ON GENERALISED CESÁRO SUMMABILITY

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(Prof. B. H. Prasad Memorial Vol.)

Reprinted from the Indian Journal of Mathematics, Vol. 9, No. 1,
January 1967, Pages 55-64
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(Received August 6, 1966)

1. Introduction. Let \(\{\lambda_n\}\) be a strictly increasing unbounded sequence with \(\lambda_0 \geq 0\), let

\[ \mu = p + \delta \quad (p = 0, 1, \ldots; 0 < \delta < 1) \]

and let \(\sum_{n=0}^{\infty} a_n\) be an arbitrary series. We write

\[ A^\mu(w) = \sum_{\lambda < w} (w - \lambda)^\mu a_{n(w)} \]

and

\[ \pi^\mu_n(t) = \begin{cases} (\lambda_{n+1} - t)^\mu & (p = 0), \\ (\lambda_{n+1} + t - n)^\mu & (p \geq 1), \end{cases} \]

\[ C^\mu_n = \sum_{\lambda \leq n} \pi^\mu_n(\lambda) a_{n(\lambda)} \]

and say that the series \(\sum a_n\) is summable to \(s\) by

(i) the Riesz method \((R, \lambda, \mu)\), if \(w^{-\mu} A^\mu(w) \to s\) as \(w \to \infty\),

(ii) the discrete Riesz method \((R^*, \lambda, \mu)\), if \(\lambda^{-\mu} A^\mu(\lambda) \to s\),

(iii) the generalised Cesàro method \((C, \lambda, \mu)\), if \(C_n^\mu[\pi^\mu_n(0)] \to s\).

The summability methods \((R^*, \lambda, \mu)\) and \((C, \lambda, \mu)\) are identical when \(0 < \mu \leq 1\). For integral values of the parameter \(\mu\), the method \((C, \lambda, \mu)\), which is essentially the same as methods defined independently by Jurkat and Burkhill, reduces to the standard Cesàro method \((C, \mu)\) when \(\lambda_n = n\). Burkhill considered only the integral case but Jurkat extended his definition to non-integral values of the parameter in a manner different from the above.

1. Jurkat (7).
2. Burkhill (4).

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The inclusions

\[(1) \quad (G, \lambda, \mu) \subseteq (R, \lambda, \mu),\]

\[(2) \quad (R, \lambda, \mu) \supseteq (G, \lambda, \mu)\]

are known to be valid under various hypotheses on the sequence \((\lambda_n)\) and the parameter \(\mu\). The most general results to date are as follows:

Russell\(^4\) has proved that (1) holds (i.e. that every series summable \((G, \lambda, \mu)\) to \(r\) is summable \((R, \lambda, \mu)\) to \(r\) without any restriction on \((\lambda_n)\) when \(\mu\) is an integer, and that (2) holds provided

\[(3) \quad \lambda_n = o(\lambda_n), \quad \lambda_n = \frac{\lambda_{n+1}}{\lambda_n}\]

when \(\mu = 3, 4, 5, \ldots\), and unrestrictedly when \(\mu = 0, 1, 2\). I have shown\(^\#\) that (2) holds if \(\mu\) is an integer and

\[(4) \quad \lambda_n = O(\lambda_n)\]

In the special case \(\lambda_n = n\), I have proved\(^2\) both (1) and (2) to hold for all \(\mu > 0\) by showing \((G, \lambda, \mu)\) to be equivalent to \((G, \mu)\), well-known to be equivalent to \((R, \lambda, \mu)\).

Since the submission of this paper for publication, it has been shown that (2) holds without restriction on \((\lambda_n)\), for \(\mu\) an integer by Meir\(^4\) and, subsequently, for all \(\mu > 0\) by Russell and myself\(^\#\) (Added May 10, 1967).

For non-integer values of \(\mu\), inclusion (1) is more difficult to deal with than (2). In the range \(0 < \mu \leq 1\), (1) is the same as the inclusion

\[(5) \quad (R^*, \lambda, \mu) \subseteq (R, \lambda, \mu)\]

and Jurkat\(^\#\) has shown (5) to hold without restriction on \((\lambda_n)\) in the said range. Peyerimhoff\(^1\) has established the validity of (5) in the range \(1 < \mu < \log 3/\log 2 = 1.5850\ldots\) subject to the conditions

\[
\begin{align*}
\lambda_{n+1} & \leq (1 + \epsilon) \lambda_n \\
\lambda_{n+1} & = o(\lambda_n)
\end{align*}
\]

and

\[
\Delta \lambda = o(\lambda_n)
\]

while Kuttner\(^\#\) has shown that, when \(\lambda_n = n\), (5) holds for \(1 < \mu < 2\) but fails

\[\text{for every } \mu > 2\]. The question whether or not (3) continues to hold for more general sequences \((\lambda_n)\) when \(\log 3/\log 2 < \mu < 2\), remains open.

