Multiplication Theorems
for Strong Nörlund Summability

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1. Introduction

In [1] a definition of strong Nörlund summability was given, and some of its properties were investigated. In this paper, some theorems concerning the strong Nörlund summability of the Cauchy product of two given series are established, which generalise known theorems about strong Cesàro summability.

2. Preliminaries

Throughout this paper \( H, H_1 \) etc will denote positive constants, which will not necessarily be the same at different occurrences. Let \( \{p_n\} \) be a sequence of real numbers with \( p_0 > 0 \) and \( p_n \geq 0 \) for all \( n > 0 \), and let

\[
P_n = \sum_{r=0}^{n} p_r,
\]

**Definition 1.** Let \( \sum_{r=0}^{\infty} a_r \) be a given series. Define

\[
t_n = \frac{1}{p_n} \sum_{r=0}^{n} p_{n-r} s_r = \frac{1}{p_n} \sum_{r=0}^{n} p_{n-r} a_r \quad (n = 0, 1, 2, \ldots),
\]

where

\[
s_n = \sum_{r=0}^{n} a_r.
\]

If \( t_n \rightarrow s \) as \( n \rightarrow \infty \), we shall write

\[
\sum_{r=0}^{\infty} a_r = s(N, p_n)
\]

or \( s_n \rightarrow s(N, p_n) \).

This is the standard definition of Nörlund summability \( (N, p_n) \). See Tamarkin [8], Nörlund [7] or Hardy [4, p. 64].

**Definition 2.** If \( t_n = O(1) \) we say \( \sum_{r=0}^{\infty} a_r \) is bounded \( (N, p_n) \), where \( t_n \) is defined by (2.1).

**Definition 3.** A method of summability is regular if it sums every convergent series to its ordinary sum.

The method \( (N, p_n) \) is regular if and only if \( p_n / R_n = o(1) \). See Hardy [4, Theorem 16, p. 64].

3 Math. Z., Bd. 107
Definition 4. If \( P \) and \( Q \) are methods of summability, \( Q \) is said to include \( P \) (written \( P \Rightarrow Q \)) if every series summable by the method \( P \) is also summable by the method \( Q \) to the same sum. If further, \( P \) includes \( Q \), \( P \) and \( Q \) are said to be equivalent (written \( P = Q \)).

If \( p_n > 0 \) for all \( n \geq 0 \) and \( \sum r^n a_r \) is a given series, we define \( t'_n \) by

\[
\sum_{r=0}^{t'} a_r = \frac{1}{p_n} \sum_{r=0}^{n} p_r \sum_{r=0}^{t'} a_r, \quad (n = 0, 1, 2, \ldots).
\]

(2.2)

Definition 5. Strong Nörlund Summability \([N, p]_s \), \( s > 0 \).

Let \((N, p)\) be a Nörlund method with \( p_n > 0 \) for all \( n \geq 0 \), and let \( \sum a_n \) be a given series. We shall say that \( \sum a_n \) is strongly summable \([N, p]_s \) with index \( s \) to \( s \), if

\[
\frac{1}{p_n} \sum_{r=0}^{n} p_r |t'_r - s|^s = O(1).
\]

We shall denote this by

\[ \sum_{r=0}^{s} a_r = s[N, p]_s \text{ or } x_s = s[N, p]_s. \]

(2.3)

This is the definition of strong Nörlund summability given in [1], except that in [1], the sequence \( p_n \) is allowed to be a sequence of complex numbers. It is proved in [1] that for \( x > 0 \) and \( s > 0 \), \((C, x)_s \), \([N, p]_x \), \((C, x)_{s, x} \), \([N, p]_{s, x} \), denotes the strong Cesàro method of summability with index \( x \), as defined in [2] for example. See also Wynn [9]. For the remainder of this paper we shall use the notation \( [N, p]_{s, x} \) to denote the method of summability \([N, p]_{s, x} \).

