On strong summability

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In this note we consider some strong Abel-type summability methods, establishing the strong summability analogues of the results proved in [4].

§ 1. Introduction

Let

\[ e_n^\lambda = \left( \begin{array}{c} n + \lambda \\ n \end{array} \right) = \frac{(\lambda + 1) (\lambda + 2) \ldots (\lambda + n)}{n!}, \quad n = 1, 2, \ldots, \]

\[ \varepsilon_0^\lambda = 1, \]

\[ s_n = \sum_{r=0}^{n} u_r, \]

\[ s_n(y) = (1 + y)^{-\lambda - 1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left( \frac{y}{1 + y} \right)^n, \]

\[ u_n(y) = (1 + y)^{-\lambda - 1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda u_n \left( \frac{y}{1 + y} \right)^n, \]

\[ U_n(y) = \lambda \int_0^y u_n(t) dt. \]

The Abel-type methods \( A_\lambda \) and \( A_{\lambda+1} \), introduced in [2] and [3] are defined as follows:

If

\[ (1 - x)^{\lambda + 1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n \]

is convergent for all \( x \) in the open interval \((0, 1)\) and tends to a finite limit \( l \) as \( x \to 1 \) in the open interval, we say that the sequence \( \{s_n\} \) is \( A_\lambda \)-convergent to \( l \) and write \( s_n \to l(A_\lambda) \).

It is evident that \( s_n \to l(A_\lambda) \) if and only if the series defining \( s_n(y) \) is convergent for all \( y > 0 \) and \( s_n(y) \to l \) as \( y \to \infty \). The \( A_0 \) method is the ordinary Abel method.

If the series defining \( u_n(y) \) is convergent for all \( y > 0 \) and \( U_n(y) \) tends to a finite limit \( l \) as \( y \to \infty \), we say that the sequence \( \{s_n\} \) is \( A_{\lambda+1} \)-convergent to \( l \) and write \( s_n \to l(A_{\lambda+1}) \).

It is known that the methods \( A_\lambda \) and \( A_{\lambda+1} \) are regular for \( \lambda > -1 \) (see [2], Theorem 1 and [5], Theorem 34).
The Hausdorff method $H_s$ and the product method $A_sH_s$ are defined as under:

Let $\chi(t)$ be a real function of bounded variation in the interval $[0, 1]$, and

$$h_n = \sum_{j=0}^{n-1} \left\{ \int_{j}^{j+1} \chi(t) dt \right\}.$$  

If $h_n \to I$ as $n \to \infty$, we say that the sequence $(s_n)$ is $H_s$-convergent to $I$ and write $s_n \to I_{H_s}$.  

If $h_n \to I(A_s)$, we say that the sequence $(s_n)$ is $A_sH_s$-convergent to $I$ and write $s_n \to I_{A_sH_s}$.

We recall that the conditions

$$x(0^+) = x(0),$$

and

$$x(1) - x(0) = 1$$

are necessary and sufficient for the regularity of the method $H_s$ (5), Theorem 298).

The absolute Abel-type summability methods, considered in [4], are defined as under (see also [7]):

If $x(t)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit $I$ as $y \to \infty$, we say that the sequence $x_n$ is absolutely $A_s$-convergent, or $|A_s|$-convergent to $I$ and write $s_n \to I |A_s|$.  

If $U_s(y)$ is of bounded variation in the range $[0, \infty)$ and tends to the limit $I$ as $y \to \infty$, we say that the sequence $x_n$ is absolutely $A_sH_s$-convergent, or $|A_sH_s|$-convergent to $I$ and write $s_n \to I |A_sH_s|$.

The following two theorems are proved in [3]:

Theorem A. For $\lambda > 0$, $s_n \to I(A_s)$ if and only if $s_n \to I(A_s)$ and $n u_n \to 0(A_s)$.  

Theorem B. For $\lambda > 0$, $s_n \to I(A_s)$ if and only if $s_n \to I(A_s)$.  

It is also known that:

Theorem C. If $\lambda > -1$, $H_s$ is a regular Hausdorff method and $s_n \to I(A_s)$, then $s_n \to I_{A_sH_s}$.

For complete references to this result, see [4].

In [4], the absolute summability analogous of these results are proved.

\section*{2. Definitions}

We now define strong summability methods based upon the Abel-type methods $A_s$ and $A_s^\prime$ and the product method $A_sH_s$ (see also [5]).

