ON RIESZ SUMMABILITY FACTORS

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1. Suppose throughout that \( a, k \) are positive numbers and that \( p \) is the integer such that \( k - 1 \leq p < k \). Suppose also that \( \phi(w), \psi(w) \) are functions with absolutely continuous \((p + 1)\)th derivatives in every interval \([a, W]\) and that \( \phi(w) \) is positive and unboundedly increasing. Let \( \lambda = \{ \lambda_j \} \) be an unboundedly increasing sequence with \( \lambda_j > 0 \).

Given a series \( \sum_{n=1}^{\infty} a_n \) and a number \( m \geq 0 \), we write

\[
A_m(w) = \begin{cases} 
\sum_{n=m}^{\infty} \frac{(w - \lambda_n)^p \lambda_n}{w} a_n & \text{if } w > \lambda_n \\
0 & \text{otherwise},
\end{cases}
\]

and \( A(w) = A_0(w) \).

If \( w^{-m} A_m(w) \) tends to a finite limit as \( w \to \infty \), \( \sum_{n=1}^{\infty} a_n \) is said to be summable \((R, \lambda, m)\).

The object of this note is to obtain conditions sufficient to ensure, when \( k \) is not an integer, the truth of the proposition

\[ P \quad \sum_{n=1}^{\infty} a_n \phi(\lambda_n) \text{ is summable } (R, \phi(\lambda), k) \quad \text{whenever} \quad \sum_{n=1}^{\infty} a_n \text{ is summable } (R, \lambda, k). \]

For integral values of \( k \), the following theorem is known \([1]\). \( T_1 \), if

\( \gamma(w) \) is positive and absolutely continuous in every interval \([a, W]\) and \( \gamma(w) = O(1) \) for \( w \geq a \).

\( w \gamma(w)^{k+1} = O \left( \frac{\gamma(w)}{w} \right)^{k+1} \); \( \gamma(w) \geq 0 \).

\( n = 0, 1, \ldots, k; \ w \geq a \).

\[
\begin{align*}
E_n & = \sum_{n=1}^{\infty} \frac{\gamma(n+1)}{\gamma(n)} \frac{\gamma(n)}{\gamma(n+1)} (n = 0, 1, \ldots, k; \ w \geq a) \\
E_n & = \sum_{n=1}^{\infty} \frac{\gamma(n+1)}{\gamma(n)} (n = 0, 1, \ldots, k; \ w \geq a).
\end{align*}
\]

then \( P \).

Other known theorems, which hold for all \( k \geq 0 \), are

\( T_2 \), if \( \phi(w) = e^w \) and \( \psi(w) = w^{-p} \), then \( P \);
It is evident that $T_n$, for non-integral $k$, is included in $T_*$, and it can readily be shown that, under the hypotheses of $T_n$, the hypotheses of $T_*$ are satisfied with $\gamma(w) = \phi(w)\phi'(w)$ and $\phi(w), \phi'(w)$ as in $T_*$.

We are indebted to the referee for valuable suggestions which led to the above formulation of the results. In the original version of our manuscript we proved that $P$ is a consequence of conditions $T_n(0)$ to $T_n(v)$ inclusive together with the condition that $k_*(w)$ is a positive monotonic non-decreasing function of $w$ in the range $w \geq 0$ for $n = 0, 1, \ldots p$. The argument in §4 is due to the referee: it shows that the conditions of $T_n$ are in fact more stringent than those of $T_*$.

2. The following lemmas are required.

**Lemma 1.** If $T_n(t)$ and $T_n(w)$, then for $n = 1, 2, \ldots, p + 1$ and $w \geq a$,

$$\int_{\gamma(t)}^{\gamma(w)} \phi^{(k)}(t) \, dt = O(\phi(w)).$$

(2.1)

and

$$\gamma(w) - \phi^{(k)}(w) = O(\phi(w)).$$

(2.2)

**Proof.** When $0 < k < 1$, (2.2) is the same as the operative hypothesis in $T_n(v)$ and (2.1) is a trivial consequence. Suppose that $k > 1$. Then (2.1) follows from the appropriate part of $T_n(v)$ by integration, hence

$$\gamma(w) - \phi^{(k)}(w) = \gamma(t) - \phi^{(k)}(t) - \int_{\gamma(t)}^{\gamma(w)} \phi^{(k+1)}(t) \, dt = O(\phi(w)),$$

since $\gamma(t) = O(1)$, and (2.2) is an immediate consequence. (Cf. [1, Lemma 2].)

