ON MULTIPLICATION OF CESARO SUMMABLE SERIES

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1. Introduction

Throughout this paper $\sum_0^\infty c_n$ denotes the Cauchy product of the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$, i.e.

$$c_n = \sum_{r=0}^n a_r b_{n-r};$$

and $(C, \alpha)$, $[C, \alpha]$, $|C, \alpha|$ denote respectively ordinary, strong and absolute Cesàro summability methods. The method $[C, \alpha]$, previously defined only for $\alpha \geq 0$, is defined in a natural way for $\alpha < 0$ in §2.

It is known (see [1] and the references there given) that if $\sum_0^\infty a_n$ is summable $(C, -\mu)$ to $A$ and $\sum_0^\infty b_n$ is summable $(C, -\mu)$ to $B$, where $\mu \geq 0$,

then $\sum_0^\infty c_n$ is summable $(C, -\mu)$ to $AB$.

As a companion to this result we prove:

THEOREM 1. If $\mu \geq 0$ and $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ are summable $(C, -\mu)$ to $A$, $B$ respectively, then $\sum_0^\infty c_n$ is summable $(C, -\mu)$ to $AB$.

The case $\mu = 0$ of this theorem has been established by Boyd [4]. We also prove the following two theorems.

THEOREM 2. There are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, respectively summable $(C, -1)$ and absolutely convergent, for which $\sum_0^\infty c_n$ is not summable $(C, 0)$.

THEOREM 3. Given $\alpha \geq -1$, there are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, respectively summable $(C, -1)$ and $(C, \alpha)$, for which $\sum_0^\infty c_n$ is not summable $(C, \alpha + 1)$.

The cases $\alpha = -1$ and $\alpha = 0$ of Theorem 3 have been proved by Boyd [4]. Immediate consequences of Theorems 2 and 3 respectively (see inclusion IV in §2) are:

COROLLARY 1. There are series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, respectively summable $(C, 0)$ and absolutely convergent, for which $\sum_0^\infty c_n$ is not summable $(C, 0)$.

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Corollary 2. Given \( t \geq 0 \), there are series \( \sum a_n \sum b_n \), respectively summable \([C, 0]\) and \([C, t]\), for which \( \sum a_n \) is not summable \([C, t]\).

We state next three known propositions, the first due to Boyd [4] and the others to Wiss [7].

(a) If \( \sum a_n \) is summable \([C, k]\) to \( A \), where \( k > 0 \), and \( \sum b_n \) is absolutely convergent with sum \( B \), then \( \sum c_n \) is summable \([C, k]\) to \( AB \).

(b) If \( \sum a_n \) is summable \([C, k]\) to \( A \) and \( \sum b_n \) is summable \([C, 0]\) to \( B \), where \( k > 0 \), \( l > 0 \), then \( \sum c_n \) is summable \([C, k+l]\) to \( AB \).

(c) If \( \sum a_n \) is summable \([C, k]\) to \( A \) and \( \sum b_n \) is summable \([C, l]\) to \( B \), where \( k > 0 \), \( l > 0 \), then \( \sum c_n \) is summable \([C, k+l]\) to \( AB \).

Corollary 1 shows that proposition (a) fails when \( k > 0 \) is replaced by \( k = 0 \). Boyd [4] has demonstrated the falsity of (b) with \( k = l = 0 \) in place of \( k > 0 \), \( l > 0 \), and of (c) with \( k = 0 \), \( l = 0 \) or \( l > 1 \) in place of \( k > 0 \), \( l > 0 \).

Corollary 2 shows that for every \( k > 0 \), (c) is false when \( k > 0 \), \( l > 0 \) is replaced by \( k = 0 \), \( l = l_0 \).

2. Notation, definitions and preliminary results.

Let \( a_n = \sum_{n} a_n \) \( (n = 0, 1, \ldots) \).

Given matrices \( M = (a_{n,m}) \), \( P = (p_{n,r}) \) \( (n, r = 0, 1, \ldots) \) with \( p_{n,0} > 0 \), the strong summability method \([P, Q]\) is defined (see [3]) as follows. Let \( a_n = Q(a_n) = \sum_{n} p_{n,r} a_r \).

