ON ABSOLUTE RIESZ SUMMABILITY FACTORS

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1. Suppose throughout that $a_k$ are positive numbers, and that $p$ is the integer such that $k-1 < p \leq k$. Suppose that $\phi(w), \psi(w)$ are functions with absolutely continuous $p$-th derivatives in every interval $[a, W]$, and that $\phi(w)$ is positive and unboundedly increasing. Let $\lambda = \{\lambda_n\}$ be an unboundedly increasing sequence with $\lambda_1 > 0$.

Given a series, $\sum_{n=1}^{\infty} a_n$, and a number, $m \geq 0$, we write

$$A_m(w) = \begin{cases} \sum_{n \leq w} (w - \lambda_n)^m a_n & \text{if } w > \lambda_1, \\
0 & \text{otherwise,} \end{cases}$$

and $A(w) = A_0(w)$.

If $w^{-m} A_m(w)$ tends to a finite limit as $w$ tends to infinity, the series, $\sum_{n=1}^{\infty} a_n$, is said to be summable $(R, \lambda, m)$; and it is said to be absolutely summable $(R, \lambda, m)$, or summable $|R, \lambda, m|$, if $w^{-m} A_m(w)$ is of bounded variation in the range $w \geq 0$.

We shall use the notation, $w D_k^k f(t)$, to denote

$$\frac{(-1)^{p+1}}{\Gamma(p+1-k)} \left( \frac{\partial}{\partial t} \right)^{p+1-k} \int_t^w (u-t)^{p-k} f(u) \, du,$$

provided this expression is defined.

The object of this note is to obtain manageable conditions sufficient to ensure, when $k$ is not an integer, the truth of the proposition

$$P: \sum_{n=1}^{\infty} a_n \phi(\lambda_n) \text{ is summable } |R, \phi(\lambda), k| \text{ whenever } \sum_{n=1}^{\infty} a_n \text{ is summable } |R, \lambda, k|.$$

The following theorems are known.

For all $k$:

T1. If $\phi(w) = e^w$ and $\psi(w) = w^{-k}$, then $P$.

For integral values of $k$:

T2. If (i) $\phi(w)$ is a logarithmico-exponential function,†

(ii) $\frac{1}{w} \leq \frac{\phi'(w)}{\phi(w)}$,

(iii) $\psi(w) = \left( \frac{\phi(w)}{w \phi'(w)} \right)^k$,

then $P$, and

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† For definitions and properties of logarithmico-exponential functions, see [6].

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Tₙ. If there is a function, γ(\(w\)), defined and positive for \(w \geq a\), such that

(i) \(γ(w) = O(\frac{1}{w})\) for \(w \geq a\),

(ii) \(w^nγ^{(n)}(w) = O\left(\frac{(\log(w))^{k-n}}{w}\right)\) for \(n = 0, 1, ..., k\) and \(w \geq a\),

(iii) \(γ(w)\frac{\phi^{(n)}(w)}{\phi'(w)} = O(φ(w))\) for \(n = 1, 2, ..., k\) and \(w \geq a\),

then \(P\).

For non-integral values of \(k\):

Tₙ. If there is a function, γ(\(w\)), defined and positive for \(w \geq a\), such that

(i) \(γ(w) = O(\frac{1}{w})\) for \(w \geq a\),

(ii) \(w^nγ^{(n)}(w) = O\left(\frac{(\log(w))^{k-n}}{w}\right)\) for \(n = 0, 1, ..., p\) and \(w \geq a\),

(iii) \(γ(w)\frac{\phi^{(n)}(w)}{\phi'(w)} = O(φ(w))\) for \(n = 1, 2, ..., p\) and \(w \geq a\), according as \(0 < k < 1\) or \(k \geq 1\),

(iv) \(\frac{\phi'(w)}{\phi(w)}\) is of uniformly bounded variation with respect to \(w\) in the range \([1, \infty)\),

then \(P\).

Tₙ is due to Tatoon [7], Tₙ to Guha [4], and Tₙ and Tₙ to Dikshita [2, 3].

Suppose, first, \(w \geq a\) is an integer. Our main theorem is:

Tₙ. If there is a function, γ(\(w\)), defined and positive for \(w \geq a\), such that

(i) \(γ(w) = O(\frac{1}{w})\) for \(w \geq a\),

(ii) \(w^nγ^{(n)}(w) = O\left(\frac{(\log(w))^{k-n}}{w}\right)\) for \(n = 1, 2, ..., p+1\)

and \(w \geq a\),

(iii) \(γ(w)\frac{\phi^{(n)}(w)}{\phi'(w)} = O(φ(w))\) for \(n = 1, 2, ..., p+1\)

and \(w \geq a\),

and \(k \geq 1\),

is of uniformly bounded variation in the range \(w \geq a\),

If \(k < 1\),

S is a logarithmic-exponential function satisfying \(\frac{1}{\phi(w)} < \frac{\phi'(w)}{\phi(w)}\).