\subsection*{Strong Abel-type summability $[A_s]$}

If

$$\int_0^1 |s_n(t) - I| dt = o(y)$$

as $y \to \infty$,

we say that the sequence $(s_n)$ is strongly $A_s$-convergent or $[A_s]$-convergent to $I$ and write $s_n \to I_{A_s}$.

\subsection*{Strong Abel-type summability $[A_s']$}

If

$$\int_0^1 |s_{n+1}(t) - I| dt = o(y)$$

as $y \to \infty$,

we say that the sequence $(s_n)$ is strongly $A_s'$-convergent or $[A_s']$-convergent to $I$ and write $s_n \to I_{A_s'}$.

\subsection*{Strong summability $[A_sH_s]$}

If $h_n \to I(A_s)$, we say that the sequence $(s_n)$ is strongly $A_sH_s$-convergent, or $[A_sH_s]$-convergent to $I$ and write $s_n \to I_{A_sH_s}$.

\subsection*{Strong boundedness}

If

$$\int_0^1 |s_{n+1}(t)| dt = O(y)$$

as $y \to \infty$,

the sequence $(s_n)$ is said to be strongly $A_s$-bounded or $[A_s]$-bounded and is written $s_n = O(1 |A_s|)$.

\section*{3. Preliminary results}

The following results are required.

\textbf{Lemma 1.} If $\lambda > \mu > -1$, $y > 0$ and $\lambda \int_0^y \left( \frac{t}{t+1} \right)^{\mu} dt$ is convergent for all $t > 0$,

then

$$\int_0^y \left( \frac{t}{t+1} \right)^{\mu} dt = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} \int_0^y (y-t)^{\lambda-1} e^{t} dt.$$  

For proof see [2], Lemma 2 (i).

\textbf{Lemma 2.} If $\lambda > -1$, $y > 0$ and $\lambda \int_0^y \left( \frac{t}{t+1} \right)^{\lambda} dt$ is convergent for all $t > 0$, then

$$\int_0^y \left( \frac{t}{t+1} \right)^{\lambda} dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1)} \int_0^y (y-t)^{\lambda-1} e^{t} dt.$$  

For proof see [2], Lemma 2 (ii).
Lemma 3. If \( \lambda > -1 \), \( \sum a_n x_n \) is convergent for \( 0 \leq x < 1 \) and \( h_n \) is defined by (1.1), then, for \( y > 0 \),

\[
\delta_n(y) = \left(1 + \frac{y}{1+y}\right)^{y-1} \sum_{n=0}^{\infty} \frac{x_n}{\delta_n(0)}. \quad (3.10)
\]

This lemma is proved in [2]. See also [1], p. 376.

Lemma 4. If \( \lambda > -1 \), \( a \) is real and \( s_n(y) = O(t) \) for \( y > 0 \) and \( (n + a)v_n = s_n \) for \( n = 0, 1, 2, \ldots \), then \( v_n \to 0 \) if \( (n + a) \).

This lemma is proved in [4] (Lemma 4).

We now also require the following result of Mikhin ([7], Theorem 6).

Lemma 5. If \( \lambda > -1 \) and \( s_n(t) \to l \), then \( s_n(t) \to l \).

In view of the above two lemmas we have the following:

Lemma 6. If \( \lambda > -1 \), \( a \) is real, \( s_n(y) = O(t) \) for \( y > 0 \) and \( (n + a)v_n = s_n \) for \( n = 0, 1, 2, \ldots \), then \( v_n \to 0 \).

An immediate consequence of the above lemma is the following:

Lemma 7. If \( \lambda > -1 \), \( p \) and \( q \) are real and \( s_n \to l \), then

\[
\frac{n + p}{n + q} \to l. \quad (3.11)
\]

§ 4. Main results

Theorem 1. If \( \lambda > 0 \) and \( s_n \to l \), then \( s_n \to l \).

Proof. We have, by (3.9),

\[
\sum_{n=0}^{\infty} \delta_n(y) - l \geq \frac{\lambda y^{-1}}{l} \left( \frac{y}{1+y} \right) \delta_n(0) dt - l dt,
\]

and, hence, we have

\[
\sum_{n=0}^{\infty} \delta_n(y) - l \geq \frac{\lambda y^{-1}}{l} \left( \frac{y}{1+y} \right) \delta_n(0) dt - l dt - o(t) \quad \text{as } y \to \infty.
\]

The theorem follows.