Definition 6. Let \((N, p)\) be a Nörlund method with \( p_n > 0 \) for all \( n \geq 0 \), and let \( \sum a_n \) be a given series. We shall say that \( \sum a_n \) is strongly bounded \([N, p] \) with index \( s \) or \( \sum a_n \) is bounded \([N, p]_s \), if

\[
\frac{1}{p_n} \sum_{r=0}^{n} p_r |t'_r| = O(1).
\]

Definition 7. Absolute Nörlund Summability \([N, p] \).

Let \((N, p)\) be a Nörlund method with \( p_n > 0 \) for all \( n \geq 0 \) and let \( \sum a_n \) be a given series. We shall say that \( \sum a_n \) is absolutely summable \([N, p] \) if

\[
\sum_{r=0}^{s} |t'_r| < \infty
\]

where \( s = \lim s \). We denote this by

\[
\sum_{r=0}^{s} a_r = s[N, p]_s,
\]

This is the definition of absolute Nörlund summability given by Mears in [5].

Definition 8. The method \((N, p) \) is absolutely regular if whenever \( \sum a_n \) is absolutely convergent, it is also summable \([N, p] \).

See Mears [5] and Nielsen [6].

3. Some Known Results

Let \( \sum a_n \) denote the Cauchy product of the series \( \sum a_n \) and \( \sum b_n \), i.e.

\[
c_{n} = \sum_{k=0}^{n} a_k b_{n-k}.
\]

(3.1)

The following propositions (a), (b) and (c) about Cesàro summability have been established; the first two by Wynn in [9], and the third by Boyd in [3].

(a) If \( \sum a_n = s[C, k] \), and \( \sum b_n = s[C, k] \),

where \( k > 0 \) and \( s = 0 \), then \( \sum c_n = s[C, k] \).

(b) If \( \sum a_n = s[C, k] \), and \( \sum b_n = s[C, k] \),

where \( k > 0 \) and \( s = 0 \), then \( \sum c_n = s[C, k] \).

(c) If \( \sum a_n = s[C, k] \), \( \sum b_n = s[C, k] \), where \( k > 0 \),

\( \sum c_n = s[C, k] \), \( \sum b_n = s[C, k] \).

Propositions (a), (b) and (c) are special cases of our Theorems 2, 4 and 6 respectively.

4. Main Theorems

Suppose that \( \{p_n\} \) and \( \{q_n\} \) are sequences of real numbers with \( p_n > 0 \), \( q_n > 0 \), \( p_n > 0 \) and \( q_n > 0 \) for all \( n \). Let

\[
R = \sum_{r=0}^{p_n} p_r, \quad Q = \sum_{r=0}^{q_n} q_r.
\]
Also let,
\[ r_n = \sum_{k=0}^{n} p_k q_k, \quad \text{and} \quad R_n = \sum_{k=0}^{n} r_k, \]
then \( a_n > 0 \) and \( q_k \geq 0 \) for \( n > 0 \).

Given any sequence \( (c_k) \) let \( \xi(x) \) denote the formal power series \( \sum_{n=0}^{\infty} c_n x^n \).

**Theorem 1.** If \( p_n > 0 \) for all \( n \), \( q_n > 0 \), \( r_n \geq 0 \) for all \( n > 0 \), \( \lambda \geq 1 \), \((N, q_n)\) is regular, then
\[
\sum_{n=0}^{\infty} a_n = 0[N, p_n],
\]
and \( \sum_{n=0}^{\infty} b_n \) is bounded \((N, q_n)\), then
\[
\sum_{n=0}^{\infty} c_n = 0(N, r_n).
\]

**Proof.** Using Hölder's inequality, it is easy to show that for \( \lambda > 1 \), \([N, p_n] \rightarrow [N, p_{n+1}]\) (see [2, Theorem 3]). Thus it is sufficient to prove Theorem 1 for the case \( \lambda = 1 \). Let
\[
w_n = \frac{1}{Q_n^{1/\lambda}} \sum_{k=0}^{n} Q_k \cdot b_k,
\]
and
\[v_n = \frac{1}{R_n} \sum_{k=0}^{n} R_k \cdot c_k.\]