**Lemma 2.** The nth derivative of $\gamma(t)$ is a sum of a number of terms like

$$A(n^{(k)}(t)),$$

where $A$ is a constant, and $a_1, a_2, \ldots, a_n$ are non-negative integers, such that

$$1 \leq \sum_{i=1}^n a_i \leq \sum_{i=1}^n a_i w_i = n.$$

This is a particular case of a theorem due to Faa di Bruno [5, 1, pp. 89–90].

**Lemma 3.** If $a_i$ is real, $a_i \leq \xi \leq w_i$, then

$$\left| \frac{\Gamma(k+1)}{\Gamma(p+1)\Gamma(k-p)} \int_{\gamma(t)}^{\gamma(w)} A_p(t)(w - r)^{k-1} \, dt \right| \leq \max_{a_i \leq \xi \leq w_i} |A_i(\xi)|.$$

A proof of this lemma has been given by Hardy and Riesz [4, 28].

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**Lemma 4.**

$$\lim_{w \to w_0} \int_{a_0}^{a_0 + t} f(w(t)) \, dt = c \text{ and } \lim_{w \to w_0} \int_{a_0}^{a_0 + t} f(w(t)) \, dt = 0$$

for every finite $\gamma > a$, and if $f(t)$ is a bounded measurable function in $(a, c)$ which tends to zero as $t \to 0$, then

$$\lim_{w \to w_0} \int_{a_0}^{a_0 + t} \frac{f(w(t))}{\phi(t)} \, dt = 0.$$

For a proof of this simple result see [3, 50] or [1, Lemma 3].

**Lemma 5.** If $T_n(v)$ and $T_*(w)$, then

$$g(t) = \frac{\phi(w) - \phi(0)}{w - t}$$

is a monotonic non-increasing function of $t$ for $t \leq t < w$.

**Proof.** We have, for $a \leq t < w$,

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\phi^{(k)}(t) - \phi^{(k-1)}(t)}{\phi^{(k-1)}(t)} = \frac{\phi^{(k-1)}(t)}{\phi^{(k)}(t)} \left( \frac{\phi(w) - \phi(0)}{w - t} - \frac{\phi'(t)}{\phi(t)} \right).$$

Since $\gamma(t) \geq 0$, the result follows.

3. **Proof of $T_*$.** We assume, without loss of generality, that

$$A(w) = 0 \text{ for } 0 \leq w \leq a$$

and

$$A(w) = o(w^p),$$

(3.1)

and note that, for $w \geq a$, it is sufficient to prove that

$$\sum_{a_i \leq \xi \leq w_i} \left| \frac{\phi^{(k)}(w)}{\phi(w)} \right| \phi(\xi) \psi(\xi) \leq 0,$$

which is equal to

$$\int_{a}^{b} \left| \frac{\phi^{(k)}(w)}{\phi(w)} \right| \psi(t) \, dt \leq 0,$$

(3.2)
tends to a finite limit as $w \to \infty$. After $p + 1$ integrations by parts, (3.2) reduces to a constant multiple of

$$\int_a^b a_k(t) \left( \phi^{(n)}(t) \right)^2 dt$$

which, by Lemma 2 and Leibnitz's theorem on the differentiation of a product, can be expressed as a sum of constant multiples of integrals of the types

$$I_1 = \phi(w)^{-1} \int_a^b a_k(t) \phi(t)^r \left( \phi(w) - \phi(t) \right)^{k-1} dt,$$

$$I_2 = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^r \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \ dt$$

and

$$I_3 = \phi(w)^{-1} \int_a^b a_k(t) \phi(t) \left( \phi(w) - \phi(t) \right)^{k-1} \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \ dt,$$

where $n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_{k-1}$ are non-negative integers such that

$$1 \leq \sum\nolimits_{i=1}^{n_i} n_i = \sigma \leq \sum\nolimits_{i=1}^{n_i} \ell_i = r \leq p,$$

$$1 \leq \sum\nolimits_{i=1}^{n_i} \ell_i = \rho \leq \sum\nolimits_{i=1}^{n_i} \ell_i = p + 1.$$

