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Strong Nörlund Summability

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1. Introduction

In this paper we give a definition of strong Nörlund summability, and show that in the case of Cesàro summability our definition is equivalent to the standard definition of strong Cesàro summability. We answer such questions as: "If one Nörlund method of summability includes another, is the same true of the associated strong methods?". We establish relations between strong Nörlund, absolute Nörlund and Nörlund summability. For a certain class of Nörlund methods of summability \((C, a)\) with \(a > -1\). Finally we consider a method of summability \((C^{\infty}, a)\), known to be equivalent to \((C, a)\) for \(a > 0\), and use our theorems to show that the associated strong methods of summability are equivalent.

2. Generalities and Definitions

Throughout this paper, \(H, H_i, \ldots\) etc. will denote positive constants, which will not necessarily take the same value at different occurrences.

Suppose throughout that

\[
s_n = \sum_{r=0}^{n} a_r, \quad s_n^* = \left(\frac{n+1}{n}\right)^n (a+1) \ldots (a+n) \frac{1}{n!}.
\]

Given an arbitrary sequence \(\{w_n\}\), we define

\[
A w_n = w_n - w_{n-1}, \quad w_{-1} = 0.
\]

Let \(\{p_n\}\) and \(\{q_n\}\) be arbitrary sequences of complex numbers, let

\[
P_n = \sum_{r=0}^{n} p_r, \quad Q_n = \sum_{r=0}^{n} q_r,
\]

and assume throughout that \(P_n\) and \(Q_n\) are non-zero for all values of \(n\). Also let

\[
P_n^* = \sum_{r=0}^{n} |p_r|, \quad Q_n^* = \sum_{r=0}^{n} |q_r|.
\]

The \(n\)-th \((N, p_n)\) and \(n\)-th \((N, q_n)\) transforms of the sequence \(\{s_n\}\) are respectively

\[
t_n = \frac{1}{p_n} \sum_{r=0}^{n} p_r s_{n-r} = \frac{1}{p_n} \sum_{r=0}^{n} p_r a_{n-r}
\]

and

\[
u_n = \frac{1}{q_n} \sum_{r=0}^{n} q_r s_{n-r} = \frac{1}{q_n} \sum_{r=0}^{n} q_r a_{n-r}.
\]

The series

\[
\sum_{n=0}^{\infty} a_n
\]

is said to be summable \((N, p_n)\) to \(s\) if \(t_n \to s\) as \(n \to \infty\). We denote this by

\[
\sum_{n=0}^{\infty} a_n = s(N, p_n).
\]

All limits in the sequel will be taken as the variable tends to infinity, unless otherwise specified.

We shall also have occasion, in the case that \(p_n \neq 0\) and \(q_n \neq 0\) for all values of \(n\), to use the notation

\[
t_n^* = \frac{1}{p_n} \sum_{r=0}^{n} p_r a_{n-r}
\]

and

\[
u_n^* = \frac{1}{q_n} \sum_{r=0}^{n} q_r a_{n-r}.
\]

We note that (2.4) and (2.5) are respectively the \(n\)-th \((N, p_n)\) and \(n\)-th \((N, q_n)\) transforms of the sequence \(\{s_n\}\).

A method of summability is regular, if it sums every convergent series to its ordinary sum.

If \(P\) and \(Q\) are methods of summability, \(Q\) is said to include \(P\) (written "\(P \supset Q\)"") if every series summable by the method \(P\) is also summable by the method \(Q\) to the same sum. \(P\) and \(Q\) are said to be equivalent (written "\(P = Q\)"") if each method includes the other.

Using standard results, [3, Theorem 2], we find that the Nörlund method \((N, p_n)\) is regular if and only if

\[
P_n^* = O(|P_n|)
\]

and

\[
p_n / P_n \to 0.
\]

See also [4].

Thus for a regular Nörlund method \((N, p_n)\), either,

\[
\sum_{r=0}^{n} |p_r| < \infty
\]
or

\( (2.9) \quad |P_n| \to \infty. \)

Since \( p_n \) and \( q_n \) are both non-zero, there exist sequences \( \{k_n\}, \{l_n\} \) and \( \{a_n\} \) such that,

\[
(2.10) \quad k_0 p_0 + \cdots + k_n p_n = q_n, \quad n = 0, 1, 2, \ldots,
\]

\[
(2.11) \quad l_0 q_0 + \cdots + l_n q_n = p_n, \quad n = 0, 1, 2, \ldots,
\]

\[
(2.12) \quad \gamma_0 p_0 + \cdots + \gamma_n p_n = 0, \quad n = 1, 2, 3, \ldots.
\]

Thus

\[
\gamma_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r a_r,
\]

if and only if

\[
r_n = \sum_{r=0}^{n} \gamma_r l_r.
\]

Therefore every Nörlund transformation is invertible.

