NOTE ON SUMMABILITY FACTORS

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1. Introduction. It is assumed throughout that $\lambda > 0$ and that all functions are real. The main object of this paper is to establish version (a) of the following theorem.

**Theorem 1.** (a) In order that $\int_1^\infty x(t) k(t) dt$ be summable $|C, \lambda|$ whenever $\int_1^\infty x(t) dt$ is summable $|C, \lambda|$ it is necessary and sufficient that, for some constant $c \geq 1$,

(i) $k(t)$ be measurable and essentially bounded in $(1, c)$,

(ii) $\frac{k(t)}{t} = \frac{1}{\Gamma(\lambda)} \int_t^\infty (u-t)^{\lambda-1} h(u) du$ p.p. in $(c, \infty)$,

where $u^{\lambda+1} h(u)$ is measurable and essentially bounded in $(c, \infty)$.

(b) Replace $|C, \lambda|$ by $(C, \lambda)$ and "essentially bounded" in (ii) by "of bounded variation".

Version (b) of the theorem has been proved by Sargent†. We shall, however, give a somewhat simpler proof of the necessity part of this result. There are results‡ similar to the above which involve the additional

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* Received 14 July, 1953; read 19 November, 1953.
† The result as stated here follows from Lemma 5, Theorem 2 and the proof of Theorem 1 in Sargent (4).
‡ For a $(C, \lambda)$ result see Borwein (2) where references are given to $(C, \lambda)$ results and to series analogues. Further references appear in Sargent (4).
hypothesis that the $\lambda$-th derivative of $k(t)$ exists and is absolutely continuous in $[1, w]$ for all $w \geq 1$.

2. Notation and some preliminary results. Let $x(u)$ be integrable $L$ in every finite interval in $(1, \infty)$. Then, for $w > 1$,

$$\int_1^w \left(1 - \frac{w}{w'}\right)^{k(u)}du = \int_1^w \left(1 - \frac{w}{w'}\right)^{k-1} \frac{w}{w'} dt$$

$$- \lambda \int_1^w t^{k-1} dt \int_1^w \left(1 - w\right)^{k-1} x\left(u_t\right) du.$$

Hence $\int_1^w x(u) du$ is

(i) summable $(C, \lambda)$ if and only if $\int_1^w t^{k-1} dt \int_1^w \left(1 - w\right)^{k-1} u x(u) du$ is convergent;

(ii) summable $[C, \lambda]$ if and only if $\int_1^w t^{k-1} dt \int_1^w \left(1 - w\right)^{k-1} u x(u) du < \infty$;

(iii) bounded $(C)$ if and only if $\int_1^w t^{k-1} dt \int_1^w \left(1 - w\right)^{k-1} u x(u) du$ is bounded in $(1, \infty)$ for some $\mu > 0$.

We shall be concerned with the following function spaces (it is to be assumed that $1 \leq w \leq b < \infty$):

$M(a, b)$: the space of functions measurable and essentially bounded in $(a, b)$.

$L(a, b)$: the normed vector space of functions $x(t)$ integrable $L$ in $(a, b)$, the norm being defined by the equation

$$\|x\| = \int_{a}^{b} |x(t)| dt.$$

The general linear functional in this space is given by an equation of the form

$$f(x) = \int_{a}^{b} x(t) s(t) dt,$$

where $s(t) \in M(a, b)$.

$BV(a, b)$: the space of functions having bounded variation in $(a, b)$.

$F$: the normed vector space of functions $x(t)$ continuous in $[1, \infty)$ and tending to finite limits as $t \to \infty$, the norm being defined by the equation

$$\|x\| = \text{bound } |x(t)|.$$

Note on summability factors.

The general linear functional in this space is given by an equation of the form

$$f(x) = \int_{a}^{b} x(t) s(t) dt + \gamma \lim_{t \to \infty} x(t),$$

where $s(t) \in BV(a, b)$ and $\gamma$ is a constant independent of $x$.

$B$: the space of functions $x(t)$ such that $\int_{a}^{b} x(t) dt$ is bounded $(C)$.

$S_{1}$: the normed vector space of functions $x(t)$ such that $\int_{a}^{b} x(t) dt$ is summable $(C, \lambda)$, the norm being defined by the equation

$$\|x\| = \text{bound } \int_{a}^{b} t^{k-1} dt \int_{a}^{b} \left(1 - w\right)^{k-1} u x(u) du.$$

$S_{2}$: the vector subspace of $S_{1}$ which consists of all functions $x(t)$ such that $x(t) = 0$ for $t < a$ and $t x(t) \in L(a, \infty)$.