Now \( R_n \cdot v_n \) is the coefficient of \( x \) in the series
\[
\sum_{n=0}^{\infty} p_n \cdot c_n x^n = \sum_{n=0}^{\infty} Q_n w_n x^n = q(x) p(x) a(s) (1 - x)^{-1} q(x) b(x) = p(x) q(x) (1 - x)^{-1} c(x) = R(x) c(x).
\]

Thus
\[R_n v_n = \sum_{n=0}^{\infty} p_n \cdot c_n \cdot Q_n \cdot w_n \to 0,
\]
and so,
\[R_n |v_n| \leq \sum_{n=0}^{\infty} |p_n| \cdot |c_n| \cdot |Q_n| \cdot |w_n| = 0.
\]

Since by hypothesis,
\[
\sum_{n=0}^{\infty} |p_n| = o(P),
\]
and
\[|w_n| = O(1),
\]
it follows that
\[R_n |v_n| \leq H \sum_{n=0}^{\infty} |p_n| \cdot |c_n| \cdot |Q_n| = H \sum_{n=0}^{\infty} q_n \cdot \sum_{n=0}^{\infty} |p_n| |c_n| = H \sum_{n=0}^{\infty} q_n \cdot o(P),
\]
and \( \sum_{n=0}^{\infty} b_n \) is bounded \((N, q_n)\), then
\[\sum_{n=0}^{\infty} c_n = o(N, r_n).
\]

Since \((N, q_n)\) is regular the final sum is \( o(R_n) \). Thus \( v_n = o(1) \), and so
\[
\sum_{n=0}^{\infty} c_n = 0(N, r_n)
\]
as required.

**Theorem 2.** If \( p_n > 0 \) for all \( n \), \( q_n > 0 \), \( r_n \geq 0 \) for all \( n > 0 \), \( \lambda \geq 1 \), \((N, p_n)\) and \((N, q_n)\) are regular,
\[
\sum_{n=0}^{\infty} a_n = s[N, p_n], \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = t(N, q_n),
\]
then \( \sum_{n=0}^{\infty} c_n = s t(N, r_n) \).

**Proof.** If \( s = 0 \) the result is an immediate consequence of Theorem 1. Suppose \( s > 0 \). Let
\[
a_0 = a_0 - s, \quad d_n = a_n + s, \quad \text{for} \quad n > 0.
\]

Thus, \( c_0 = d_0 - s b_0 \),
by hypothesis, we have
\[
\sum_{n=0}^{\infty} c_n = 0(N, r_n),
\]
by Theorem 1. Further, since \((N, p_n)\) is regular, \((N, q_n)\to (N, r_n)\) (see Hardy [4, Theorem 17, p. 65]). Thus
\[
\sum_{n=0}^{\infty} b_n = t(N, r_n).
\]
Hence, since
\[
\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n + s \sum_{n=0}^{\infty} b_n \quad \text{for} \quad m = 0, 1, 2, \ldots,
\]
it follows that
\[
\sum_{n=0}^{\infty} c_n = s t(N, q_n).
\]
This completes the proof.