The following propositions give necessary and sufficient conditions for inclusion or equivalence relations to hold between two Nörlund methods. The proofs are closely modelled on proofs given by Hardy [3, Theorems 19 and 21], but are applicable to a larger class of Nörlund methods than are considered by Hardy. In connection with the second proposition see also [4, Corollary 1].

**Proposition 1.** For Nörlund methods \((N, p_n)\) and \((N, q_n)\), (not necessarily regular), necessary and sufficient conditions that \((N, p_n) \approx (N, q_n)\) are

\[
(2.14) \quad |k_0| |P_0| + \cdots + |k_n| |P_n| \leq H |Q_n|
\]

where \( H \) is independent of \( n \), and

\[
(2.15) \quad k_{r+1} q_r - q_{r-1} = 0, \quad \text{for each } r.
\]

**Proof.** Referring to (2.2), we have

\[
(2.16) \quad \alpha_n = \sum_{r=0}^{n} \epsilon_r l_r,
\]

with \( \epsilon_r = -k_r, q_{r+1} \gamma_r \), for \( r \leq n \) and \( \epsilon_r = 0 \) for \( r > n \), cf. [3, Theorem 19].

Since a Nörlund method is invertible \((N, p_n) \approx (N, q_n)\) if and only if the sequence to sequence transformation (2.16) is regular, A standard result now yields (2.14) and (2.15). See for example [3, Theorem 2].

**Proposition 2.** For regular Nörlund methods \((N, p_n)\) and \((N, q_n)\), necessary and sufficient conditions that \((N, p_n) \approx (N, q_n)\) are

\[
(2.17) \quad \sum_{r=0}^{n} |k_r| < \infty \quad \text{and} \quad \sum_{r=0}^{n} |l_r| < \infty.
\]

**Proof.** Necessity.

By Proposition 1 both \(|P_n|/|Q_n|\) and \(|Q_n|/|P_n|\) are bounded. Also by (2.14),

\[
|k_0| + |k_1| |P_1| + \cdots + |k_n| |P_n| \leq H |Q_n|/|P_n|
\]

for \( r \leq n \). Now fixing \( n \), letting \( r \) tend to infinity, and using the fact that for a regular Nörlund method \((N, p_n)\), \( P_n \to P_0 \) for each \( r \), we have

\[
|k_0| + \cdots + |k_r| \leq H \limsup \left( |Q_n|/|P_n| \right) = H < \infty,
\]

so that

\[
\sum_{r=0}^{n} |k_r| < \infty.
\]

Similarly

\[
\sum_{r=0}^{n} |l_r| < \infty,
\]

and, hence we have (2.17).

**Sufficiency.**

We now have \( k_{r+1} q_r - q_{r-1} = 0 \) for each \( r \), and, because of the regularity of \((N, q_n)\),

\[
|Q_n| \leq H |Q_{n+2}| \leq H^2 |Q_{n+1}| > 0; \text{ it follows that } (2.15) \text{ holds. Also, by } (2.6) \text{ and its}
\]

analogue for the method \((N, q_n)\),

\[
|P_n| \leq |Q_n| |l_1| + \cdots + |Q_n| |l_n| \leq H |Q_n| \sum_{r=0}^{n} |l_r|,
\]

and

\[
|k_0| |P_0| + \cdots + |k_n| |P_n| \leq H |Q_n| |\sum_{r=0}^{n} |k_r| |l_r|.
\]

Thus (2.14) holds, and hence \((N, p_n) \approx (N, q_n)\). Similarly, \((N, q_n) \approx (N, q_n)\), and the proof is complete.

The \( n \)-th \((N, p_n)\) transform of the sequence \( \{x_n\} \) is

\[
(2.18) \quad x_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r x_r.
\]

Associated with this transform is a method of summability, \((\bar{N}, p_n)\), defined in the same way as in the case of \((N, p_n)\).

Using a standard result [3, Theorem 2], we find that \((N, p_n)\) is regular if and only if (2.6) and (2.9) hold.