The main object of this paper is to prove that inclusion (1) holds whenever \(1 < \mu < 2\), provided the sequence \((\lambda_n)\) satisfies

\[(6) \quad \lambda_{n+1} = o(\lambda_n)\]

and

\[(7) \quad \Delta \lambda = o(\lambda_n)\]

Note that (6) implies both (3) and (4).

It is hoped that, by extending the arguments employed below, it will be possible to establish (1) for non-integer \(\mu > 2\) under reasonably light restrictions on \((\lambda_n)\).

2. Auxiliary results. Let \((c_n)\) be a normal matrix (i.e. \(c_n = 0\) for \(n > n\) and \(c_n \neq 0\)) and let

\[a_n = \sum_{n=0}^{\infty} c_n \cdot t_n \quad (m = 1, 2, \ldots)\]

The following three lemmas, incorporating key results required in the rest of the paper, are due essentially to Jurkat and Peyerimhoff\(^2\). The proof of Lemma 2 is straightforward\(^1\) and Lemma 3 is an immediate consequence of Lemmas 1 and 2.

**Lemma 1.** If

\[(8) \quad c_{n+1} > 0 \quad (0 \leq n \leq \eta)\]

\[(9) \quad \frac{c_{n+1}}{c_n} \leq \frac{\max_{0 \leq m \leq n} |t_m|}{\min_{0 \leq m \leq n} |t_m|} \quad (1 \leq \mu < n - 1)\]

then the matrix \((c_{n+1})\) satisfies the "mean value" condition

\[(10) \quad \left| \sum_{n=0}^{\infty} c_n \cdot t_n \right| \leq \max_{0 \leq m \leq n} \left| t_m \right| \quad (0 \leq m \leq n)\]

**Lemma 2.** If \(\xi_n = 0\), then \((\xi_n)\) and

\[(11) \quad a_n = o(\xi_n)\]

\[(12) \quad 0 \leq \sum_{n=0}^{\infty} c_n \cdot t_n \quad (0 \leq c_n \leq n, M a positive constant)\]

1. Jurkat and Peyerimhoff (9, Satz 1 and Satz 3; see also Peyerimhoff, 11, p. 71).
2. Jurkat and Peyerimhoff (8, p. 98).
then \[ \max_{x \leq \xi} \frac{\xi^2}{\nu^2} \left| v_x \right| = \omega(\xi). \]

**Lemma 3.** If \( \lambda > 0 \), \( \nu = o(\lambda) \) and the matrix \((a_{x,y})\) satisfies conditions (10), (11) and (12), then \[ \nu_x = \left( \frac{\xi}{\nu} \right)^{\nu_x}. \]

We require one additional lemma.

**Lemma 4.** The method \((C, \lambda, \mu)\) is regular for every \( n \geq 0 \).

The case \( 0 < \nu < 1 \) of this lemma is well-known, and Russel\(^1\) has proved it for \( \mu \) an integer. Let \[ \nu_x = \frac{\xi}{\nu} \]
then \[ \frac{C_{x,y}}{\nu_x} = \frac{\nu_x}{\nu_x} \]
where \[ \nu_x = \frac{1}{\nu^2} \left( \left| v_x^2 \right| \right) \left( \left| v_x^2 \right| \right) \geq 0 \quad (0 \leq \nu \leq 9). \]

Now, for any fixed \( \nu > 0 \), \[ \frac{\nu_x}{\nu_x} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty, \]
so that \[ \nu_x \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty; \]
and \[ \frac{\xi}{\nu} \nu_x \rightarrow \frac{\xi}{\nu} \nu_x \rightarrow \frac{\nu_x}{\nu_x} \rightarrow 1. \]

It follows, by a standard result, that \( C_{x,y}/\nu_x \) converges whenever \( \nu_x \rightarrow \), i.e. that \((C, \lambda, \mu)\) is regular.