Remark. If we assume only \( [A_{1}] \)-boundedness in the hypothesis of the theorem, we obtain \( A \)-boundedness as the conclusion. This remark also applies to other theorems.

Theorem 2. If \( \lambda > 0 \) and \( s_n \to l \), then \( s_n \to l \).

Proof. We have, by hypothesis, that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } t \to \infty.
\]

It follows, by the regularity of the (C. 1) method, that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } y \to \infty;
\]

i.e., \( s_n \to l \).

The next theorem gives necessary and sufficient conditions for \( [A_{1}] \)-convergence.

Theorem 3. For \( \lambda > 0 \), the necessary and sufficient conditions for the \( [A_{1}] \)-convergence of the sequence \( \{s_n\} \) to \( l \) are:

\[
(4.1) \quad s_n \to l \quad \text{and} \quad s_n \to l \quad \text{as } y \to \infty.
\]

Proof. \( \text{(i) Necessity.} \)

(4.1) follows by Theorem 1.

Now, by (3.8), we have that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l \geq \frac{\lambda y^{-1}}{l} \left( \frac{y}{1+y} \right) \delta_n(0) dt - l dt - o(t) \quad \text{as } y \to \infty.
\]

By Theorem 2.

\( \text{(ii) Sufficiency.} \)

By (4.1) and Theorem 2, we have that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } y \to \infty.
\]

Hence, it follows by (4.1) and (3.8) that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } y \to \infty.
\]

This completes the proof of the theorem.

Theorem 4. For \( \lambda > 0 \), \( s_n \to l \) if and only if \( s_n \to l \).

Proof. \( \text{(i) Suppose that } s_n \to l \), i.e.,

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } y \to \infty.
\]

By (3.4), we have that

\[
\sum_{n=0}^{\infty} \delta_n(y) - l = o(t) \quad \text{as } y \to \infty.
\]

Hence, the result follows by (4.1) and (3.8).
where
\[
I(y) = \frac{\lambda}{\alpha} \int_0^\beta \left(1 + t\right)^{\alpha-1} dt \int_0^z \left(1 + z\right)^{\alpha} s_{\alpha+1}(z) - l \, dz
\]
\[
= \frac{\lambda}{\alpha} \beta \left(1 + \alpha\right) s_{\alpha+1}(\beta) - l \int_0^\beta \left(1 + t\right)^{\alpha-1} dt
\]
\[
= o(y) - \frac{\lambda}{\alpha} \frac{1}{\alpha} \frac{1}{2} (1 + \alpha)^{-1} \frac{1}{2} \int_0^\beta \left(1 + z\right)^{\alpha} s_{\alpha+1}(z) - l \, dz
\]
\[
= o(y) \quad \text{as} \quad y \to \infty.
\]
i. e., \( s_{\alpha} \to [A_{\alpha+1}]. \)

Further, by (3.5), we have that
\[
(1 + t) s_{\alpha+1}(t) = s_{\alpha+1}(t) - U_{\alpha+1}(t).
\]

But
\[
(1 + t) s_{\alpha+1}(t) = (1 + t)^{\alpha-1} \sum_{n=1}^\infty \frac{\lambda + 1 + n}{\alpha} \frac{n}{\alpha} \frac{1}{\alpha+1} \frac{1}{\alpha+1} \frac{1}{\alpha+1} u_{\alpha+1}(t) + o_1((1 + t)^{-1}).
\]

Thus we have
\[
\int_0^\beta \left(1 + t\right)^{\alpha-1} \sum_{n=1}^\infty \frac{\lambda + 1 + n}{\alpha} \frac{n}{\alpha} \frac{1}{\alpha+1} \frac{1}{\alpha+1} \frac{1}{\alpha+1} u_{\alpha+1}(t) + o_1((1 + t)^{-1}) dt
\]
\[
\leq o(y) \quad \text{as} \quad y \to \infty.
\]
i. e., \( s_{\alpha+1} \to [A_{\alpha+1}]. \)

Consequently, by Lemma 7
\[
u_{\alpha+1} \to 0[A_{\alpha+1}].
\]

(ii) Suppose that \( s_{\alpha} \to [A_{\alpha+1}] \) and \( u_{\alpha+1} \to 0[A_{\alpha+1}]. \) It follows from the last part of (i) that
\[
\int_0^\beta \left(1 + t\right)^{\alpha-1} dt = o(y) \quad \text{as} \quad y \to \infty.
\]
i. e., \( s_{\alpha} \to [A_{\alpha+1}]. \)

This completes the proof of the theorem.