**Proposition 3.** If \((N, p_n)\) is regular, \( p_n > 0 \) for all \( n \) and either

\[
|p_n| \text{ is non-decreasing and } \sum_{n=0}^{\infty} |p_n| < \infty
\]

or

\[
|p_n| \text{ is non-increasing and } \sum_{n=0}^{\infty} |p_n| < \infty
\]

then

\[
(\bar{N}, p_n) = (\bar{N}, 1).
\]
Proof. This result is an immediate consequence of a theorem given by Hardy [3, Theorem 4].

Definitions. 1. Strong summability \([N, p_\lambda, \lambda > 0]\).

Let \((N, p_\lambda)\) be a Nörlund method with \(p_\lambda \rightarrow 0\) for all values of \(n\). We shall say that

\[
\sum_{n=0}^{\infty} a_n = s(N, p_\lambda)
\]

is strongly summable \((N, p_\lambda)\) with index \(\lambda\) to \(s\), if

\[
\frac{1}{p_\lambda} \sum_{n=0}^{n} |p_\lambda| |s - s|^\lambda = o(1).
\]

We shall denote this by

\[
\sum_{n=0}^{\infty} a_n = s(N, p_\lambda),
\quad \text{or} \quad
s_n \rightarrow s[N, p_\lambda].
\]

Remark. Whenever (2.6) holds,

\[
\sum_{n=0}^{\infty} a_n = s(N, p_\lambda),
\]

if and only if

\[
\frac{1}{p_\lambda} \sum_{n=0}^{n} |p_\lambda| |s - s|^\lambda = o(1).
\]

We shall take advantage of this result without further comment.

2. Absolute summability \([N, p_\lambda, \lambda > 0]\).

We shall say that

\[
\sum_{n=0}^{\infty} a_n
\]

is absolutely summable \((N, p_\lambda)\) with index \(\lambda\), or summable \([N, p_\lambda]\), if

\[
\sum_{n=0}^{\infty} |p_\lambda| |s_n - s_{n-1}| < \infty.
\]

When \(\lambda = 1\), this definition reduces to the customary definition of absolute Nörlund summability, as given by Mears [5] for example. See also [1].

We recall now the standard definition of strong Cesàro summability \([C, \alpha + 1]\), and show that it is equivalent to our definition. For \(\lambda > 0, \alpha > -1\), the series

\[
\sum_{n=0}^{\infty} a_n
\]

is said to be summable \([C, \alpha + 1]\) to \(s\), if

\[
\frac{1}{s + 1} \sum_{n=0}^{n} |s - s|^\alpha = o(1)
\]

where

\[
\alpha = \frac{1}{\alpha + 1} \sum_{n=0}^{n} |s - s|^\alpha.
\]

Since the Cesàro method of summability \([C, \alpha + 1]\) is the method \((N, p_\lambda)\) with \(p_\lambda = \infty\), in order to show that our definition of strong Cesàro summability as strong Nörlund summability is equivalent to the standard definition of strong Cesàro summability it suffices to observe that \((N, 1) \Rightarrow (N, \alpha)\) for \(\alpha > -1\).

This follows from Proposition 3, because \([\alpha]_n\) is non-increasing when \(-1 < \alpha \leq 0\), non-decreasing when \(\alpha > 0\), and

\[
\alpha_{n+1}^\alpha = n(\alpha + 1) \quad \text{as} \quad n \to \infty.
\]

3. Inclusion Theorems

In this section we shall prove certain theorems giving sufficient conditions for one strong Nörlund method of summability to include another. Before doing so however, we make the following simplifying remark.

Remark. If \((N, p_\lambda)\) is a Nörlund method with \(p_\lambda \rightarrow 0\) for all values of \(n\), and \(s_n\) is a sequence, then \(s - s\) is the \(r\)-th \((N, d\lambda, \lambda)\) transform of the sequence \(s_n - s\).

Thus, we have

\[
\sum_{n=0}^{\infty} a_n = s[N, p_\lambda],
\]

if and only if \(s_n \rightarrow 0\) \([N, p_\lambda]\). Hence in order to prove that \([N, p_\lambda]\) \(\Rightarrow [N, q_\lambda]\), it is sufficient to prove that \(s_n \rightarrow 0\) \([N, p_\lambda]\) implies \(s_n \rightarrow 0\) \([N, q_\lambda]\). For the remainder of this paper, we shall assume that if \((N, p_\lambda)\) is a Nörlund method, then \(p_\lambda \rightarrow 0\) for all values of \(n\), unless mention is made to the contrary.

Theorem 1. If \((N, p_\lambda) \Rightarrow (N, q_\lambda)\) then \([N, p_\lambda] \Rightarrow [N, q_\lambda]\).