Also, Tₙ implies that \(P\) is true when \(φ(\infty) = w\) and \(φ(w)\) is a logarithmic-exponential function tending to a non-zero finite limit. Using now a result due to Chandrasekharan [1], we can easily deduce that Tₙ is also true when \(w \geq a\) is non-integer. We have not investigated the exact relation between our theorems and Tₙ, but each condition of Tₙ is simple to verify in particular cases, whereas the unwieldy condition (iv) of Tₙ is not.

2. The following lemmas are required.

Lemma 1. The \(n\)-th derivatives of \(f(w)\) are a sum of constant multiples of a finite number of terms like

\[f^{(n)}(w) = \sum_{\mu=1}^{n} (f^{(\mu)}(w))^{x_{\mu}}\]

where \(x_{1}, x_{2}, ..., x_{n}\) are non-negative integers such that

\[1 \leq \sum_{\mu=1}^{n} x_{\mu} = \mu \leq \sum_{\mu=1}^{n} x_{\mu} = n.\]

This is a particular case of a theorem due to Faa di Bruno. See [8; I, pp. 89-90].
Lemma 2. For $w > t > 0$

$$\Delta u \left( \left( \frac{\phi(t)}{\phi(u)} \right)^{1/\alpha} \phi(u) \right) = \frac{1}{\Gamma(p-1+1)} \int_0^w \left( u - \phi(t) \right)^{p-1} \left( \frac{1}{\phi(u)} - \frac{\phi(t)}{\phi(u)} \right) \psi(u) \, du.$$ 

This is similar to Lemma 5 (first part of proof) in [5].

Lemma 3. For $w > 0$

$$\int_0^w \left( u - \phi(t) \right)^{p-1} \phi(u) \psi(u) \, du = \frac{1}{\Gamma(k+1)} \int_0^w \phi(t) \left( \frac{1}{\phi(u)} - \frac{\phi(t)}{\phi(u)} \right) \psi(u) \, du.$$ 

This is similar to Lemma 6 in [5].

Lemma 6. 

(i) If $\alpha > \mu > 0$ and $\phi(u) > 0$ for $u > 0$, then

$$\int_0^w \left( u - \phi(t) \right)^{p-1} \phi(u) \psi(u) \, du = 0.$$ 

(ii) If $\alpha > \mu > 0$ and $\phi(u) > 0$ for $u > 0$, then

$$\phi(t)^{p-1} \left( u - \phi(t) \right)^{p-1} \phi(u) \psi(u) \, du = 0.$$ 

Proof. (i) Denoting $\frac{x^p}{\phi(t+x)}$ by $F_t(x)$, we have for $x > 0$, $t > 0$ and $0 < e < 1$,

$$\frac{F_t(x)}{F_t(x)} = \frac{e^{\mu u} \left( t + 1 + 1 \right)}{e^{\mu u} \left( t + 1 + 1 \right)} + \frac{1}{\phi(t+x) - \phi(t)}$$ 

where

$$F_t(x) = \phi(t+x) - \phi(t+x),$$

and so

$$F_t(x) = -e^{\mu u} \phi(t+x).$$

Hence

$$\frac{e^{\mu u} \left( t + 1 + 1 \right)}{e^{\mu u} \left( t + 1 + 1 \right)} = \frac{1}{\phi(t+x) - \phi(t+x)} \frac{1}{\phi(t+x) - \phi(t+x)} \frac{1}{\phi(t+x) - \phi(t+x)}$$

$$\leq 0,$$

since $\frac{e^{\mu u} \phi(t+x)}{\phi(t+x)}$ and $\frac{t}{t+x}$ are non-negative monotonic non-decreasing functions of $u$ for $u > 0$. Since $F_t(x)$ is non-negative, and $\lim_{x \to 0} F_t(x) = 1$, the result follows.