Theorem 5. For \( \lambda > 0, \) \( s_{\alpha} \to [A_{\alpha+1}] \) if and only if \( s_{\alpha-1} \to [A_{\alpha-1}]. \)

Proof. (i) Suppose that \( s_{\alpha} \to [A_{\alpha+1}], \) i. e.,
\[
\int_0^\beta U_{\alpha+1}(t) - l \, dt = o(y) \quad \text{as} \quad y \to \infty.
\]

By (3.6), we have that
\[
\int_0^\beta \left| s_{\alpha}(t) - l \right| dt \leq \int_0^\beta \left| U_{\alpha+1}(t) - l \right| dt + \int_0^\beta \left| u_{\alpha+1}(t) \right| dt = o(y) + I(y)
\]
as \( y \to \infty. \) Now, by (3.7), it follows that
\[
I(y) = \frac{\lambda}{\alpha} \int_0^\beta \left(1 + t\right)^{\alpha-1} dt \int_0^\beta \left| s_{\alpha}(t) - l \right| dt + \int_0^\beta \left| U_{\alpha+1}(t) - U_{\alpha+1}(l) \right| dt + \int_0^\beta \left| u_{\alpha+1}(t) \right| dt = o(y) \quad \text{as} \quad y \to \infty,
\]
in view of Theorems 3 and 2.

Hence \( s_{\alpha} \to [A_{\alpha-1}], \) i. e.,
\[
\int_0^\beta \left| s_{\alpha}(t) - l \right| dt = o(y) \quad \text{as} \quad y \to \infty.
\]

Again, by (3.6), it suffices to show that
\[
\int_0^\beta \left| u_{\alpha+1}(t) \right| dt = o(y) \quad \text{as} \quad y \to \infty.
\]

But, by (3.2), we have that
\[
\int_0^\beta \left| u_{\alpha+1}(t) \right| dt \leq \int_0^\beta \left(1 + t\right)^{\alpha-1} \left| s_{\alpha}(t) - l \right| dt + \int_0^\beta \left(1 + t\right)^{\alpha-1} \int_0^\beta \left(1 + t\right)^{\alpha-1} \left| s_{\alpha}(t) - l \right| dt + \int_0^\beta \left(1 + t\right)^{\alpha-1} \left| s_{\alpha}(t) - l \right| dt = o(y) + I(y)
\]
where
\[
I(y) = \frac{\lambda}{\alpha} \int_0^\beta \left(1 + t\right)^{\alpha-1} dt \int_0^\beta \left(1 + t\right)^{\alpha-1} \int_0^\beta \left(1 + t\right)^{\alpha-1} \left| s_{\alpha}(t) - l \right| dt = o(y) \quad \text{as} \quad y \to \infty.
\]

The theorem follows.

Theorem 6. If \( \lambda > 0, \) \( H_{\alpha} \) is a regular Hausdorff method and \( s_{\alpha} \to [A_{\alpha}] \), then
\( s_{\alpha} \to [A_{\alpha+1}]. \)

Proof. We have, by (3.10), that
\[
h_{\alpha+1}(y) = (1 + y)^{\alpha-1} \sum_{n=1}^\infty \frac{n}{\alpha} \left( \frac{y}{1+y} \right)^n h_{\alpha}(y) dy = \int_0^\beta g_{\alpha+1}(y) dy
\]
Since $H_2$ is regular, it follows that
\[
\int_0^y |h_{k+1}(z) - l| \, dz = \int_0^y dz \left| \int_0^1 \{s_{k+1}(zt) - l\} \, d\chi(t) \right| \leq \int_0^y dz \int_0^1 \{s_{k+1}(zt) - l\} \, d\chi(t) = y \int_0^1 d\chi(t) \frac{1}{yt} \int_0^y s_{k+1}(x) - l \, dx.
\]

Hence
\[
\frac{1}{y} \int_0^y |h_{k+1}(z) - l| \, dz \leq \int_0^1 f(yt) \, d\chi(t)
\]
where
\[
f(t) = \frac{1}{t} \int_0^t |s_{k+1}(x) - l| \, dx = o(1) \quad \text{as } t \to \infty.
\]

It follows by the regularity of the continuous Hausdorff transformation ([5], Theorem 217) that
\[
\int_0^y |h_{k+1}(z) - l| \, dz = o(y) \quad \text{as } y \to \infty.
\]

The theorem follows.

References


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