CONVERGENCE CRITERIA FOR BOUNDED SEQUENCES

by D. BORWEIN
(Received 26th July 1971)

1. Introduction

Let \( \{K_n\} \) be a sequence of complex numbers, let

\[
K(z) = \sum_{n=0}^{\infty} K_n z^n
\]

and let

\[
k_0 = K_0, \quad k_n = K_n - K_{n-1} \quad (n = 1, 2, \ldots).
\]

Let \( D \) be the open unit disc \( \{z : |z| < 1\} \), let \( \overline{D} \) be its closure and let \( \partial D = \overline{D} - D \).

The primary object of this paper is to prove the two theorems stated below, the first of which generalises a result of Copson (1).

Theorem 1. If

\[
\sum_{n=0}^{\infty} |K_n| < \infty, \quad (1)
\]

and if

\[
K(z) \neq 0 \text{ on } \partial D, \quad (2)
\]

\( \{a_n\} \) is a bounded sequence \( \quad (3) \)

such that, for some positive integer \( N \),

\[
\sum_{r=0}^{n} k_r a_{n-r} \geq 0 \quad (n = N, N+1, \ldots), \quad (4)
\]

then \( \{a_n\} \) is convergent.

In essence, Copson's theorem is the above result with conditions (1) and (2) replaced by the single condition

\[
-1 = K_0 < K_1 < \ldots < K_{N-1} < K_N = K_{N+r} = 0 \quad (r = 1, 2, \ldots). \quad (C)
\]

If (C) holds, then (1) is trivially satisfied, and \( K(z) \) is a polynomial satisfying (2), since \( K(1) < 0 \) and, for \( z = e^{i\theta} \), \( 0 < \theta < 2\pi \),

\[
\Re (1-z)K(z) = - \sum_{r=1}^{N} k_r (1-\cos r\theta) < 0.
\]

The next theorem shows that condition (2) is necessary for the validity of Theorem 1 when \( K(z) \) is subject to certain additional conditions: in particular, it shows that (2) is necessary when \( K(z) \) is analytic on \( \overline{D} \) and \( K(1) \neq 0 \).
D. BORWEIN

Theorem 2. If \( K(z) = \frac{1}{q(z)} \) where \( q(z) \) is a polynomial and

\[ q(z) = \sum_{n=0}^{\infty} a_n z^n, \]

and if

\[ \sum_{n=0}^{\infty} |a_n| < \infty, \]  \hspace{1cm} (5)

\[ q(z) \neq 0 \quad \text{on} \quad B, \]  \hspace{1cm} (6)

\[ K(z) = 0, \quad \zeta \neq 1, \quad |\zeta| = 1, \]  \hspace{1cm} (7)

then there is a bounded divergent sequence \( \{a_n\} \) and a positive integer \( N \) such that

\[ \sum_{n=N}^{\infty} k_n a_n = 0 \quad (n = N, N+1, \ldots). \]  \hspace{1cm} (8)

2. Proof of Theorem 1

By (1), \( K(z) \) is analytic on \( D \) and continuous on \( \overline{D} \). Hence, by (2), \( K(z) \) can have at most a finite number of zeros in \( D \); and consequently

\[ K(z) = \frac{1}{q(z)} \]  \hspace{1cm} (9)

where \( p(z) \) is a polynomial with no zeros in the complement of \( D \), and \( q(z) \) is analytic on \( D \) and continuous and non-zero on \( \overline{D} \).

Let

\[ a(z) = \sum_{n=0}^{\infty} a_n z^n, \]

and let

\[ u(z) = q(z) a(z), \]

\[ v(z) = p(z) a(z). \]  \hspace{1cm} (10)

Since, by (3), \( a(z) \) is analytic on \( D \), so also are \( u(z) \) and \( v(z) \).

Let \( \{a_n\} \), \( \{a_n\} \), \( \{k_n\} \) be the sequences such that

\[ q(z) = \sum_{n=0}^{\infty} a_n z^n, \quad u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=0}^{\infty} v_n z^n \]

for all \( z \) in \( D \).

Since \( a(z) = K(z) a(z) \), we have that

\[ v_n = \sum_{n=0}^{\infty} k_n a_n, \]

and hence, by (1) and (3), that \( \{v_n\} \) is bounded. Further, by (4), we have that

\[ v_n - v_{n-1} = \sum_{n=0}^{\infty} k_n a_n \geq 0 \quad (n = N, N+1, \ldots). \]  \hspace{1cm} (12)

It follows that

\[ v_n \rightarrow v \]  \hspace{1cm} (13)

where \( v \) is finite.

We prove next that \( \{a_n\} \) satisfies (5), and that

\[ u_n \rightarrow u \]  \hspace{1cm} (14)

where \( u \) is finite.