Then \( \sum a_n \) is summable \([P, Q]\) to \( s \) and we write \( a_n \rightarrow_s [P, Q] \).

is defined for each \( n \) and tends to \( 0 \) as \( n \rightarrow \infty \).

We use the notation:

\[
\xi_n = \left( \frac{a_n}{n} \right), \quad \Delta^n \eta_n = \sum_{k=0}^{n-1} \xi_{n-k} \quad (n = 0, 1, \ldots) \quad \text{any real } s.
\]

Denote by \( C_{\alpha, \beta} \) the matrix of the linear transformation from \( [a_n] \) to \( [c_n] \) given by

\[
c_n = \sum_{m=0}^{n} \frac{\xi_{n-m}}{\eta_m} \xi_m \Delta^\alpha \xi \xi_m \Delta^\beta \xi_m (\alpha = -1, \beta = -1);
\]

It is known (see [2], Theorem 8) that \( C_{\alpha, \beta} \) is the Haudorff matrix generated by the sequence \( (\xi_n, \Delta^n \eta_n) \).

Define \( C_{\alpha, \beta} \) to be the matrix \( C_{\alpha, \beta} \) when \( \alpha > 0 \), \( \beta > 0 \), \( C_{\alpha, \beta} \) when \( \alpha < 0 \), \( \beta < 0 \).

Then, for any real \( s \), the statement

\[
\sum a_n \text{ is summable \([C, \alpha]\) to } A
\]

can be interpreted (see [1], 443) as

\[
a_n = C_{\alpha, \beta} a_n \rightarrow A.
\]

We define, for every real \( s \), the strong Cesàro method \([C, \alpha]\) to be \([C, \alpha + 1] \). The definition is standard for \( \alpha > 0 \); for \( \alpha < 0 \), the method \([C, \alpha]\) does not appear to have been defined explicitly before. The following proposition, which is a special case of a known result (3, III) with \( X = C_{\alpha - 1} \), shows that our definition of \([C, 0]\) is equivalent to one framed by Hyslop [6].

3. The series \( \sum a_n \) is summable \([C, 0]\) to \( A \) if and only if it is convergent with sum \( A \) and

\[
\sum_{n=0}^{\infty} p_{n,r} a_r = o(n).
\]

Given summability methods \( X, Y \) we say that \( X \) is included in \( Y \) and write \( X \preceq Y \) if every series summable \( X \) is also summable \( Y \) to the same sum; \( X \) and \( Y \) are said to be equivalent and we write \( X \asymp Y \) if each is included in the other.

We list next some inclusions, which hold for every real \( s \), together with references to results of which they are immediate consequences.

II. \([C, s] \preceq [C, \alpha] (\alpha > 0, \beta > 0) \).

\((3, 3) \); and \([2], \text{ Theorem } 9) \).

III. \([C, s] \preceq [C, \alpha + s] (\alpha > 0) \).

\((3, 3) \); and \([2], \text{ Theorem } 6) \).

IV. \([C, s] \preceq [C, \alpha] \preceq [C, \alpha] \).

\((3, 3) \); and \([2], \text{ Theorem } 7) \).

V. \([C, s] \preceq [C, \alpha] \).

\((3, 3) \); and \([1], \text{ Theorem } 7) \).

That III, IV and V hold for \( s > 0 \) was known before (see [4], [6], [7]). Inclusion V is listed for interest only and is not used in the rest of this paper.
3. **Proofs of the theorems.**

In order to prove Theorem 1 we require a lemma which is similar to one proved by Winn (17), 483-484.

**Lemma.** If \( W_n = \sum_{i=0}^{n} w_i = o(n) \) then, for \( a < 1 \), \( \sum_{n=0}^{\infty} e_n^{-a} w_n = o(e_n^{-a} + 1) \).

**Proof.** By partial summation we have

\[
\sum_{n=0}^{\infty} e_n^{-a} w_n = \sum_{n=0}^{\infty} W_n (e_n^{-a} - e_{n+1}^{-a}) = \sum_{n=0}^{\infty} e_n^{-a} W_{n+1} + o(e_n^{-a} + 1).
\]

Since \( W_n/(r+1) \rightarrow 0 \) and \( \sum_{n=0}^{\infty} e_n^{-a} = e_n^{-a} + 1 \), the required result can now be obtained by an application of Toeplitz's theorem.