$V_{1}$: the normed vector space of functions $x(t)$ such that $\int_{a}^{b} x(t) dt$ is summable $[C, \lambda]$, the norm being defined by the equation

$$\|x\| = \int_{a}^{b} t^{k-1} dt \int_{a}^{b} \left(1 - w\right)^{k-1} u x(u) du.$$

$V_{2}$: the vector subspace of $V_{1}$ which consists of all functions $x(t)$ such that $x(t) = 0$ for $t < a$ and $t x(t) \in L(a, \infty)$.

We shall require the following lemmas.

**Lemma 1.** (a) For $t \geq 1$, the general linear functional in the space $V_{r}$ is given by an equation of the form

$$f(x) = \frac{1}{(r+1)(a)} \int_{a}^{b} u x(u) du \int_{a}^{b} \left(1 - w\right)^{r+1} h(t) dt,$$

where $t^{r+1} h(t) \in M(a, \infty)$.

(b) Replace $V$ by $S$ and $M$ by $BV$.

**Proof of (a).** It is easily seen that the equation

$$y(t) = \int_{a}^{b} \left(1 - w\right)^{r+1} u x(u) du (t \geq 1)$$

* Banach (1), 65; see also Sagert (4), Lemma 1.

$1$ It is implicit in the definition of this space and of $S_{1}$ and $V_{1}$ that they are contained in $L(1, w)$ whenever $1 < w < \infty$.

$\int_{a}^{b} \left(1 - w\right)^{r+1} u x(u) du (t \geq 1)$
defines a linear and isometric transformation between all functions \( x \) of \( V_\varepsilon \) and a vector subspace of functions \( y \) of \( L(1, \infty) \). Hence, by the Hahn-Banach extension theorem, the general linear functional in \( V_\varepsilon \) is given by an equation of the form

\[
f(x) = \int_1^\infty a(t) t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du,
\]

where \( a(t) \in M(1, \infty) \). Since \( \int_1^\infty |x(u)| \, du < \infty \) when \( x(u) \in \Gamma_\varepsilon \), we can change the order of integration and then obtain the required result by putting

\[
h(t) = \Gamma(h) t^{-1-\varepsilon} u x(u) \, du \quad (t \geq 1).
\]

Proof of (b). The equation

\[
y(w) = \int_1^w t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \quad (w \geq 1)
\]
defines a linear and isometric transformation between all functions \( x \) of \( S_\varepsilon \) and a vector subspace of functions \( y \) of \( F \). Hence the general linear functional in \( S_\varepsilon \) is given by an equation of the form

\[
f(x) = \int_1^\infty a(u) \int_1^{(t-u)^{-1}} u x(u) \, du
\]

\[
+ \gamma \int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du,
\]

where \( a(u) \in BF(1, \infty) \) and \( y \) is a constant. Since \( \int_1^\infty |x(u)| \, du < \infty \) when \( x(u) \in \Sigma_\varepsilon \), we can change the order of integration and then obtain the required result by putting

\[
h(t) = \Gamma(h) t^{-1-\varepsilon} \int_1^\infty a(u) \, du \quad (t \geq 1).
\]

Lemma 2. (a) If \( x(t) \in BF \) whenever \( x(t) \in V_\varepsilon \), then (i) \( h(t) \in M(1, \infty) \), (ii) the functional

\[
f(x) = \int_1^\infty x(t) h(t) \, dt
\]
is linear in \( V_\varepsilon \) for some \( c \geq 1 \).

(b) Replace \( V \) by \( S \).

Since \( S_\varepsilon \supseteq V_\varepsilon \), result (b) follows from a(i) which has been established elsewhere.

Proof of a(ii). Since \( k(t) \in M(1, \infty) \), \( f(x) \) is defined and additive in \( V_\varepsilon \) for all \( \varepsilon \geq 1 \). Suppose there is no \( c \geq 1 \) for which \( f(x) \) is linear in \( V_\varepsilon \).

