \leq |C_V C_V Q|_{\gamma}.
\]

Proposition (V) follows directly from the case \( s = 0 \) of a theorem on strong Cesàro summability given by Fleit (Theorem 2 in [8]), where the notation \( C_s \) is used with the same meaning as \( C_s \) in the present paper. The case \( s > -1/2 \) of this theorem is a corollary of an earlier result on strong Cesàro summability due to Gladsief ([6, Theorem 8]); see also [7 on p. 130 and the references there given]. Proposition (VI) can be immediately deduced from a result due to Fleit ([4, Theorem 1]).

To indicate the scope of Theorems 10 and 11 we shall employ them, together with (II) and Theorem 6 (ii), to give alternative proofs of (V) and (VI). Parts (ii) of propositions (V) and (VI) cannot be deduced from the general theorems of the present paper; the proofs of Fleit and Gladsief, pertaining to these parts of the propositions, depend ultimately on a deep but special inequality of Hardy, Littlewood and Polya ([9] see also [3, 230]).

Proof of (V) (i). The case \( \lambda = \mu \) is a direct consequence of result (II). Suppose therefore that \( \mu > \lambda > 1 \) and let \( 1/p = 1 - 1/\mu - 1/\lambda \). Now \( C_{\lambda} = (b, 1, \infty)^{\alpha} \) and

\[
1/\lambda^\alpha = \int_0^1 \rho \phi(\rho) d\rho,
\]

where \( \phi(\rho) = \rho(1 - \rho)^{\gamma-1} \). Further, \( p - 1 > 1 - 1/\mu + 1/\lambda = -1/\mu \) so that \( \phi(\rho) > 1 \). Since \( \phi(\rho) \in L^p(0, 1) \), and the required inclusion follows by Theorem 10.

Proof of (V) (ii). Note that \( C_{\gamma} = C_\epsilon \epsilon' C_{\lambda} X_{\lambda} \), where \( X = (b, \epsilon' \epsilon'^{\gamma+2}) \), and that \( \epsilon' \epsilon'^{\gamma+2} = \int_0^1 \rho \phi(\rho) d\rho \), where

\[
\phi(\rho) = (\gamma + 2)/(\gamma + 1) \left[ (1 - \rho)^{\gamma} - 1 \right].
\]

Suppose first that \( \lambda = \mu \). Then, since \( \gamma - 1 > 0 \), \( p > 0 \), we see that \( \phi(\rho) \in L^p(0, 1) \), and so, by Theorem 6 (ii), \( |C_{\gamma} y|_{\gamma} \leq |C_\epsilon y|_{\gamma} \). The required inclusion is then immediate consequence.

Suppose now that \( \mu > \lambda > 1 \) and let \( 1/p = 1 - 1/\mu + 1/\lambda \). Then, as above, \( \phi(\rho) \in L^p(0, 1) \), and, since \( \epsilon' + 1/\lambda > 0 \), \( p(\epsilon' + 1/\lambda - 1/\rho) < 1 \). Hence \( \phi(\rho) \in L^p(0, 1) \) and

\[
\phi(\rho) \in L^p(0, 1),
\]

and the required inclusion follows by Theorem 10 (ii).

Many special inclusions can be established with the aid of the above results. As an illustration, we prove the following (cf. [5, Theorem 2]):

\[
|H_s|_{\gamma} \leq |H_s|_{\lambda}.
\]

If either \( \mu > \lambda > 1 \), \( \beta > 1/\mu = 1/\lambda \), or \( \mu > \lambda > 1 \), \( \beta > 1/\mu + 1/\lambda \), then \( |H_s|_{\gamma} \leq |H_s|_{\gamma} \).

By (13), \( C_V H_{\mu \lambda} \approx C_V H_{\mu \lambda} \) (\( p > -1 \)), and the result is therefore a consequence of (II) and (V). Note that \( s \) can be any real number.
6. Relations between summability processes of different types. We first prove

THEOREM 12. If $\lambda > 1$, $2 > \rho > -1$, $X$ is a Hausdorff matrix, and if $\sum_{n=0}^{N} u_n$ is (i) summable
$|C_1, X, 0|$, and (ii) summable $AC_{\rho} X$ to $s$, then the series is summable $|C_1, X|$, to $s$.

When $\lambda = 1$ condition (ii) is not required.