**Proof of Theorem 1.**

**Case (i).** Suppose \( A = B = 0 \).

Let \( \mu = m + n \), where \( m \) is a non-negative integer and \( 0 \leq a < 1 \); and let

\[
s_n = \sum_{b=0}^{n} a_b, \quad t_n = \sum_{b=0}^{n} b_b.
\]

It has been shown (11, 447) that a necessary and sufficient condition for \( \sum_{n=0}^{\infty} s_n \) to be summable \((C, -\mu)\) to 0 is that

\[
X + Y + Z = o(1),
\]

where

\[
X_n = \sum_{i=0}^{n} \frac{1}{e_{i+1}^{-a} - e_i^{-a}} \Delta^{n+1} x_i (e_i^{-a} - e_{i+1}^{-a}),
\]

when \( m \geq 1 \) and \( X_n = 0 \) when \( m = 0 \), and

\[
Y_n = \frac{1}{e_{n+1}^{-a} - e_n^{-a}} \sum_{b=0}^{n} b_b \Delta^{n+1} (e_n^{-a} - e_{n+1}^{-a}),
\]

and

\[
Z_n = \frac{1}{e_{n+1}^{-a} - e_n^{-a}} \sum_{b=0}^{n} a_b \Delta^{n+1} (e_n^{-a} - e_{n+1}^{-a}).
\]

By hypothesis, \( s_n \rightarrow 0 \) \((C, -\mu)\), \( t_n \rightarrow 0 \) \((C, -\mu)\), so that by the second inclusion in IV (12),

\[
s_n \rightarrow 0 \quad (C, -\mu), \quad t_n \rightarrow 0 \quad (C, -\mu);
\]

and a known consequence (11, 447-448) is that

\[
X_n = o(1).
\]

Now let

\[
y_n = \Delta^{n+1} (e_n^{-a} - e_{n+1}^{-a}) = e_n^{-a} C_{a+1} - e_{n+1}^{-a} (e_n^{-a}).
\]

From the hypothesis \( s_n \rightarrow 0 \) \((C, -\mu)\) we deduce, by II, that

\[
\sum_{n=0}^{\infty} \frac{1}{e_{n+1}^{-a} - e_n^{-a}} y_n = o(1)
\]

and hence, by the Lemma, that

\[
\sum_{n=0}^{\infty} \frac{1}{e_{n+1}^{-a} - e_n^{-a}} y_n = o(1).
\]

Next, since \( t_n \rightarrow 0 \) \((C, -\mu)\) we have, by III, that \( t_n = o(1) \) and it follows that

\[
Y_n = \frac{1}{e_{n+1}^{-a} - e_n^{-a}} t_n \rightarrow Z_n = o(1).
\]

Similarly \( Z_n = o(1) \); and the proof of Case (i) is complete.

**Case (ii).** Suppose now that there are no restrictions on \( A, B \). Let \( a'_n = a_n - A, b'_n = b_n - B, a'_n - a_n, b'_n - b_n \), \((r > 0)\) and let

\[
c'_n = \sum_{r=0}^{n} a'_r b'_r.
\]

Since \( \sum_{n=0}^{\infty} a'_n, \sum_{n=0}^{\infty} b'_n \) are summable \((C, -\mu)\) to \( A, B \) respectively, it is readily seen that \( \sum_{n=0}^{\infty} b'_n \) and \( \sum_{n=0}^{\infty} b'_n \) are summable \((C, -\mu)\) to 0, from which it follows, by Case (i), that \( \sum_{n=0}^{\infty} c'_n \) is summable \((C, -\mu)\) to 0.

But

\[
\sum_{n=0}^{\infty} c'_n = \sum_{n=0}^{\infty} a_n b_n + \sum_{n=0}^{\infty} A b_n + \sum_{n=0}^{\infty} b_n A - AB,
\]

and \( \sum_{n=0}^{\infty} b_n \) are summable \((C, -\mu)\) to \( A, B \) respectively. Hence \( \sum_{n=0}^{\infty} c_n \) is summable \((C, -\mu)\) to \( A + B \). This completes the proof.