Then we can define by induction a sequence of functions \( \{x_n\} \) and an increasing unbounded sequence of real numbers \( \{c_n\} \) as follows:

Let \( c_1 = 1 \) and suppose that \( c_1, c_2, \ldots, c_{n-1}, x_1, x_2, \ldots, x_{n-1} \) have been defined and that \( x_r \in F_{c_r} \) for \( r = 1, 2, \ldots, n-1 \). Since \( f(x) \) is not linear in \( F_{c_{n-1}} \), there is a function \( x_n \) such that

\[
x_n \in F_{c_{n-1}}, \quad ||x_n|| < 2^{-n} \quad \text{and} \quad f(x_n) > 1.
\]

Let

\[
c_n = 2c_{n-1} + \sum_{r=1}^{n-1} ||x_r|| \cdot k(t) \, du.
\]

Now define a function \( x(t) \) by putting

\[
x(t) = x_1(t) + x_2(t) + \ldots + x_n(t)
\]

when \( 1 \leq t \leq c_n \) and \( n = 1, 2, \ldots, \).

Then, for any \( \mu \geq 1 \) and \( n = 1, 2, \ldots, \)

\[
\int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
= \int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
= \int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
\geq \frac{1}{\mu} \int_1^\infty x_1(t) \, dt \quad \text{for any } \mu \geq 1 \quad \text{and so } x(t) \, dk(t)
\]

\[
is not bounded (C, \mu) \text{ for any } \mu \geq 1 \quad \text{and so } x(t) \, dk(t)
\]

is not in \( B \).

On the other hand, for \( n = 1, 2, \ldots, \)

\[
\int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
= \int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
\leq \sum_{r=1}^{n-1} \int_1^\infty t^{-1-\varepsilon} \int_1^{(t-u)^{-1}} u x(u) \, du \, dk(t) \, du
\]

\[
\leq \sum_{r=1}^{n-1} ||x_r|| \cdot k(t) \, du < 1,
\]

and hence \( x(t) \in F_{c_n} \).

Since this contradicts the hypothesis, the required result is established.

* Cf. Sargents (4), Lemma 3.
Proof of (b). We can proceed as above, with $S$ in place of $V$, up to and including the statement:

$$x(u)k(u) \text{ is not in } B;$$

and then, to complete the proof, obtain a contradiction as follows.

Let $s$ be an arbitrary positive integer. Then

$$\lim_{s \to \infty} \left| \int_{s}^{s+1} t^{-\alpha-1} d t \left( u-t \right)^{-\alpha} w \left( u \right) d u \right|$$

$$\leq \lim_{s \to \infty} \left( \frac{s}{s+1} \right) \left| \int_{s}^{s+1} t^{-\alpha-1} d t \left( u-t \right)^{-\alpha} u \left( u \right) d u \right|$$

$$+ \frac{s}{s+1} \left( \int_{s}^{s+1} t^{-\alpha-1} d t \left( u-t \right)^{-\alpha} w \left( u \right) d u \right)$$

$$\leq \frac{2}{s} \sum_{r=1}^{s} \left| x(r) \right| < 2 \sum_{r=1}^{s} \left| x(r) \right| < 2 s^{2-\alpha}.$$  

Hence $x(t) \in S$.  

Lemma 3. If $\alpha \geq 1$, $r > 0$ and $x(t) \in L(1, w)$ for all $w > 1$, then a necessary and sufficient condition for $x(t)$ to be in $V_{r}$ is that

$$\int_{t}^{t+1} t^{-\alpha-1} d t \left( u-t \right)^{-\alpha} w \left( u \right) d u < \infty.$$  

This follows from a result established elsewhere.

Lemma 4. If $\alpha > 1$ and $x(t) \in V_{r}$, then $t^{\alpha} \int_{t}^{t+1} \left( u-t \right)^{\alpha} x(u) d u \in V_{r-1}$.  

This also follows from the above-mentioned result.

Lemma 5. If $x(t) \in V_{r}$, then $x(t) = x(1) / C \cdot t^{\alpha-1}$ as $t \to \infty$.  

This well-known result follows from the identity

$$t^{-\alpha-1} \int_{t}^{t+1} \left( u-t \right)^{\alpha} x(u) d u = t^{-\alpha} \int_{t}^{t+1} \left( u-t \right)^{\alpha} x(u) d u - t^{\alpha} \int_{t}^{t+1} \left( u-t \right)^{\alpha-1} x(u) d u$$

$$\left( t \geq 1 \right).$$

We shall now prove two theorems; the first of these includes both necessity parts of Theorem 1 and the second is simply a restatement of the sufficiency part of Theorem 1(a).

\* Borwein (3), Theorem 1 with $\rho = -r$, $\alpha = \lambda$.