**Theorem 3.** If \( p_n > 0 \) for all \( n \), \( \lambda \geq 1 \), \((N, q_n)\) is regular,
\[
\sum_{n=0}^{\infty} a_n = 0[N, p_n],
\]
and \( \sum_{n=0}^{\infty} b_n \) is bounded \((N, q_n)\), then
\[
\sum_{n=0}^{\infty} c_n = 0(N, r_n).
\]
Proof. Let
\[ v_n^{\ast} = \frac{1}{8} \sum_{k=0}^{n} c_k, \] and
\[ w_n^{\ast} = \frac{1}{8} \sum_{k=0}^{n} q_k, \] Now
\[ \sum_{n=0}^{\infty} a_n^{\ast} x^n = \sum_{n=0}^{\infty} p_n^{\ast} x_n q_n w_n^{\ast} x^n. \] Thus
\[ c_n^{\ast} = \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} \] and so, we have
\[ (c_n^{\ast}|c_n^{\ast}|)^{\frac{1}{2}} \leq \left( \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} \right)^{\frac{1}{2}}. \] Using Hölder's inequality, we find that
\[ (c_n^{\ast}|c_n^{\ast}|)^{\frac{1}{2}} \leq \left( \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} \right)^{\frac{1}{2}}. \] Thus
\[ \sum_{n=0}^{\infty} c_n^{\ast}|c_n^{\ast}|^2 \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_k^{\ast} q_k w_k^{\ast} |x_n|^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} q_k w_k^{\ast} |x_n|^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} q_k w_k^{\ast} |x_n|^2. \] Now, by hypothesis
\[ \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} = O(Q_n) \] and \[ \sum_{n=0}^{\infty} q_n^{\ast} w_n^{\ast} = O(Q_n). \] Thus
\[ \sum_{n=0}^{\infty} c_n^{\ast}|c_n^{\ast}|^2 \leq \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} = O(Q_n). \] since \((N,q_n)\) is regular. Thus
\[ \sum_{n=0}^{\infty} c_n^{\ast} = 0[N,r_1] \] as required.

Theorem 4. If \(p_n > 0, q_n > 0\) for all \(n, \lambda \geq 1, (N,p_n)\) and \((N,q_n)\) are regular,
\[ \sum_{n=0}^{\infty} c_n^{\ast} = s[N,r_1] \] and \[ \sum_{n=0}^{\infty} b_n^{\ast} = t[N,r_1], \] then
\[ \sum_{n=0}^{\infty} c_n^{\ast} = s[N,r_1] \] and so
\[ (c_n^{\ast}|c_n^{\ast}|)^{\frac{1}{2}} \leq \left( \sum_{n=0}^{\infty} p_n^{\ast} q_n w_n^{\ast} \right)^{\frac{1}{2}}. \]
Using Hölder's inequality, we find that
\[
(p_n |c_n|)^{\lambda} \leq \left( \sum_{n=0}^N p_n |c_n|^{\lambda} \right)^{\lambda} \left( \sum_{n=0}^N |b_n| \right)^{1-\lambda}.
\] (4.1)

Since by hypothesis \( \sum_{n=0}^\infty b_n \) is absolutely convergent and \( (p_n) \) is non-decreasing, we find
\[
p_n |c_n|^{\lambda} \leq H \left( \sum_{n=0}^N p_n |c_n|^{\lambda} \right)^{\lambda} \left( \sum_{n=0}^N |b_n| \right)^{1-\lambda}.
\]

We do not need to use the fact that \( (p_n) \) is non-decreasing in the case \( \lambda = 1 \), since the last sum in (4.1) does not appear when \( \lambda = 1 \). Now, we have
\[
\sum_{n=0}^N p_n |c_n|^{\lambda} \leq H \left( \sum_{n=0}^N p_n |c_n|^{\lambda} \right)^{\lambda} \left( \sum_{n=0}^N |b_n| \right)^{1-\lambda}
\]
\[
= H \sum_{n=0}^N p_n |c_n|^{\lambda} \sum_{n=0}^N |b_n|
\]
\[
\leq H \sum_{n=0}^N p_n |c_n|^{\lambda},
\]

since \( \sum_{n=0}^\infty b_n \) is absolutely convergent. The final term is \( a(\mathcal{C}) \) by hypothesis, hence
\[
\sum_{n=0}^N p_n |c_n|^{\lambda} = a(\mathcal{C})
\]
so
\[
\sum_{n=0}^N c_n = 0(N, p_n),
\]
for \( \lambda \geq 1 \).

This completes the proof.

