SCALES OF LOGARITHMIC METHODS OF SUMMABILITY

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1. Introduction. We suppose throughout that \( p \) is a non-negative integer, and use the following notations:

\[
\pi_p(x) = \begin{cases} 
\frac{1}{\log_0 x \cdot \log_1 x \cdot \cdots \cdot \log_p x} & \text{for } x \geq e_p, \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( \log_0 x = x \) for \( x \geq e_0 = 1 \), and \( \log_n x = \log(\log_{n-1} x) \) for \( x \geq e_{n+1} = e^n \) \( \{n = 0, 1, 2, \cdots\} \);

\[
\sigma_p(x) = \sum_{n=0}^{\infty} \pi_p(n) x^n \quad (-1 < x < 1);
\]

\[
s_n = \sum_{k=0}^{n} a_k \quad (n = 0, 1, 2, \cdots);
\]

\[
t_p(n) = \frac{1}{\log_{p+1} n} \sum_{k=0}^{n} \pi_p(k) s_k \quad (n \geq e_{p+1}).
\]

The series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \( L_p \) to \( s \), and we write \( \sum_{n=0}^{\infty} a_n = s \) \( (L_p) \) or \( s \rightharpoonup s \) \( (L_p) \), if

\[
\lim_{x \to 1^{-}} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n = s.
\]

If \( t_s(n) \rightarrow s \) as \( n \rightarrow \infty \) the series \( \sum_{n=0}^{\infty} a_n \) is said to be summable \( f_p \) to \( s \), and we write \( \sum_{n=0}^{\infty} a_n = s(t_p) \) or \( a_n \sim s(t_p) \) (see [5]).

Given two summability methods \( A, B \) we write \( A \supseteq B \) if any series summable \( B \) is summable \( A \) to the same sum; if in addition there is a series summable \( A \) but not summable \( B \) we write \( A \supset B \). If \( A \supseteq B \) and \( B \supseteq A \) the two methods are said to be equivalent and we write \( A \cong B \). It is known [3] that the \( L_0 \) and \( f_p \) methods are regular and that \( L_0 \supseteq f_0 \supseteq f \) where \( L \) and \( f \) are standard logarithmic methods (for definitions see [1]). The aim of this paper is to establish various inclusion theorems for the two scales of methods.

2. Lemmas. We require four lemmas.

**Lemma 1.** If \( s_n = s(t_p) \), then \( s_n = \sum_{n=0}^{\infty} \frac{n}{p} \) and
\[
a_n = \sum_{n=0}^{\infty} \frac{1}{p}.
\]

Proof. The case \( p = 0 \) of this lemma is due to Ishiguro [3, Theorem 4]. For \( n \geq e_p + 1 \) we have that
\[
s_n = \sum_{n=0}^{\infty} \frac{n}{p} = \sum_{n=0}^{\infty} \log_{p+1}(n+1) - \log_{p+1}(n) = \sum_{n=0}^{\infty} \log_{p+1}(n+1);
\]

hence
\[
\pi_p(t_p) = \sum_{n=0}^{\infty} \frac{n}{p} - \sum_{n=0}^{\infty} \log_{p+1}(n+1)
\]
and so,
\[
\pi_p(t_p) = \pi_p(t_p) = \sum_{n=0}^{\infty} \frac{n}{p} - \sum_{n=0}^{\infty} \log_{p+1}(n+1).
\]

**Lemma 2.** \( L_p \supseteq f_p \).

Proof. Since \( f_p \supseteq (\pi_0, q_0) \) with \( q_0 = (n, p) \), the lemma follows from a known result [4, Theorem 1].

**Lemma 3.** If \( x \geq e_p \), \( y > 0 \), then
\[
(\log_0 x)^{-1} = \int_0^x e^{-st} \lambda_{\pi_0} (t) dt,
\]
where \( \lambda_{\pi_0} (t) \) is defined by the recursive formulae:
\[
\lambda_{\pi_0}(t) = \frac{1}{(t+1)^2} \int_0^t e^{-st} \lambda_{\pi_0}(s) ds,
\]
\[
\lambda_{\pi_0}(t) = \frac{1}{(t+1)^2} \int_0^t e^{-st} \lambda_{\pi_0}(s) ds.
\]

Proof. The lemma is true for \( p = 0 \), since, when \( x \geq e_0 = 1 \),
\[
(\log_0 x)^{-1} = \int_0^x e^{-st} \lambda_{\pi_0}(t) dt = \int_0^x e^{-st} \lambda_{\pi_0}(t) dt.
\]

Assume the lemma is true for \( p = r \). Then, for \( x \geq e_r \), we have
\[
(\log_{r+1} x)^{-1} = \int_0^x e^{-st} \lambda_{\pi_0}(t) dt = \int_0^x e^{-st} \lambda_{\pi_0}(t) dt.
\]
the inversion in the order of integration being justified by Fubini's theorem since all the functions concerned are non-negative and Lebesgue measurable. The lemma is thus established by induction.

The case \( p = 1 \) of the next lemma is due to Hardy [2, page 268].

**Lemma 4.** If \( n \geq e^p \), \( y > 0 \), then

\[
\left( \log n \right)^y = \int_0^1 e^{\psi(t)} \, dt,
\]

where the function \( \psi \) is non-negative and independent of \( n \).