Proof. Referring to (2.4) and (2.5), we have, for a given sequence \(s_n\),

\[
q_n s_n = \sum_{n=0}^{n} k_{n-1} p_\lambda |s_n|,
\]

Thus

\[
|q_n| |s_n|^\alpha = \sum_{n=0}^{n} |k_{n-1}| |p_\lambda| |s_n|^\alpha
\]

and hence

\[
\sum_{n=0}^{n} |q_n| |s_n|^\alpha = \sum_{n=0}^{n} \sum_{k_{n-1}} |p_\lambda| |s_n|^\alpha
\]

Setting

\[
q_n = \sum_{k_{n-1}} |p_\lambda| |s_n|^\alpha.
\]
we see that
\[ \sum_{r \geq 0} |s_r| |t_r| = \sum_{r \geq 0} |k_{r,-r}| |p_r| = \sum_{r \geq 0} |k_{r,-r}| |p_r| |t_r|/|p_r|. \]

Supposing now, that \( z_n \to 0 \), we have \( \varphi_\omega \equiv |P_r| = o(1) \). Thus
\[ \sum_{r \geq 0} |s_r| |P_r| = o(1), \]

provided
\[ |k_{r,-r}| |P_r| = O(1), \]

and
\[ |k_{r,-r}| = O(1), \quad \text{for each } r. \]

But, by Proposition 1 this is equivalent to our hypothesis \( (N, p_n) \to (N, q_n) \). It follows from (a) and (b) that \( z_n \to 0 \).

Thus
\[ \sum_{r \geq 0} |k_{r,-r}| |P_r| = O(1), \]

and the proof is complete.

**Corollary 1.** If \( (N, p_n) \approx (N, q_n) \), then \( (N, p_n) \approx (N, q_n) \).

**Theorem 2.** If \( (N, p_n) \to (N, q_n) \) and
\[ \sum_{r \geq 0} |k_{r,-r}| |P_r| = O(1), \]

then
\[ \sum_{r \geq 0} |P_r| = O(1), \quad \text{for } \lambda > 1. \]

**Proof.** Using Hölder's inequality, we obtain
\[ |q_r| \leq H |q_{\lambda^{-1}} | + \sum_{r \geq 0} |k_{r,-r}| |P_r| |t_r|^\lambda |. \]

Thus
\[ \sum_{r \geq 0} |s_r| |t_r| \leq \sum_{r \geq 0} |k_{r,-r}| |P_r| |t_r| |. \]

To complete the proof we now set
\[ \varphi_\omega = \sum_{r \geq 0} |p_r| |t_r|, \]

and proceed as in the proof of Theorem 1.
Theorem 3. If $(N, q_a)$ is a regular Nörlund method, $(p_n)$ satisfies (3.3), $q_a > 0$, $q_a/q_{a-1} \leq p_a/p_{a-1}$ for $n > 0$, and $p_n = O(q_a)$, then $(N, p_n)$ is regular, (3.1) holds, and $(N, p_n) \Rightarrow (N, q_a)$.

Proof. \[
\frac{1}{q_a} \leq P_n \leq \frac{1}{q_{a-1}}
\]
and
\[
p_n \leq H q_a
\]
so that $p_n \leq q_a q_{a-1}/q_{a-2}$ and hence $p_n \to 0$.

The condition $P_n = O(q_{a-1})$ is satisfied because $p_n > 0$. Thus $(N, p_n)$ is regular.

We write $c_n = -c_n$ for $n > 0$, so that, in view of the above mentioned result of Hardy, $c_n \geq 0$ for $n > 0$.

Now
\[
p_n - c_n p_{n-1} - \cdots - c_1 p_0 = 0
\]
for $n > 0$
and by (2.10) and (2.11)
\[
q_a - c_n q_{a-1} - \cdots - c_1 q_0 = k_n
\]
for $n \geq 0$.

Hence
\[
\frac{k_n}{q_a} = 1 - c_n - \cdots - c_1
\]
and thus
\[
k_n \leq 0
\]
for $n > 0$.

So we have
\[
-k_n p_{a-1} + k_{a-1} p_{a-2} - \cdots + k_0 p_0 = -k_0 p_a - k_1 p_{a-1} - \cdots - k_{a-1} p_0
\]
for $n > 0$.