Case (i). \( p(z) = cz^m \) (\( m = 0, 1, \ldots \)).

It is evident that (5) and (14) hold in this case.

Case (ii). \( p(z) = z - \beta, \quad |\beta| < 1 \).

By (9), \( K(z) = 0 \) and \( q(z) = (z - \beta)^{-1} K(z) \). Hence

\[ a_n = \frac{1}{\sum_{n=0}^{\infty} x_n z^n} = \sum_{n=0}^{\infty} \frac{1}{z^n} \sum_{n=0}^{\infty} x_n z^n, \]

and so, by (1), we have that

\[ \sum_{n=0}^{\infty} |a_n| \leq \sum_{n=0}^{\infty} |K_n| \sum_{n=0}^{\infty} |x_{n+1} - x_n| \leq \frac{1}{1 - |x|} \sum_{n=0}^{\infty} |K_n| < \infty. \]

Also, by (11), \( e(z) = 0 \) and \( u(z) = (z - \beta)^{-1} e(z) \). Hence, by (13), we have that

\[ u_n = \sum_{n=0}^{\infty} x_n \rightarrow u \quad \text{as} \quad n \rightarrow \infty. \]

Thus, (5) and (14) hold in Case (ii).

Application of Case (i) followed by repeated applications of Case (ii) establishes (5) and (14) in the remaining case:

\[ p(z) = cz^m (z - \beta_1 - \beta_2 - \ldots - \beta_k - \gamma), \quad 0 < |\beta_1| < 1, \quad 0 < |\beta_2| < 1, \ldots, \quad 0 < |\beta_k| < 1. \]

Finally, since \( q(z) \) has no zeros on \( D \) and (5) holds, we have, by the Wiener-Lévý Theorem (2), p. 246, that there is a sequence \( \{e_n\} \) such that

\[ \frac{1}{q(z)} = \sum_{n=0}^{\infty} e_n z^n \quad (z \in D) \]  \hspace{1cm} (15)

and

\[ \sum_{n=0}^{\infty} |e_n| < \infty. \]  \hspace{1cm} (16)

By (10), \( a(z) = u(z)/q(z) \), and hence, by (14) and (15), we have that

\[ a_n = \sum_{n=0}^{\infty} c_n e_n \rightarrow u \quad \text{as} \quad n \rightarrow \infty. \]

3. Proof of Theorem 2

Define a sequence \( \{a_n\} \) and a function \( a(z) \) by

\[ a(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{q(z) K(z)} \quad (z \in D); \]

(17)
and let

\[ w_n = \sum_{k=n}^{\infty} k a_{n+k}, \]

\[ w(z) = \sum_{n=0}^{\infty} w_n z^n. \]

Then

\[ w(z) = (1-z)R(z)z^s = \frac{(1-z)p(z)}{\zeta - z} \]

and, by (6) and (7), \( \zeta - z \) is a factor of the polynomial \( p(z) \). Consequently \( w(z) \) is a polynomial, of degree \( N - 1 \) say, and (8) follows.

Further, by the Wiener-Lévy Theorem, hypotheses (5) and (6) imply conditions (15) and (16). Hence, by (17), we have that

\[ \frac{1}{n!} a_n = -\sum_{k=0}^{\infty} \frac{1}{k!} e^{-k} \rightarrow \frac{1}{g(0)} \quad \text{as} \quad n \rightarrow \infty. \]

Since \( q(0) \neq 0 \), it follows that \( \{a_n\} \) is bounded but not convergent.

4. Remarks

1. The proof of Theorem 1 shows that conditions (1) and (2) imply that \( k(\zeta) \) must satisfy all the hypotheses of Theorem 2 preceding hypothesis (7).

2. The following theorem is a corollary of Theorems 1 and 2.

Theorem 3. If \( k(\zeta) \) is analytic on \( \mathbb{D} \) and \( k(1) \neq 0 \), then condition (3) is necessary and sufficient for every bounded sequence \( \{a_n\} \) satisfying (4), for some positive integer \( N \), to be convergent.

A direct proof of Theorem 3 that avoids the Wiener-Lévy theorem and other complications can readily be constructed from parts of the proofs of Theorems 1 and 2.

3. Theorem 1 remains valid when condition (4) is replaced by

\[ \sum_{k=n}^{\infty} k a_{n+k} \in Q \quad (n = N, N+1, \ldots) \]  

(18)

where \( Q \) is any closed quadrant of the plane.

To establish this we need only modify the proof of Theorem 1 to the extent of changing \( \zeta \geq 0 \) in (12) to \( \zeta \in Q \).

Condition (18) is slightly more general than (4) and somewhat more appropriate in the context of complex sequences.