3. The main results. Suppose throughout this section that \( 0 < s < 1 \).

In addition to the notations introduced in 3.1, we shall also use the following:
[laTeX code for notations]

1. Russel\(^1\), (12).
2. Hardy\(^2\), p. 49.
Differentiating (18) with respect to \( t \), we get
\[
\frac{\partial}{\partial t} \phi(a, t) = \frac{1}{a - t} \phi(a, t) - \frac{\phi(a, t)}{a} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)}
\]

Since
\[
\lim_{t \to a^-} \frac{\phi(a, t)}{\phi(a, t)} = \phi(a, t), \quad (\lambda_s < h_t < \lambda_{s+1}, 0 < t < a),
\]

inequality (9) will be established if we can show that
\[
\frac{\phi'(a, t)}{\phi(a, t)} \leq \frac{\phi'(a, t)}{\phi(a, t)}(0 < t < \lambda_s, \ s \geq 1).
\]

In view of (17), (18), and (19), we have
\[
\frac{\phi'(a, t)}{\phi(a, t)} = -\frac{\phi'(a, t)}{\phi(a, t)} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

Now, for fixed \( f \geq 0 \),
\[
\frac{\phi'(a, t)}{\phi(a, t)} \text{ decreases with increasing } u > \min (t, \lambda_s).
\]

since, when
\[
\frac{\phi'(a, t)}{\phi(a, t)} = \frac{\phi'(a, t)}{\phi(a, t)} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right) \left( \frac{a}{t} - 1 \right)
\]

and, in virtue of (6),
\[
\frac{\phi'(a, t)}{\phi(a, t)} \text{ is decreasing.}
\]

Equation (20) follows from (7), (21) and (22); and it remains to establish that the matrix satisfies (11) and (12).

In view of (17) and (18), we have
\[
\frac{\phi(a, t)}{\phi(a, t)} = \frac{\phi(a, t)}{\phi(a, t)} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

where
\[
\left( \lambda_s < h_t < \lambda_{s+1}, 0 < t < \lambda_s, \ s \geq 1 \right).
\]

Since
\[
\lambda_s > h_t + \lambda_{s+1} - \lambda_s > h_t + \lambda_{s+1} - \lambda_s > \lambda_{s+1}, \ s \geq 1, \frac{a}{t} > 1
\]

it follows from (22) and (23) that
\[
\frac{\phi(a, t)}{\phi(a, t)} \leq \frac{\phi(a, t)}{\phi(a, t)} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

Hence, by (7), (16), and (24),
\[
\phi(a, t) = \phi(a, t) \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

i.e. the matrix \( (\epsilon_{a, t}) \) satisfies (11).

Finally, by (6), (7) and (16),
\[
\lambda_s > h_t + \lambda_{s+1} - \lambda_s > \lambda_{s+1} - \lambda_s > \frac{a}{t} > 1
\]

and so it follows from (24) that, for \( 0 < t < a - 1 \),
\[
\epsilon_{a, t} \leq \epsilon_{a, t} \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

showing that the matrix \( (\epsilon_{a, t}) \) satisfies (12).

In what follows, we suppose that
\[
x = x(a) \quad (a > \lambda_s)
\]

is the integer such that
\[
\lambda_s < x < \lambda_{s+1}.
\]

**Theorem 2.** If (6), (7) and (16) hold, if \( \lambda_s > 1 \), and if
\[
\phi(a, t) = \phi(a, t) \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

then
\[
\epsilon_{a, t} = \phi(a, t) \left( 1 - t \right) \left( a - t \right) \frac{\phi'(a, t)}{\phi(a, t)} \left( \frac{a}{t} - 1 \right)
\]

**Proof.** Note first that, by (16), \( x < a \).
In virtue now of (13) and (23), we have

\begin{equation}
\alpha_\nu = \sum_{k=1}^{\infty} c_k \frac{\Gamma(k+\gamma)}{\Gamma(k+1)} = c_k \frac{\Gamma(k+\gamma)}{\Gamma(k+1)},
\end{equation}

from which it follows, by Theorem 1, Lemmas 1 and 3, and (14), that

\[ x_k = \left( \frac{\Gamma(k+\gamma)}{\Gamma(k+1)} \right) = \left( \frac{\Gamma(k+\gamma)}{\Gamma(k+1)} \right), \]

i.e. that (26) holds.