Since $p_n = O(q_a)$. This proves (3.1). It follows by summing both sides of (3.1) that
\[
\sum_{n=0}^{\infty} k_n p_{a-1} = O(q_a)
\]
Also we have
\[
|k_n| p_a + |k_{a-1}| p_{a-1} + \cdots + |k_0| p_0 \leq O(q_a) = O(q_a)
\]
since $(N, q_a)$ is regular. Further $0 < q_a \leq q_{a-1}$ for $r > 0$ and so
\[
k_n q_a \to 0
\]
as $n \to \infty$.

It follows now from Proposition 1 that $(N, p_n) \Rightarrow (N, q_a)$.

Corollary 4. If $(N, p_n)$ and $(N, q_a)$ satisfy the hypotheses of Theorem 3, then
\[
[N, p_n] \Rightarrow [N, q_a]
\]
for $\lambda > 1$.

Proof. This result is an immediate consequence of Theorems 2 and 3.

Theorem 4. If $(N, p_n)$ and $(N, q_a)$ are regular Nörlund methods, $(p_n)$ satisfies (3.3), and $q_a > 0$, $p_n p_{n-1} \leq q_a p_{a-1}$ for $n > n_0$, then
\[
[N, p_n] \Rightarrow [N, q_a]
\]
for $\lambda > 1$.

Proof. In the case $n_0 = 0$ the result is an immediate consequence of Theorem 2 and Hardy's Theorem 23 in [3], which yields $(N, p_n) \Rightarrow (N, q_a)$ and $k_n \geq 0$ for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
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for all $n$, so that
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\]
for all $n$, so that
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|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n$, so that
\[
|k_0| p_a + \cdots + |k_n| p_0 = k_0 p_a + \cdots + k_n p_0
\]
for all $n
To prove that \((N, q_n) \Rightarrow (N, p_\nu)\) let \(s_n\) be defined as in the proof of Theorem 4. 

Now, \((N, q_n) \Rightarrow (N, n)\) by Theorem 3, \((N, n) \Rightarrow (N, p_\nu)\) by HÃ¥RDWÃ¶RNE's Theorem 23 in [3], and so \((N, q_n) \Rightarrow (N, p_\nu)\). 

Thus \((N, p_\nu) \Rightarrow (N, q_n)\) and consequently, by Corollary 1, \([N, p_\nu] \equiv [N, q_n]\). 

To show that \([N, p_\nu] \equiv [N, q_n]\), for \(k > 1\), we observe that \([N, p_\nu] \equiv [N, q_n]\), by Theorem 4, \([N, n] \equiv [N, p_\nu]\), by the case \(n_\nu = 0\) of Theorem 4, and \([N, q_n] \equiv [N, q_n]\) by Corollary 4 and hence \([N, q_n] \equiv [N, p_\nu]\). This completes the proof.

4. Relations Between Strong NÃ¶rlund, Absolute NÃ¶rlund, and NÃ¶rlund Summability Methods

Theorem 6. \([N, p_\nu] \equiv (N, p_\nu)\). 

Proof. Suppose \(s_n \rightarrow s\) \([N, p_\nu]\). 

Now 

\[
\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} p_\nu (t^\nu - s) \leq \frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|.
\]

Thus 

\[
\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} p_\nu t^\nu \rightarrow s.
\]

But 

\[
\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} p_\nu t^\nu = \frac{1}{|P_\nu|} \sum_{j=0}^{\nu} p_\nu s_{\nu j},
\]

and so 

\[
s_n \rightarrow s(N, p_\nu).
\]

Theorem 7. If \(P_\nu = O(|P_\nu|)\) then \([N, p_\nu] \Rightarrow (N, p_\nu)\) for \(k > 1\). 

Proof. Using the fact that \(P_\nu = O(|P_\nu|)\) in conjunction with Theorem 1 in [1], we find that \([N, p_\nu] \Rightarrow [N, p_\nu]\), for \(k > 1\). The result now follows from Theorem 6.

Theorem 8. If \((N, p_\nu)\) is regular, and \(k \geq 1\) then, \(s_n \rightarrow s\) \([N, p_\nu]\) if and only if, 

\[
s_n \rightarrow s(N, p_\nu).
\]

The proof of this theorem follows closely the proof of BORWEIN's Theorem 7 in [1].

Proof. We have to prove that 

\[
\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|^4 = o(1)
\]

if and only if 

\[
t_n \rightarrow s
\]

and 

\[
\left(\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|^4\right) = o(1).
\]

(i) Suppose that (4.2) holds. Then, by Theorem 7, (4.3) holds, and so 

\[
\left(\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|^4\right) = o(1)
\]

since \((N, p_\nu)\) is regular. Hence, by MINKOWSKI's inequality and (4.2), 

\[
\left(\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|^4\right)^{1/4} = o(1)
\]

and (4.4) follows.