Consider first $I_1$. Integrate it by parts to obtain

$$I_1 = -\int_0^1 \rho^k I_{12} \ dt,$$

where

$$I_{11} = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^r dt$$

and

$$I_{12} = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^{r-1} dt.$$

Now, by a standard result [4, 28] and (3.1),

$$A_{p+1}(w) = o(w^{p+1}).$$

Hence, using (3.3) and $\mathcal{A}_p$, we obtain

$$\int_a^b \left( \phi^{(n+2)}(t) \right)^2 \ dt < \infty,$$

and so, by Lebesgue's theorem on dominated convergence, $I_{11}$ tends to

$$l = \int_a^b \left( \phi^{(n+2)}(t) \right)^2 \ dt$$

as $w \to \infty$. 

\[ \frac{1}{m} \quad \text{being finite.} \]

For $I_{12}$, consider the function

$$f_1(w, t) = \phi(w)^{-1} \phi^{(r+1)}(t) \phi(t) \left( \phi(w) - \phi(t) \right)^{r-1}.$$

Using $\mathcal{T}_p$ (iii), we note that, for $w > \epsilon > n$, \nexists $M_1$ such that $f_1(w, t) < M_1 \phi(w)^{-1} \phi(t) \left( \phi(w) - \phi(t) \right)^{r-1}$.

where $M_1$ is a constant. Hence $f_1(w, t)$ satisfies the hypotheses of Lemma 4, and so

$$\int_a^b f_1(w, t) \phi^{(r+1)}(t) \ dt \to 0 \quad \text{as} \quad w \to \infty.$$

That is, $I_{12} = 0$ and so

$$\lim_{w \to \infty} I_1 = L.$$

Considering now $I_2$, we see, on integrating by parts, that it is equal to the sum of constant multiples of integrals of the types

$$I_{21} = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^r \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \ dt,$$

$$I_{22} = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^{r-1} \phi'(t) \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \ dt$$

and

$$I_{23} = \phi(w)^{-1} \int_a^b a_k(t) \phi^{(r+1)}(t) \left( \phi(w) - \phi(t) \right)^{r-1} \phi''(t) \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \ dt,$$

where $n_1, n_2, \ldots, n_r, \ell_1, \ell_2, \ldots, \ell_{k-1}$ are non-negative integers, such that

$$1 \leq \sum\nolimits_{i=1}^{n_i} n_i = \sigma \leq \sum\nolimits_{i=1}^{n_i} \ell_i = r \leq p,$$

$$1 \leq \sum\nolimits_{i=1}^{n_i} \ell_i = \rho \leq \sum\nolimits_{i=1}^{n_i} \ell_i = p + 1.$$

For $I_{21}$, consider

$$f_2(w, t) = \phi(w)^{-1} \phi^{(r+1)}(t) \phi^{(r+2)}(t) \left( \phi(w) - \phi(t) \right)^{r-1} \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r.$$

Suppose that the non-vanishing $n_i$ of highest suffix is $n_r$. Then

$$f_2(w, t) = \phi(w)^{-1} \phi^{(r+1)}(t) \phi^{(r+2)}(t) \left( \phi(w) - \phi(t) \right)^{r-1} \prod\nolimits_{\ell=1}^{k-1} (\phi^{(\ell)}(t))^r \phi^{(n_r)}(t) \left( \phi(w) - \phi(t) \right)^{n_r-1}$$