(ii) Denoting $\frac{d(t+x)}{\phi(t+x)}$ by $F_t(x)$, we have for $x > 0$, $t > 0$ and $0 < e < 1$,
In order to show that $I(t)$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, it is sufficient, in view of (1) and Lemma 4, to prove that

$$
\int_{t}^{\infty} d_u q_t(u, t) = O(1) \text{ for } t \geq a.
$$

Now, by Lemma 2, for $w \geq t \geq a$,

$$
(-1)^{n+1} \Gamma(p+1-k) q_t(u, t) = \int_{u}^{\infty} (-w)^{-p-k} \left( \frac{\theta}{\phi(w)} \right)^{p+1} \left( 1 - \frac{\phi(u)}{\phi(w)} \right) \psi(u) du.
$$

Since

$$
\int_{0}^{\infty} (1 - \phi(u))^{p+1-k} du = \Gamma(p+1-k) \frac{\Gamma(k+1)}{\Gamma(p+1)},
$$

in order to prove that $I(t)$ is of bounded variation with respect to $w$ in the range $[a, \infty)$, it is sufficient, in view of Lemma 4 (6), to prove that

$$
\int_{0}^{a} [d_u q_t(u, t)] = O(1) \text{ for } u \geq 0.
$$

Integration by parts yields, for $w \geq y \geq a$

$$
g_t(w, y) = b_0 \psi(w) \left( 1 - \frac{\phi(y)}{\phi(w)} \right)^{p+1} + \sum_{i=1}^{n} b_i \psi\left( \frac{2i}{\phi(w)} \right) \left( 1 - \frac{\phi(y)}{\phi(w)} \right)^{p+1} \psi(y),
$$

where $b_0, b_1, \ldots, b_n$ are constants.

By $T_n$ (6) and (9), $\psi(y) = O(1)$ for $y \geq a$, and hence the term with coefficient $b_0$ is of uniformly bounded variation with respect to $w$ in the range $[y, \infty)$ for $y \geq a$. Also, by Lemma 1 and Leibnitz's theorem on the differentiation of a product, the other terms can be expressed as sums of constant multiples of terms like

$$
g_t(w, y) = \sum_{i=1}^{n} b_i \psi\left( \frac{2i}{\phi(w)} \right) \left( 1 - \frac{\phi(y)}{\phi(w)} \right)^{p+1} \psi(y).
$$
where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n+1}$ are non-negative integers such that
\[
0 \leq \sum_{i=1}^{n} \alpha_i = \mu \leq \sum_{i=1}^{n} \beta_i = n \leq r < p, \tag{1}
\]
and
\[
\Phi_{\nu}(w, y) = \left( \frac{\Gamma(w)}{\Gamma(y)} \right)^{\frac{\nu}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}}.
\]
Now, in view of Lemma 5, $g_{\nu}(w, y)$ is of uniformly bounded variation with respect to $w$ in the range $[y, \infty)$ for $y \geq a$, since we take $m = p + 1$ when $r > n$ and $m = k$ when $r = n$, we have, by Lemma 4 (i) and (ii),
\[
g_{\nu}(w) = \frac{\Gamma(w)}{\Gamma(y)} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} = O\left( \frac{\Gamma(w)}{\Gamma(y)} \right) = O(1),
\]
$m > 0$ being positive.

Hence $I_1$ is of bounded variation with respect to $w$ in the range $[a, \infty)$.

Consider now $I_4$. In view of (i) and Lemma 5, in order to prove that $I_4$ is of bounded variation with respect to $w$ in the range $[0, \infty)$, it is sufficient to prove that
\[
\int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} = O(1)
\]
\[
is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geq a$.

Now, in view of Lemma 1 and Leibnitz's theorem on the differentiation of a product, it is sufficient to prove that each of the following integrals is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geq a$:
\[
I_m = \left( \frac{\phi(u)}{\phi(w)} \right) \int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1}.
\]
\[
I_m = \left( \frac{\phi(u)}{\phi(w)} \right) \int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1},
\]
and
\[
I_m = \left( \frac{\phi(u)}{\phi(w)} \right) \int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1},
\]
\[
where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n+1}$ are non-negative integers such that
\[
0 \leq \sum_{i=1}^{n} \alpha_i = \mu \leq \sum_{i=1}^{n} \beta_i = n \leq r < p, \tag{1}
\]
and
\[
1 \leq \sum_{i=1}^{n} \beta_i = \sigma \leq \sum_{i=1}^{n} \alpha_i = r < p + 1.
\]
Now, for $t \geq a$, by Theorem 4 (i) and (ii),
\[
\int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1} = O(1)
\]
and the latter integral is finite and independent of $t$. Hence, in view of Lemma 4 (i), it is clear that $I_4$ is of uniformly bounded variation with respect to $w$ in the range $[y, \infty)$ for $y \geq a$.