(ii) Suppose that (4.3) and (4.4) hold. Again, (4.5) holds. Hence, by MINKOWSKI's inequality and (4.4), 

\[
\left(\frac{1}{|P_\nu|} \sum_{j=0}^{\nu} |p_\nu| |t^\nu - s|^4\right)^{1/4} = o(1),
\]

so that (4.2) holds. The proof is thus complete.

Remark. 

If \(\sum_{n=0}^{\infty} a_n\) is summable \([N, p_\nu]\), then \(\sum_{n=0}^{\infty} a_n = s(N, p_\nu)\) where 

\[
s = \sum_{n=0}^{\infty} (a_n - a_{n-1}) + a_0.
\]

Theorem 9. If \((N, p_\nu)\) is regular and 

\[
\sum_{n=0}^{\infty} a_n\] is summable \([N, p_\nu]\), then 

\[
\sum_{n=0}^{\infty} a_n = s(N, p_\nu),
\]

where \(s\) is given by (4.6).
Proof. By Theorem 8 it suffices to prove that,

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| = o(1).
\]

Now, for \( r > 0 \),

\[
\frac{1}{P_n} \sum_{l=0}^{n} A_{p_1} t_{l} - A_{p_2} t_{l-1} = \frac{P_n \sum_{l=0}^{n} p_{r} t_{l} - P_n \sum_{l=0}^{n-1} p_{r} t_{l-1}}{P_n} = \frac{P_n \sum_{l=0}^{n} p_{r} t_{l} - P_n \sum_{l=0}^{n-1} p_{r} t_{l-1}}{P_n} = \frac{P_n \sum_{l=0}^{n} p_{r} t_{l} - P_n \sum_{l=0}^{n-1} p_{r} t_{l-1}}{P_n}.
\]

so that

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| = o(1).
\]

and hence

\[
\left| t_{l} - t_{l-1} \right| \leq \frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right|.
\]

Consequently, since \( t_{0} = t_{0} \),

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| \leq \frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right|.
\]

Let

\[
b_{l} = t_{l} - t_{l-1}, \quad B_{l} = \sum_{l=0}^{n} b_{l},
\]

then

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| = \frac{1}{P_n} \sum_{l=0}^{n} \sum_{l=0}^{n-1} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| = o(1)
\]

by the regularity of \( (N, p_r) \). The required conclusion follows.

**Theorem 10.** If \( r > 1 \), \( (N, p_r) \) is regular and

\[
P_{n+1} = o(1).
\]

and if the series

\[
\sum_{l=0}^{n} a_{l},
\]

is summable \((N, p_r)\) to \( s \) and is summable \( |N, p_r| \) then the series is summable \([N, p_r]\) to \( z \).

**Proof.** Using (4.8), we find that

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| \leq \frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right|.
\]

Using the same technique as in the proof of Theorem 9 we find that the final term is \( o(1) \), so that

\[
\frac{1}{P_n} \sum_{l=0}^{n} \left| p_{r} \right| \left| t_{l} - t_{l-1} \right| = o(1).
\]

Finally, using Theorem 8, we obtain the desired conclusion.

**Remark.** Condition (4.9) is satisfied when \( \{p_r\} \) is non-decreasing, and also if \( (N, p_r) \) is the \((C, \infty)\) method of summability with \( \alpha = -1 \).

5. Construction of a Scale of Nörlund Methods

We shall restrict ourselves now to Nörlund methods \((N, p_r)\) for which \( p_n > 0 \) and \( p_n \geq 0 \).

Given any sequence \( \{a_n\} \) we use the notation

\[
a_{n} = \sum_{r=0}^{n} a_{r}^{n+1}.
\]

so that

\[
b_{n} = a_{n-1}.
\]

The following identities are immediate:

\[
\sum_{r=0}^{n} a_{r}^{n+1} = \sum_{r=0}^{n} b_{r} = \sum_{r=0}^{n} p_{r}.
\]

We are going to consider the family of Nörlund methods \((N, p_r)\) for \( \alpha > -1 \), and, when \( p_n > 0 \) for all values of \( n \), we shall allow \( \alpha = -1 \). In the special case \( p_0 = 1 \), \( p_n = 0 \) for \( n > 0 \) we have \( p_n = e_{n+1} \), so that \( (N, p_0) \) is the Cesàro method \((C, \infty)\).