and

$$1 \leq \sum\nolimits_{i=1}^{n_i} n_i = \sigma \leq \sum\nolimits_{i=1}^{n_i} \ell_i = r.$$
Using (2.2) and \( T_\beta (v) \), we find that, for \( w > t \geq a \),
\[
I'(w, t) \leq M_3 \gamma(t)^{r-1} \left( \frac{\gamma(w)}{\gamma(t)} \right)^{r-1} \int f(w, t) \, dt \to 0 \quad \text{as} \quad w \to \infty.
\]
That is, \( \lim_{w \to \infty} I_1 = 0 \). Similarly \( \lim_{w \to \infty} I_2 = 0 \), and \( \lim_{w \to \infty} I_{12} = 0 \) in the case \( k - \sigma - 1 > 0 \). The remaining case of \( I_{12} \) is that in which \( r = \sigma = \rho \), and we write the integral as
\[
\left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \int \delta_{\psi+1}(t) \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt.
\]
Consider
\[
I_2(w, t) = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt.
\]
Using (2.2) and \( T_\beta (v) \), we find, for \( w > t \geq a \),
\[
I_2(w, t) \leq M_3 \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \int f(w, t) \, dt \to 0 \quad \text{as} \quad w \to \infty.
\]
where \( M_3 \) is a constant. Hence \( I_2(w, t) \) satisfies the hypotheses of Lemma 4, and so
\[
\int_0^w \int f(w, t) \, dt \to 0 \quad \text{as} \quad w \to \infty.
\]
That is, \( \lim_{w \to \infty} I_{12} = 0 \). Hence \( I_2 = 0 \). Finally, consider \( I_3 \), which can be written in the form
\[
I_3 = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt,
\]
where
\[
g(t) = \frac{\psi(t) - \psi(w)}{\gamma(\psi(t))^{r-1} \gamma(t)^{r-1}} \quad \text{for} \quad 0 \leq t < w, \quad g(w) = 1
\]
and
\[
H(t) = \sum_{i=1}^{\infty} \left( \frac{\gamma(t)^{r-1} \gamma(t)^{r-1}}{\gamma(t)^{r-1}} \right)^{\rho_i}.
\]
where \( \beta_1, \beta_2, \ldots, \beta_{p+1} \) are non-negative integers such that
\[
1 \leq \sum_{i=1}^{p+1} \beta_i = p \leq \sum_{i=1}^{p+1} \gamma_i = p + 1.
\]
Then \( H(t) \) is of bounded variation in \([a, \infty)\), because of \( T_\beta (v) \), and so can be expressed as the difference between two bounded monotonic non-increasing functions. Consequently, we can assume, without loss of generality, that \( H(t) \) is bounded and monotonic non-increasing. Also, \( \psi(t) - \psi(w) \lVert \psi \rVert_1 \), \( \gamma(t) \), and \( \gamma_{\beta+1} \) are monotonic functions of \( t \) in the range \( a \leq t \leq w \), the first being non-increasing since \( p+1-p \geq 0 \) and the second non-decreasing by Lemma 5. Using the second mean-value theorem for integrals twice, we now see that
\[
I_3 = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt,
\]
where \( w \geq t \geq a \). Hence, by Lemma 3 and (3.1),
\[
I_3 = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt.
\]
Now, by (2.2), and \( T_\beta (v) \),
\[
G(w, t) = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt,
\]
and, by Lemma 3 and (3.1),
\[
G(w, t) = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt.
\]
Hence, by Lemma 3 and (3.1),
\[
I_3 = \int \left( \frac{\gamma(\psi)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(t)}{\gamma(t)} \right)^{r-1} \left( \frac{\psi(w)}{\gamma(t)} \right)^{r-1} \, dt.
\]
Since \( p \geq 1 \). Hence
\[
\lim_{w \to \infty} I_3 = 0.
\]
Because of (3.4) and (3.5) and (3.6) we can deduce that (3.2) tends to a finite limit as \( w \) tends to infinity. This completes the proof of \( T_\beta (v) \).

4. Proof of \( T_\beta (v) \). Suppose that \( T_\beta (0) T_\beta (0)(v) \) and \( T_\beta (v) \) hold. It is clearly sufficient to show that \( T_\beta (v) \) is a consequence.

It follows from \( T_\beta (v) \) that, for \( w \geq a \),
\[
\psi(w) \lVert \psi \rVert_1 \lVert \psi \rVert_1 > c,
\]
where \( c \) is a positive constant; and hence, by \( T_\beta (v) \),
\[
\gamma(w) = O\left( \left( \frac{\gamma(w)}{\gamma(t)} \right)^{r-1} \right),
\]
and
\[
\gamma(w) = O\left( \left( \frac{\gamma(w)}{\gamma(t)} \right)^{r-1} \right).
\]
Consequently, by \( T_\beta (v) \).
and so
\[ w^p = O\left(\{\psi(w)\}^{\lambda - p}\right). \]

Hence, for \( w \geq a, \psi'(w) > bw^\psi(a - p) \), where \( b \) is a positive constant, and \( T_A \) (viii) follows by integration.

REFERENCES