**Proof.** By Lemma 3,

\[
\left( \log n \right)^y = \int_0^1 e^{\frac{\psi(t) - \frac{1}{2} \log \psi(t)}} \, dt = \int_0^1 \Phi(t) \, dt,
\]

where \( \Phi(t) = \frac{1}{t} \log \left( \frac{1}{t} \right) \).

3. Inclusion Theorems.

**Theorem 1.** There is a series summable \( f_{p+1} \) but not summable \( L_p \) i.e., \( L_p \nsubseteq f_{p+1} \)

**Proof.** Let \( N \) be the integer such that \( N - 1 < e^{p+1} \leq N \), and, with \( i = \sqrt{N} \), let

\[
a_n = \begin{cases} p \left( \log_{p+1} \right)^{-i} & \text{for } n \geq e^{p+1}, \\ 0 & \text{for } n < e^{p+1}. \end{cases}
\]

Then

\[
\sum_{n=1}^{N} a_n = \left( \log_{p+1} \right)^{-i} \left( \log_{p+1} \right)^{-i}
\]

\[
= \sum_{k=1}^{N} \frac{\left( \log_{p+1} \right)^{-i}}{p} + \left( \log_{p+1} \right)^{-i} - \int_{N}^{n} \left( \log_{p+1} \right)^{-i} \, dt
\]

\[
= \sum_{k=N}^{N} \frac{\left( \log_{p+1} \right)^{-i}}{p}.
\]

where

\[
\frac{1}{k} = \int \left( \int_{x} \frac{dx}{x} \left( \log_{p+1} \right)^{-i} \right) \, dt
\]

\[
= \int \left( \int_{x} \frac{dx}{x^2} \left( \log_{p+1} \right)^{-i} \right) \, dt
\]

\[
= \int \left( \int_{x} \frac{dx}{x^2} \right) \, dt
\]

\[
= C \left( \frac{1}{k} \right).
\]
Hence $\sum_{k=1}^{n} \frac{1}{k}$ converges, and so $s_n = \sum_{k=1}^{n} \frac{1}{k} \left( \log_{p+1} n \right)^{-1}$ tends to a finite limit as $n \to \infty$. Since $s_n = s_{n-1} + \frac{1}{p} \sum_{k=1}^{n} \frac{1}{k},$

we have that $s_n = \frac{1}{p} \sum_{k=1}^{n} \frac{1}{k} + k$, where $k_n$ tends to a finite limit as $n \to \infty$.

Consequently $\{s_n\}$ is bounded but does not converge, and as $n = O(\log_{p+1}(n))$, it follows from a known tauberian theorem [4, Corollary] that $\sum a_n$ is not $L_1$-summable.

We now show that $\sum_{n=0}^{\infty} a_n$ is $L_{p+1}$-summable. For $m \geq N$, we have that

$$
\frac{1}{p+1} \sum_{n=N}^{m} \frac{1}{n} = \frac{1}{p} \sum_{n=N}^{m} \frac{1}{n} \left( \log_{p+1} n \right)^{-1} + \sum_{n=N}^{m} \frac{1}{p+1} \frac{1}{n} + \sum_{n=N}^{m} \frac{1}{p+1} \frac{1}{n} \left( \log_{p+1} n \right)^{-1}.
$$

and hence $t_{p+1}(m)$ tends to a finite limit as $m \to \infty$.

**Theorem 2.** $L_{p+1} \supseteq L_p$.

**Proof.** By Lemma 4, for $n \geq p+1$,

$$
\frac{n^{p+1}}{\varphi(n)^{p}} \sum_{n} \frac{1}{n} \left( \log_{p+1} n \right)^{-1} \int_{0}^{1} \frac{1}{n} \left( \log_{p+1} n \right)^{-1} dt
$$

where $\varphi(t)$ is non-negative and independent of $n$, and hence, by a result due to Borwein [1, Theorem A], $L_{p+1} \supseteq L_p$. The stronger inclusion follows immediately from Theorem 1 and Lemma 2.

**Theorem 3.** $L_p \supseteq L_p'$

**Proof.** We consider a series used to show the existence of a sequence summable by the Abel method A, but not summable by any Cesàro method [2, Theorem 56].

Let

$$
e^{-\gamma(t+x)} = \sum_{n=0}^{\infty} a_n x^n.
$$

It is known that $a_n$ is not $O(n^r)$ for any $r$, and hence, by Lemma 1, $\sum a_n$ is not summable $L_p$. Since the series is summable $L_p$ and $\sum_{n=0}^{\infty} a_n \gamma(t+x)$ the theorem can now be deduced from Lemma 2.

**Theorem 4.** $L_{p+1} \supseteq L_p'$

**Proof.** The inclusion $L_{p+1} \supseteq L_p'$ follows immediately from a known theorem for $L_1$ methods [2, Theorem 56]. The stronger inclusion may be deduced from Theorem 1. However, a direct proof is easy.

Consider

$$
\sum_{n=1}^{\infty} a_n = \left\{ \gamma(t+x) \frac{1}{n^{p+1}} \right\}_{n=0}^{\infty} \quad (n \geq p+1).
$$
Then $s_n \to 0$ (i.e. $\sum_{n=0}^{\infty} a_n$ is summable $\ell_{p+1}$,
but $s_n \neq o\left(\frac{1}{\pi_{p+1}(n)}\right)$; hence, by Lemma 1, $\sum_{n=0}^{\infty} a_n$ is not $\ell_p$ summmable.

REFERENCES


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