**Theorem 11.** If either (i) \( \beta > \alpha > -1 \) for \( \beta = -1, \alpha > 0 \) and \( p_0 \rightarrow \infty \), then \( (N, p_0) = (N, p_0) \).

**Proof.** We use Proposition 1 with \( p_0 \) in place of \( p_r \) and \( q_r \).

Let \( \beta - \alpha = \delta > 0 \).
Now the appropriate $k_n = n^{-2} > 0$ and, by (5.3),
\[ e_{2n}^{-1} P_{2n} + \cdots + e_{n}^{-1} P_{n} = e_{1}^{n-1}. \]
So the appropriate form of (2.14) in Proposition 1 holds.

Since in this case $k_{n-1} \sim k_n$, in order to verify the appropriate form of (2.15) it suffices to show that
\[ e_{2n}^{-1} P_{2n} = o(1). \]
Now, for $x > -1$,
\[ e_{2n}^{-1} P_{2n} = e_{2n}^{-1} \sum_{n=0}^{\infty} e_{n}^{-1} P_{n} = e_{2n}^{-1} (1/(n+1)) = o(1). \]
Finally when $x = -1$ and $P_n \to \infty$, we may suppose without loss in generality that $0 < \delta \leq 1$ and then obtain
\[ e_{2n}^{-1} P_{2n} = e_{2n}^{-1} \sum_{n=0}^{\infty} e_{n}^{-1} P_{n} = 1/P_n = o(1). \]
This completes the proof of the theorem.

Corollary 5. If $(N, p_n)$ is a regular Nörlund method, then so also is $(N, p_n^2)$ for $x > 0$.

Theorem 12. If either (i) $x > 0$ or (ii) $x = 0$, $p_n > 0$ and $P_n \to \infty$, then $(N, p_n^2) \Rightarrow [N, p_n^2]$ for $\lambda > 1$, $x = 0$.

Proof. Let $s_n \to s(N, p_n^2)$, i.e. let
\[ w_n = n^{-1} \sum_{n=0}^{\infty} e_{n}^{-1} s_n \to s. \]
Then
\[ \frac{1}{n} \sum_{n=0}^{\infty} e_{n}^{-1} P_n = o(1) \]
if $(N, p_n^2)$ is regular, which is the case when either (i) or (ii) is satisfied. Since (5.6) is equivalent to $s_n \to s(N, p_n^2)$, this completes the proof.

Theorem 13. For $x \geq 0$ and $\lambda > 1$,
\[ [N, p_n^2] \Rightarrow (N, p_n^2) \]
provided $p_n > 0$ when $x = 0$.

Proof. This result follows immediately from Theorems 6 and 7, and $p_n^2$ in place of $p_n$.

Theorem 14. If $(N, p_n) \Rightarrow (N, q_n)$, then $(N, p_n^2) \Rightarrow (N, q_n^2)$ for $x > 0$.

Proof. We have, by hypothesis and Proposition 1,
\[ |k_n| P_n + \cdots + |k_n| P_n \leq H Q_n. \]
Now
\[ |k_n| P_n + \cdots + |k_n| P_n = \sum_{n=0}^{\infty} e_{n}^{-1} P_n \]
\[ = \sum_{n=0}^{\infty} e_{n}^{-1} (1/(n+1)) = o(1). \]
Also $|k_n| (1/2)|k_n| (1/2) = o(1)$ by hypothesis and Proposition 1. The conclusion now follows from Proposition 1.

Corollary 6. If $(N, p_n)$ is regular, then, for $x > 0$,
\[ (C, x) \Rightarrow (N, p_n^2) \quad \text{and} \quad [C, x] \Rightarrow [N, p_n^2] \quad \text{for} \quad \lambda > 1. \]

Theorem 15. If $\beta > x \geq 0$ and $\lambda \geq 1$ then $[N, p_n^2] \Rightarrow [N, p_n^2]$, provided $p_n > 0$ when $x = 0$.

Proof. This follows immediately from Theorems 1 and 11 in the case $\lambda = 1$, and from Theorems 2 and 11 in the case $\lambda > 1$, because
\[ \sum_{n=0}^{\infty} e_{n}^{-1} P_n = p_n^2 \quad \text{and} \quad e_{n}^{-1} > 0. \]

6. An Application

The method $(C^*, \mu)$ is defined by Borwein [2] as follows: Let $\mu = m + \delta$, where $m$ is a non-negative integer, and $0 \leq \delta < 1$, and let $\pi_0(s) = m! (s + m + \delta)!^{(s + 1)} \cdots (s + m + \delta)!^s$.