Suppose now that \(\lambda > \lambda_0\) so that \(x = x(\omega) > \lambda_0\). Then

\begin{equation}
A^t(u) = 0 \int_{x-1}^{x} (u - t)^{-1} A(t) dt = 0 \int_{x-1}^{x} (u - t)^{-1} A(t) dt
\end{equation}

where

\[ b_{r+1} = \int_{x-1}^{x} (u - t)^{-1} dt \quad (0 < r < \infty). \]

In view of (26), we have

\begin{equation}
0 \int_{x-1}^{x} (u - t)^{-1} A(t) dt \leq 0 \int_{x-1}^{x} (u - t)^{-1} dt 
\end{equation}

where

\[ b_{r+1} = \int_{x-1}^{x} (u - t)^{-1} dt \quad (0 < r < \infty). \]

Hence, by (33), (34) and (35),

\begin{equation}
\int_{x-1}^{x} (u - t)^{-1} A(t) dt = \int_{x-1}^{x} (u - t)^{-1} dt \quad (u = \infty). \end{equation}

Next we observe that

\begin{equation}
\frac{b_{r+1}}{\alpha_{r+1}} = \frac{\delta(u-t)}{\alpha_{r+1}} \quad (0 < r < \infty, t > 0). \end{equation}

As \(t\) increases from 0 to \(\lambda_0\),

\[ x(t) = \frac{\delta(u-t)}{\alpha_{r+1}} \]

decreases, since, by (21),

\[ x'(t) = \frac{1}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} < 1 \quad (u = \infty). \]

and, therefore, it follows from (32) that

\[ \frac{b_{r+1}}{\alpha_{r+1}} \preceq \frac{\delta(u-t)}{\alpha_{r+1}} \quad (0 < r < \infty). \]

Consequently, by (30) and partial summation,

\begin{equation}
\delta \int_{x-1}^{x} (u - t)^{-1} A(t) dt \leq \frac{\delta(u-t)}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} \quad (u = \infty). \end{equation}

Now, by (24) and (32),

\begin{equation}
\delta \int_{x-1}^{x} (u - t)^{-1} A(t) dt \leq \frac{\delta(u-t)}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} \quad (u = \infty). \end{equation}

Also, by Theorem 1 and Lemmas 1 and 2, it follows from (29) that

\begin{equation}
\max_{0 < s < \infty} s \frac{\delta(u-t)}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} \leq \delta(u-t) \quad (u = \infty). \end{equation}

Hence, by (33), (34) and (35),

\begin{equation}
\int_{x-1}^{x} (u - t)^{-1} A(t) dt \leq \int_{x-1}^{x} (u - t)^{-1} dt \quad (u = \infty) \end{equation}

and conclusion (27) follows from (30), (31), and (36).

We are now in a position to establish (28), the one outstanding conclusion. By (35) and (27),

\[ A^{\infty}(\lambda) = (\lambda - 1)^{\frac{1}{\gamma}} \preceq \frac{\delta(u-t)}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} \quad (u = \infty) \]

and consequently, by (27) again,

\[ A^{\infty}(\lambda) = (\lambda - 1)^{\frac{1}{\gamma}} \preceq \frac{\delta(u-t)}{\alpha_{r+1}} \frac{\delta(u-t)}{\alpha_{r+1}} \quad (u = \infty) \]

**Theorem 3.** If \(\delta(u-t)\) and \(\gamma(t)\) hold, and \(C_1 = c(q, \lambda, \gamma, 0)\), then

\[ A^{\infty}(\lambda) = (\lambda - 1)^{\frac{1}{\gamma}} \preceq \delta(u-t) \quad (u = \infty). \]

**Proof.** Take \(t = \lambda_0 + \delta(1)\); then

\[ \frac{\delta}{\lambda_0 + \delta(1)} \preceq \delta(u-t) \quad (u = \infty). \]

so that \(\delta(u-t)\) satisfies (16). Also, by (5), \(\delta(1)\). Consequently, by Theorem 2,

\[ A^{\infty}(\lambda) = (\lambda - 1)^{\frac{1}{\gamma}} \preceq \delta(u-t) \quad (u = \infty). \]
since, in view of (6),

$$\frac{\lambda_{x+1}^{*+1}}{\lambda_{x+1}^{*}} < \frac{\lambda_{x+1}^{*}}{\lambda_{x+1}^{*}} \left( \frac{\lambda_{x+1}^{*}}{\lambda_{x+1}^{*}} \right)^{\delta} = O(1).$$

Since \((R, \lambda, \mu)\) is known to be regular and \((C, \lambda, \mu)\) is regular, by Lemma 4, the following theorem is an immediate consequence of Theorem 3.

**Theorem 4.** If \(\frac{\lambda_{x+1}^{*}}{\lambda_{x+1}^{*}} \downarrow, \frac{\lambda_{x+1}^{*+1}}{\lambda_{x+1}^{*}} \downarrow\) and \(1 < \mu < 2\), then

\((C, \lambda, \mu) \subseteq (R, \lambda, \mu)\).

**REFERENCES**