Here $A$ denotes the Abel method of summability and summable $AC_{\rho} X$ to be inter-
preted as follows: $s_n \rightarrow s (AC_{\rho} X)$ means that $s_n = C_1 (X) s_n \rightarrow s(A)$, i.e. that

$$\lim_{n \rightarrow \infty} (1 - x) \sum_{r=0}^{n} x^r s_n = s.$$ 

It is known (see [1] and the references there given) that

$$C_1 = AC_{\rho} = AC_{\rho} \quad (s > -1, \gamma > 0, \beta > -1).$$

Proof. Let $s_n = \sum_{r=0}^{n} x^r s_n = C_1 (X) s_n$. Then, by hypothesis (i),

$$\frac{1}{n+1} \sum_{r=0}^{n} |r^\mu| = \frac{1}{n+1} \sum_{r=0}^{n} (s+1-r)^\mu \left(\frac{1}{r^\mu} - \frac{1}{(n+1-r)^\mu}\right) = \Theta(1),$$

so that

$$s_n \rightarrow 0 |C_1, X|.$$ 

Hence, by result (III), we have only to show that

$$s_n \rightarrow s |C_1, X|.$$ 

in order to complete the proof.

When $\lambda = 1$, (20) is an immediate consequence of hypothesis (i), and so hypothesis (ii) is redundant in this case.

Suppose now that $\lambda > 1$ and that $2 > \rho > 1 + 1/\lambda$. In view of (19) the additional restric-

tion of $\rho$ can be imposed without loss in generality. Let

$$C_1 (X) s_n = u_n = \sum_{r=0}^{n} u_r,$$

so that, by (5),

$$s_n \rightarrow s |C_1, X|.$$ 

Then, by (ii),

$$u_n \rightarrow s (A),$$

i.e., $\sum_{n=0}^{\infty} u_n$ is summable $A$ to $s$.

Further, by result (VI),

$$|C_1, X, 0| = |C_1, X, 0| \quad (\mu > \lambda) \quad \text{since} \quad \rho - 1 > 1/\lambda - 1/\mu.$$

Hence, by (i),

$$\sum_{n=0}^{\infty} n^\mu |u_n|^r < \infty.$$ 

Now by a Tauberian theorem of Hardy and Littlewood [8] (see also Flett [3, Theorem 4]),

a consequence of (21) and (22) is that, for every $\delta > 1/\mu + 1$, $\sum_{n=0}^{\infty} u_n$ is summable $\{C, \delta\}$ to $s$, i.e. that

$$C_1 (X) s_n \rightarrow s.$$
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Proof. Let $\delta$ be a positive number such that $2 \geq \delta > \rho + 1 - \alpha$. Then, by (13), $H_1 = H_2 \simeq C_{\rho+1}$, and so, by a result due essentially to Hausdorff (9); see also

[Theorem 4],

\[AH_1 = AG_1.\]

Since $H_1 = C_{\rho+1}$, we obtain the required result by applying first Theorem 12 (with $\delta$ in place of $\rho$) and then Inclusion (24).

In the same way we can prove

(VII'). If $\lambda > 1, 1 + \alpha + \rho > 0, \beta > \alpha - 1 + 1/\lambda$, and if $\sum_1^\infty x_n$ is (i) summable \[C, \alpha, 0, \lambda\],

and (ii) summable $AC_0$, to $\alpha$, then the series is summable \[(H, \beta)\] to $\alpha$.

The case $\alpha = 0, \beta = 0$ of this result is effectively the theorem of Hardy and Littlewood used in the above proof of Theorem 12. The case $\beta = 1, \rho = 0, \alpha > -1/\lambda$, is due to Zygmund [18], and Flett [4] has established the case $\alpha > -1/\lambda, \rho = 0$.

(VIII). If $\lambda > 1, \gamma > 0, \beta > \alpha - 1 - \gamma + 1/\lambda$, then

\[|H, n, \gamma| = |H, x - y| \simeq (H, \beta).\]

Proof. Let $X = C_{\rho+1}H_1$ where $\rho > \gamma$. Then $C_{\rho+1}X - H_1$ and, by (13),

\[C_{\rho+1}X \simeq H_{\rho+1+1}\].

Consequently, by Theorem 13 and results (II) and (24),

\[|H, n, \gamma| = C_{\rho+1}X, |H| = |C_{\rho+1}X, |H| = |H, H_{\rho+1+1}| \simeq (H, \alpha - \gamma) \simeq (H, \beta).\]

A similar proof shows that

(VIII'). If $\lambda > 1, \alpha < -1, \gamma > 0, \beta > \alpha - 1 - \gamma + 1/\lambda$, then

\[C, \alpha, \gamma, \gamma = (H, \beta).\]

The case $\alpha > -1/\lambda$ of this result has been proved by Flett [4].

REFERENCES

5. T. M. Flett, Some more theorems concerning the absolute summability of Fourier series and power series. Ibid. (3), 8 (1958), 357–387.