A series
\[ \sum_{n=0}^{\infty} \pi_0(s) \]
is said to be summable $(C^*, \mu)$ to $s$ if,
\[ \sigma_s = \frac{1}{\pi_0(s)} \sum_{n=0}^{\infty} \pi_0(s - n) a_n \to s. \]

The method $(C^*, \mu)$ is the Nörlund method $(N, p_n^2)$ with
\[ p_n = \pi_0(s - n) \pi_0(n - 1). \]

Borwein [2] has proved that
\[ (C^*, \mu) \Rightarrow (C, \mu) \quad \text{for} \quad \mu \geq 0. \]
We now define the strong method \([C^*, \mu]\), to be the method \([N, p_n]\) with \(p_n\) given by (6.1), and prove the following theorem.

**Theorem 16.** For \(\mu > 0, \lambda \geq 1, [C^*, \mu]\) \(\Rightarrow [C, \mu]\).

**Proof.** The case \(\lambda = 1\) follows immediately from (6.2) and Theorem 1. Suppose therefore that \(\lambda > 1\), and let \(q_n = c_n^{-1}\). We consider two cases, (i) \(\mu \geq 1\) and (ii) \(\mu < 1\).

Case (i) \(\mu > 1\). Now \(\pi_n(n) \sim \Gamma(\mu + 1) c_n^\lambda\).

Also

\[
\frac{\pi_n(n)-\pi_n(n-1)}{\pi_{n-1}(n)} \to \delta = \mu \text{ as } n \to \infty.
\]

Thus

\[
p_n = \frac{\pi_n(n) - \pi_n(n-1)}{\mu \pi_{n-1}(n)} \to \Gamma(\mu + 1) c_n^\lambda = \Gamma(\mu + 1) q_n.
\]

Now \(q_n \leq c_n^\lambda \leq q_{n+1}\), since \(\mu \geq 1\), and, by (6.2) and Proposition 2,

\[
\sum_{n=1}^\infty |k_n| < \infty \quad \text{and} \quad \sum_{n=1}^\infty |k_n| q_n < \infty.
\]

Hence

\[
|k_n| p_n + \cdots + |k_l| q_{n+n-l} = O(k_1 q_n + \cdots + k_l q_n) = O(q_n)
\]

and

\[
|k_n| q_n + \cdots + |k_l| q_n = O(q_n) = O(p_n).
\]

The desired result now follows from Theorem 2.

Case (ii) \(\mu < 1\) i.e. \(\mu = \delta < 1\),

Now

\[
q_{n+1}/q_n = q_{n+1}/q_{n-1} \quad \text{for } n > 0,
\]

because

\[
q_{n+1}/q_n = n^\delta + n \delta - 1 \geq 1 \quad \text{for } n > 0.
\]

Also

\[
p_{n+1}/p_n = p_{n+1}/p_n \quad \text{for } n > 0,
\]

because

\[
p_{n+1}/p_n = \frac{n^\delta + n \delta - 1}{n^\delta + n \delta - 1} = \frac{1}{n^\delta + n \delta - 1} \geq \frac{1}{n^\delta + n \delta - 1} = p_{n+1}/p_n.
\]

We show next that there is an integer \(n_0\) such that

\[
(6.5) \quad p_{n+1}/p_n = q_{n+1}/q_n \quad \text{for } n > n_0.
\]

Let

\[
f(x) = \sum_{n=0}^\infty f_n x^n
\]

\[
= \left(1 + (2x)^{1-\delta} + (1 + (2x)^{1-\delta} + 1 + \delta - 1 + (1 + x))^{1-\delta} - (1 - 1 + x)\right).
\]

An elementary computation shows that

\[
f_0 = f_1 = f_2 = 0 \quad \text{and} \quad 2f_3 = \delta (1 - 1/2)^2 > 0.
\]

Hence, for \(\delta\) sufficiently large,

\[
\delta (1 + (2x)^{1-\delta} + (1 + (2x)^{1-\delta} + 1 + \delta - 1 + (1 + x))^{1-\delta} - (1 - 1 + x) > 0
\]

and (6.5) follows, because

\[
p_{n+1}/p_n = q_{n+1}/q_n \quad \text{for } n > n_0.
\]

Since \(p_0 = q_0 = 1, p_n = O(q_n)\) and (6.3), (6.4) and (6.5) hold, we obtain the desired conclusion in Case (ii) by appealing to Theorem 5.

**References**


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