** Theorem** 2. (a) If $x(t)k(t) \in B$ whenever $x(t) \in V_{r}$, then there is a number $c \geq 1$ such that

$$\left( i \right) k(t) \in M(1, c),$$

$$\left( ii \right) \frac{k(t)}{t} = \frac{1}{\Gamma(\alpha)} \int_{t}^{t+1} \left( u-t \right)^{\alpha-1} h(u) d u \text{ p.p. in } (c, \infty),$$

where $w^{\alpha+1} h(u) \in M(c, \infty)$.  

(b) Replace $V$ by $S$ and $M$ by $M(1, \infty)$.

Parts (a) and (b) follow from the corresponding parts of Lemma 2.

Proof of (a(ii)). In view of Lemmas 1(a) and 2(a) there is a number $c \geq 1$ such that, for all $x(t) \in V_{r}$,

$$\int_{c}^{\infty} x(t)k(t)d t = \frac{1}{\Gamma(\alpha)} \int_{c}^{\infty} \frac{x(t)k(t)d t}{t} \left( u-t \right)^{\alpha-1} h(u) d u,$$

where $w^{\alpha+1} h(u) \in M(1, \infty)$. The required result follows since, for arbitrary $w > c$, $V_{r}w$ contains the characteristic function of the interval $[c, w]$.  

Proof of (b). Replace (a) by (b), $V$ by $S$ and $M$ by $M(1, \infty)$ in the above proof.

** Theorem** 3. If $x(t) \in V_{r}$ and, for some number $c \geq 1$,

$$\left( i \right) k(t) \in M(1, c),$$

$$\left( ii \right) \frac{k(t)}{t} = \frac{1}{\Gamma(\alpha)} \int_{t}^{t+1} \left( u-t \right)^{\alpha-1} h(u) d u \text{ p.p. in } (c, \infty),$$

where $w^{\alpha+1} h(u) \in M(c, \infty)$, then $x(t)k(t) \in V_{r}$.

Write, for $t \geq c, p > 0$,

$$y(t) = \varphi(t),$$

$$\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{t+1} (u-t)^{\alpha-1} y(u) d u,$$

and let

$$H = \text{ess. bound } \left| x(t) \right|.$$  

Note that $x(t)k(t) \in L(1, w)$ for all $w > 1$ and so, by Lemma 3, it is sufficient to prove that

$$\int_{c}^{\infty} x(t)^{-1} d t \left| \int_{c}^{\infty} \left( u-t \right)^{\alpha-1} \frac{x(t)}{x(u)} \left| k(t) \right| d u \right|$$

\* Version (b) of this theorem is slightly more general than Theorem 1 in Segregat (4).
is finite. Further, since \( x(t) \in V_a \), we have, by Lemma 3, that
\[
\int t^{\lambda-2} |g(t)| dt < \infty.
\]

**Case 1.** Suppose that \( \delta < \lambda < 1 \), and write, for \( t > e > 0 \),
\[
Q(v, t) = \frac{1}{\Gamma(\lambda)} \int_t^e (u-t)^{\lambda-1} (u-v)^{\lambda-1} g(u) du.
\]
It has been shown* that, for almost all \( v \) in \((s, t)\),
\[
|Q(v, t)| \leq (t-v)^{\lambda-1}|g(v)| + (t-e)^{\lambda-1} \int_e^t (u-t)^{\lambda-1} |g(u)| du.
\]
Hence
\[
\int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} g(u) \frac{d\lambda}{u} \right] \leq \frac{1}{\Gamma(\lambda)} \int_t^e (u-t)^{\lambda-1} g(u) du \int_t^e (u-v)^{\lambda-1} |g(v)| dv + \int_t^e (u-t)^{\lambda-1} |g(v)| dv \int_t^e (u-e)^{\lambda-1} |g(u)| du
\]
\[
 \leq H \int_t^e (u-t)^{\lambda-1} |g(u)| du + \frac{1}{\Gamma(\lambda-1)} \int_t^e (u-t)^{\lambda-1} Q(v, t) dt + H \int_t^e (u-t)^{\lambda-1} \int (u-v)^{\lambda-1} |Q(u, v)| dv dt
\]
\[
 \leq 2H \int_t^e (u-t)^{\lambda-1} |g(u)| du + \frac{1}{\Gamma(\lambda-1)} \int_t^e (u-t)^{\lambda-1} |Q(u, v)| dv dt
\]
\[
 \leq 2H \int_t^e (u-t)^{\lambda-1} |g(u)| du \left[ (t-e)^{\lambda-1} + \frac{1}{\Gamma(\lambda)} \right]
\]
\[
 + 2H \int_t^e (u-t)^{\lambda-1} (u-v)^{\lambda-1} \int (u-v)^{\lambda-1} |g(v)| dv dt
\]
\[
 = 2H \frac{\lambda}{\lambda-1} (\lambda-1) (1-\lambda) \int \int \left[ (u-t)^{\lambda-1} (u-v)^{\lambda-1} |g(v)| dv \right] dt
\]
\[
+ 2H \left[ (\lambda-1) + B(1+1, \lambda) \right] B(1+1, \lambda) \int (u-v)^{\lambda-1} |g(v)| dv dt < \infty.
\]