**Proof of Theorem 2.** For convenience we divide the proof into three parts.

**Part (i).** Let \( s_n \geq 0, s_n \geq 0, U_n = \sum_{n=0}^{m} s_n, V_n = \sum_{n=0}^{m} s_n (n = 0, 1, \ldots) \), and let \( a_n = (-1)^n a_n, b_n = (-1)^n b_n \). Then

\[
c_n = \sum_{n=0}^{m} a_n b_n = (-1)^n \sum_{n=0}^{m} s_n s_{n-r},
\]

and hence (14, 30)

\[
\sum_{n=0}^{m} |c_n| = \sum_{n=0}^{m} \sum_{n=0}^{m} s_n s_{n-r} = \sum_{n=0}^{m} r_n U_{m-r} + \sum_{n=0}^{m} r_n U_{2n-r}
\]

\[
\geq \sum_{n=0}^{m} r_n U_{2n-r} \geq U_n \sum_{n=0}^{m} r_n.
\]
Part (ii). We show now that given any unbounded sequence of positive numbers \( \{U_n\} \), there is a sequence \( \{v_n\} \) such that
\[
v_n \geq 0, \sum_n v_n = \infty \quad \text{and} \quad U_n \sum_n v_n \neq o(n).
\] (2)

Let \( \{\beta_n\} \) be a sequence not converging to 0 such that
\[
\beta_n > 0 \quad \text{and} \quad \sum_n \frac{\beta_n}{U_n} < \infty;
\]
a suitable sequence can be constructed by first defining an increasing sequence of positive integers \( \{n_k\} \) for which
\[
U_{n_k} > \beta_k^2,
\]
and then taking \( \beta_n \) to be 1 whenever \( n = n_k \) and 0 otherwise.

Let \( x_n = 0, x_n = \frac{\beta_{n-1}}{U_n} - \frac{1}{n} \beta_{n-1} \quad (n \geq 1). \)

Then
\[
U_N n_n \sum_n x_n = n_n \beta_n
\]
and
\[
\sum_n |x_n| < \infty.
\]

Setting \( v_n = |x_n| \), we have
\[
U_n \sum_n v_n \geq n \beta_n
\]
and so the sequence \( \{v_n\} \) satisfies (2) as required.

Part (iii). To prove our theorem take \( a_n = (-1)^n w_n \) where \( w_n > 0 \), \( n w_n = o(1) \) and \( \sum_n (-1)^n w_n \) is conditionally convergent; e.g. \( w_n = 1 / (n + 2) \log (n + 2) \). Then \( U_n = \sum_n w_n \) is positive and tends to infinity, and \( \sum v_n = \sum_n a_n \) is summable \((C,-1)\). Let \( b_n = (-1)^n v_n \) where \( \{v_n\} \) is a sequence satisfying (2); then \( \sum b_n \) is absolutely convergent.

In virtue of I, the Cauchy product \( \sum_n c_n \) of the above series \( \sum_n a_n \sum_n b_n \) is not summable \((C,0)\), since, by (1) and (2),
\[
\sum_{n=0}^\infty |c_n| \neq o(n).
\]
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It follows that
\[ \sum_{n=0}^{m} \epsilon_n^2 |\sigma_n| \neq o(m^{m+1}) \]

and hence, by our Lemma, that
\[ \sum_{n=0}^{\infty} |\sigma_n| \neq o(m) \]

Consequently \( \sum_0^\infty c_n \) is not summable \([C, \alpha+1]\) to 0. However, by a standard result ([5], Theorem 164), \( \sum_0^\infty c_n \) is summable \([C, \alpha+1]\) to 0 and so, by the second inclusion in IV (§2), the series cannot be summable \([C, \alpha+1]\) to any number other than 0. Hence \( \sum_0^\infty c_n \) is not summable \([C, \alpha+1]\).

Remark. It is known ([5], Theorem 166) that, given \( \alpha \geq -1 \), there are series \( \sum_0^\infty a_n \), \( \sum_0^\infty b_n \), respectively summable \([C, -1]\) and \([C, \alpha]\), for which \( \sum_0^\infty c_n \) is not summable \([C, \alpha]\).

Our Theorem 3 is stronger than this result, since \([C, \alpha]\) is included in, but is not equivalent to, \([C, \alpha+1]\).

References.


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