Further, for $w \geq t > a$, we can write
\[
I_{4a} = \int_{0}^{t} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1}
\]
By Theorem 4 (i) and (ii), for $t \geq a$,
\[
\int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1} = O(1)
\]
and the latter integral is finite and independent of $t$. Hence, in view of Lemma 4 (i) and 5, it is clear that $I_4$ is of uniformly bounded variation with respect to $w$ in the range $[t, \infty)$ for $t \geq a$.

Finally, for $w \geq t > a$, we can write
\[
I_{4a} = \int_{t}^{a} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1}
\]
Since $h + 1 - \sigma$ is a positive integer (or zero), we can expand $(\phi(w) - \phi(u))^{p+1-\sigma}$ by the binomial theorem, giving, for $w \geq t \geq a$,
\[
I_4 = \sum_{\rho = 1}^{p+1-\sigma} c_{\rho}(\phi(w) - \phi(u))^{p+1-\sigma} \int_{0}^{\infty} (u - t)^{-1} \left( \frac{\beta}{\alpha} \right)^{\frac{h}{2}} \left( \frac{\beta - 1}{\alpha - 1} \right)^{\frac{h-1}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \left( \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \right)^{\frac{h}{2}} \frac{d\nu}{\nu+1} \frac{d\nu}{\nu+1},
\]
where $c_0, c_1, \ldots, c_{p+1-\sigma}$ are constants.
To the typical integral of the above sum, apply the transformation
\[ u = t - \epsilon x; \quad v = t + x \]
to obtain a constant multiple of \( q_\epsilon(x, t) \), where, for \( t \geq a \), \( q_\epsilon(0, t) = 0 \) and for \( x \geq 0 \) and \( t \geq a \)
\[ q_\epsilon(x, t) = \rho \left( \frac{f(t + x)\eta^{p-1} - \eta}{x(1 - \epsilon)} \right)^{r-1} \left( \frac{f(t + x)}{t + x} \right)^{p-1} \]
\[ \times \phi(t + x) \prod_{j=1}^{\frac{\rho t}{2}} (\phi(t + jx))^2 \quad e^{\frac{\rho t}{2} - 1} \frac{v^r}{r} \frac{dt}{dt + v} . \]

It is sufficient to prove that \( q_\epsilon(x, t) \) is of uniformly bounded variation with respect to \( x \) in the range \([0, \infty)\) for \( t \geq a \).

Now, for \( x > 0 \) and \( t > a \), we can write
\[ q_\epsilon(x, t) = \int_0^x e^{t-1} (1 - \epsilon)^{p-1} \left( \frac{t}{t + v} \right)^{1} \left( \frac{1}{t + v} \right)^{p-1} H(t, \epsilon, t) \]
\[ \times \left( \frac{\partial f(t + x)}{\partial x} \right)^{p-1} \frac{\partial f(t + x)}{\partial x} \left( \frac{1}{t + x} \right)^{p-1} \left( Q(t, \epsilon) \right)^{p-1} \frac{dt}{dt + v} \]
where
\[ H(t, \epsilon, x) = \left( \frac{t}{t + v} \right)^{p-1} \left( \frac{1}{t + x} \right)^{p-1} \left( \frac{1}{t + v} \right)^{p-1} \left( \frac{1}{t + x} \right)^{p-1} \]
and
\[ Q(t, \epsilon) = \left( \frac{\partial f(t + x)}{\partial x} \right)^{p-1} \left( \frac{\partial f(t + x)}{\partial x} \right)^{p-1} \left( \frac{\partial f(t + x)}{\partial x} \right)^{p-1} \left( \frac{\partial f(t + x)}{\partial x} \right)^{p-1} \]
\[ \int_0^x e^{t-1} (1 - \epsilon)^{p-1} \frac{dt}{dt + v} = \Gamma(p+1-2) \Gamma(k-p) H(t, \epsilon, x)^{p-1} \left( \frac{Q(t, \epsilon)^{p-1}}{t} \right) \]
\[ = O(1). \]
Consequently, \( q_\epsilon(x, t) \) is of uniformly bounded variation with respect to \( x \) in the range \([0, \infty)\) for \( t \geq a \), and so, \( I_{23} \) is of uniformly bounded variation with respect to \( x \) in the range \([t, \infty)\) for \( t \geq a \).

Hence \( I_4 \) is of bounded variation with respect to \( x \) in the range \([a, \infty)\), and it follows that (2) is of bounded variation for \( x \geq a \), this completing the proof of \( T_B \).

4. Proof of \( T_A \). In view of Lemma 6, \( T_A \) follows immediately from \( T_B \).

References


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