The result in this case follows.

* Borwein (2), Inequality (6.7), note that this differs from the required inequality by a factor \( t^{-\lambda} \) in the left-hand side and that a suitable value for the constant \( \mathcal{M} \) is \( \max \{ t^{-\lambda}, (1-\lambda)/(\lambda-1) \} = t^{-\lambda} \) \( (\delta < \lambda < 1) \).

**Case 2.** Suppose that \( \lambda = 1 \). The required result is now obtained from the following inequality:
\[
\int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} g(u) \frac{d\lambda}{u} \right] = \int t^{\lambda-1} dt \left[ \int g(u) du \right] \frac{1}{\lambda} \int g(v) dv
\]
\[
 \leq H \int t^{\lambda-1} dt \int (u-t)^{\lambda-1} |g(v)| dv + 2H \left[ (\lambda-1) + B(1+1, \lambda) \right] B(1+1, \lambda) \int (u-v)^{\lambda-1} |g(v)| dv dt < \infty.
\]

**Case 3.** Suppose that \( \lambda > 1 \), and assume the result with \( \lambda \) replaced by \( \lambda-1 \). Suppose further, without any loss in genericity, that \( x(t) = 0 \) for \( 1 < t < \infty \).

Let \( p(t) = \frac{1}{\Gamma(\lambda-1)} \int (u-t)^{\lambda-1} |g(u)| du \) when \( t \geq e \), \( p(t) = 0 \) when \( 1 < t < e \), and note that, for almost all \( t > e \),
\[
k(t) = \frac{1}{\Gamma(\lambda-1)} \int (u-t)^{\lambda-1} |g(u)| du
\]

Then it is easily verified that, for \( t > e \),
\[
\int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} g(u) \frac{d\lambda}{u} \right] = \int t^{\lambda-1} dt \left[ \int p(u) u^{-\lambda} k(u) du \right]
\]
\[
 \leq t^{\lambda-\lambda} \int t^{\lambda-1} dt \left[ \int p(u) u^{-\lambda} k(u) du \right] + \frac{1}{\Gamma(\lambda-1)} \int t^{\lambda-1} dt \left[ \int u^{\lambda-1} |g(u)| du \right] \int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} |k(u)| du \right] \int t^{\lambda-1} dt \left[ \int (u-v)^{\lambda-1} |g(v)| dv \right] \int t^{\lambda-1} dt \left[ \int (u-v)^{\lambda-1} |k(v)| dv \right] \int t^{\lambda-1} dt \left[ \int (u-v)^{\lambda-1} |g(u)| dv \right] \int t^{\lambda-1} dt \left[ \int (u-v)^{\lambda-1} |k(v)| dv \right]
\]

Now \( w, u, v, s \in \mathcal{M}(c, \infty) \), \( w, u, v, s \in \mathcal{M}(c, \infty) \) and so both \( k(t) \) and \( p(t) \) satisfy the hypotheses of \( k(t) \) with \( \lambda \) replaced by \( \lambda-1 \). Further, since \( x(t) \in V_a \), we have, by Lemma 4, that \( w^{\lambda-\lambda} \int t^{\lambda-1} dt \left[ \int p(u) u^{-\lambda} k(u) du \right] \int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} |k(u)| du \right]
\]

Thus, by the assumption, \( w^{\lambda-\lambda} \int t^{\lambda-1} dt \left[ \int p(u) u^{-\lambda} k(u) du \right] \int t^{\lambda-1} dt \left[ \int (u-t)^{\lambda-1} |k(u)| du \right]
\]

It follows that \( x(t) k(t) \in V_a \) and the result in this case is thus established by induction from the two previous cases.

This completes the proof of the theorem.

**References.**


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