Theorem 6. If \( p_n > 0 \) for all \( n, \mathcal{P} \to \infty, (N, p_n) \) is absolutely regular,\n\[
\sum_{n=0}^\infty a_n = s(N, p_n),
\]
and \( \sum_{n=0}^\infty b_n \) is absolutely convergent with sum \( s \), then
\[
\sum_{n=0}^\infty c_n = s(N, p_n).
\]

Proof. If \( s = 0 \), the result is an immediate consequence of Theorem 5. Suppose \( s > 0 \), and let \( a_0 = a_0 - s, c_0 = c_0 \) for \( n > 0 \) then
\[
\sum_{n=0}^\infty c_n = 0(N, p_n).
\]

Let
\[
\zeta_0 = \sum_{n=0}^{\infty} a_n b_n,
\]
then
\[
\zeta_0 = c_0 = c_n + s b_n,
\]
and
\[
\sum_{n=0}^\infty c_n = 0(N, p_n),
\]
by Theorem 5. Further \( \sum_{n=0}^\infty b_n \) is summable \( (N, p_n) \), by the absolute regularity of \( (N, p_n) \). Thus
\[
\sum_{n=0}^\infty b_n = \Pi(N, p_n),
\]
by Theorem 9 in [1]. Hence
\[
\sum_{n=0}^\infty c_0 = s(N, p_n),
\]
This completes the proof.

5. Corollaries to Theorems 2, 4 and 6

Let \( a \) be real. Define
\[
a_0 = 1, \quad a_n = \frac{(x+1) \ldots (x+n)}{n!}, \quad n = 1, 2, \ldots.
\]

Given any sequence \( (c_n) \) we use the notation
\[
s^*_n = \sum_{n=0}^\infty c_n^{-1} n_{a_n},
\]
so that
\[
(A_n) = c_n^{-1}.
\]
The following identities are immediate:
\[
\sum_{n=0}^\infty a_n^{-1} c_n = a_n^{-1},
\]
\[
I_{\mathcal{P}} = p_n^{-\alpha} x_{\mathcal{P}}^\alpha = \sum_{n=0}^\alpha p_n.
\]
We shall now suppose that \( a \geq 1 \), and consider the Norlund methods \( (N, p_n) \) where \( p_n > 0 \) for all \( n \) when \( a = 1 \). These methods of summability were studied in some detail in [1]. The corollaries stated below include as special cases propositions (a), (b) and (c) about strong Cesaro summability.

Corollary 2.1. If \( a > 0 \) or \( a = 0 \) and \( p_n > 0 \) for all \( n, \beta > 0, \lambda \geq 1, (N, p_n) \) is regular,
\[
\sum_{n=0}^\infty a_n = s(N, p_n), \quad \text{and} \quad \sum_{n=0}^\infty b_n = r(C, \beta), \quad \text{then} \quad \sum_{n=0}^\infty c_n = s(N, p_n^{\alpha}).
\]

Corollary 2.2. If \( \beta \geq 0, \alpha > 0, \lambda \geq 1, (N, p_n) \) is regular,
\[
\sum_{n=0}^\infty a_n = s(C, \beta), \quad \text{and} \quad \sum_{n=0}^\infty b_n = r(N, p_n), \quad \text{then} \quad \sum_{n=0}^\infty c_n = s(N, p_n^{\alpha}).
Corollary 3.1. If $\beta > 0$ or $\beta = 0$ and $p_n > 0$ for all $n$, $\alpha > 0$, $\lambda \geq 1$

\[ \sum_{n=0}^{\infty} a_n = s[N, p_n^\alpha]_{\lambda} \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = t[C, \alpha]_{\lambda}, \quad \text{then} \quad \sum_{n=0}^{\infty} c_n = s[t[N, p_n^\alpha + \beta]]_{\lambda}. \]

Corollary 4.1. If $(N, p_n)$ is absolutely regular, $P_n \to \infty$, $\alpha > 0$, 

\[ \sum_{n=0}^{\infty} a_n = s[N, p_n^\alpha]_{1} \]

and \[ \sum_{n=0}^{\infty} b_n \] is absolutely convergent with sum $t$, then

\[ \sum_{n=0}^{\infty} c_n = s[t[N, p_n^\alpha]]_{1}. \]

To prove Corollary 4.1, one notes that if $(N, p_n)$ is absolutely regular, then $(N, p_n^\alpha)$ is absolutely regular for each $\alpha > 0$, then applies Theorem 4.

References

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