TAUBERIAN THEOREMS FOR BOREL-TYPE METHODS OF SUMMABILITY

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1. Introduction. Suppose throughout that $s$, $a_n$ ($n=0, 1, 2, \ldots$) are arbitrary complex numbers, that $\alpha > 0$ and $\beta$ is real and that $N$ is a non-negative integer such that $\alpha N + \beta \geq 1$. Let

$$s_n = \sum_{n=0}^{\infty} a_n, \quad S_{a, \beta}(x) = \alpha e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}, \quad A_{a, \beta}(z) = \alpha e^{-z} \sum_{n=0}^{\infty} a_n \frac{z^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)}$$

where $z = x + iy$ is a complex variable and the power $z^\alpha$ is assumed to have its principal value.

We shall be concerned with the Borel-type method of summability $(B, \alpha, \beta)$ defined as follows (see [1]): we write $s_n \to s(B, \alpha, \beta)$, or $\sum \alpha a_n = s(B, \alpha, \beta)$, if $S_{a, \beta}(x)$ exists for all $x \geq 0$ and tends to $s$ as $x \to \infty$. Further, we write $s_n = 0(1)(B, \alpha, \beta)$ if $S_{a, \beta}(x)$ exists and is bounded on $[0, \infty)$.

The actual choice of the integer $N$ in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods $(B, \alpha, \beta_r)$ ($r=1, 2, \ldots, k$) are under consideration that $N$ is such that $\alpha N + \beta_r \geq 1$ ($r=1, 2, \ldots, k$).

The following result is known (see [2]):

(A) If $\beta > \mu$ and $\sum a_n = s(B, \alpha, \mu)$, then $\sum a_n = s(B, \alpha, \beta)$.

This result is “abelian” in character. Our object is to establish the “tauberian” results listed in the next section.

One of our tauberian conditions involves the notion of “slow decrease” defined as follows: a real-valued function $f(x)$, with domain $[0, \infty)$, is slowly decreasing if for every $\epsilon > 0$ there exist positive numbers $X$, $\delta$ such that $f(x) - f(y) > -\epsilon$ whenever $x - y > X$ and $x - y < \delta$.

2. Statements of the main results

**Theorem 1.** If $\sum a_n = s(B, \alpha, \mu)$ and $a_n \to 0(B, \alpha, \beta)$, then $\sum a_n = s(B, \alpha, \beta)$.

**Theorem 2.** If $s_n \to s(B, \alpha, \beta + \epsilon)$ for some $\epsilon > 0$ and $s_n = 0(1)(B, \alpha, \beta)$, then $s_n \to s(B, \alpha, \beta + \delta)$ for any $\delta > 0$.

**Theorem 2*.** If $\sum a_n = s(B, \alpha, \beta + \epsilon)$ for some $\epsilon > 0$ and $a_n = 0(1)(B, \alpha, \beta)$, then $\sum a_n = s(B, \alpha, \beta + \delta)$ for any $\delta > 0$.

**Theorem 3.** If $s_n \to s(B, \alpha, \beta + \epsilon)$ for some $\epsilon > 0$ and $S_{a, \beta}(x)$ is slowly decreasing, then $s_n \to s(B, \alpha, \beta)$.
Theorem 3. If $\sum a_n = \sigma(B, a, (a, b))$ for some $c > 0$ and $A_{a,b}(x)$ is slowly decreasing, then $\sum a_n = \sigma(B, a, (a, b))$.

Theorem 4. If $s_n = 0(1)(B, a, (a, b))$, then $s_n \geq -K$ for all $n \geq 0$ where $K$ is a positive constant, then $s_n = 0(B, a, (a, b))$.

Theorem 5. If $s_n \to (B, a, (a, b))$, then $s_n \geq -K$ for all $n \geq 0$ where $K$ is a positive constant, then $s_n \to (B, a, (a, b))$.

Theorem 6. If $\sum a_n = (B, a, (a, b))$ and $a_n \geq K$ for all $n \geq 0$ where $K$ is a positive constant, then $\sum a_n = (B, a, (a, b))$.

The following theorems are extensions of a result due to Gaier [3].

Theorem 7. If $\sum a_n = (B, a, (a, b))$ and $a_n \leq K$ for all $n \geq 0$ where $K$ is a positive constant, then $\sum a_n = (B, a, (a, b))$.

3. Preliminary results. It is known that the $(B, a, (a, b))$ method is regular (see [2]). Also, using the root test and a known result [1, Lemma 4], it can readily be shown that if either $S_{a,b}(x)$ or $A_{a,b}(x)$ exists for all $x > 0$ then both $S_{a,b}(x)$ and $A_{a,b}(x)$ exist for all $x > 0$.

Lemma 1. Let $S_{a,b}(x)$ exist for $x > 0$. Then, for $x > 0$,

(i) $S_{a,b}(x) = \int_0^x h(u) du$ where $\delta > 0$ and $h(u) = u^{\delta - x - 1}$ for $u < \delta$,

(ii) $A_{a,b}(x) = S_{a,b}(x) + \int_0^x h(u) du$ as $x \to \infty$.

The proof of Lemma 1 is straightforward, and Lemma 2 follows immediately from Lemma 1(ii) and result (A).

Theorem 8. Let $f(t)$ be Lebesgue integrable on every finite interval of $[0, \infty)$ and $F(x) = \int_0^x e^{-t} f(t) dt$ is such that $F(x) \to x$ as $x \to \infty$, then $f(t) \to x$ as $x \to \infty$.

Proof. Since

$$F(x) = \frac{1}{w} \int_0^w g(u) du$$

where $w = e^x$ and

$$g(u) = \begin{cases} 0, & 0 \leq u < 1, \\ f(u), & u \geq 1, \end{cases}$$

Theorem 8 follows from a known result [4, p. 126]. (Observe that $g(u)$ is "slowly decreasing" in the sense given on p. 124 of [4].

Lemma 3. If $f(t)$ is bounded on every finite interval of $[0, \infty)$ and is slowly decreasing, then there exist positive numbers $M_n$ and $M_m$ such that

$$f(x) - f(y) \geq -M_n(x - y) - M_m$$

whenever $x \geq y \geq 0$.

Proof. Since $f(t)$ is slowly decreasing, there exist positive numbers $X, \delta$ such that $f(x) - f(y) > 1$ if $x \geq y \geq X$ and $x - y \leq \delta$. Hence, if $x \geq y \geq X$ and $m$ is the smallest positive integer such that $(x - y)/m \leq \delta$, then

$$f(x) - f(y) = \sum_{j=0}^m \left( f(y + j \frac{x - y}{m}) - f(y + (j - 1) \frac{x - y}{m}) \right)$$

$$\geq -m = -(m-1) - 1,$$

$$\geq \frac{x - y}{\delta} - 1.$$ That is, $M = \sup_{x \geq X} \left| f'(x) \right|$, then

$$f(x) - f(y) \geq \frac{1}{\delta} (x - y) - 2M - 1$$ whenever $x \geq y \geq 0$.

Theorem 9. Let $h(u)$ be a real-valued, non-negative, Lebesgue measurable function such that

$$0 < \int_0^\infty h(u) du < \infty$$ and $\int_0^\infty uh(u) du < \infty$.

Let $f(t)$ be a real-valued function such that, for some positive numbers $M_1$ and $M_2$,

$$f(x) - f(y) \geq -M_1(x - y) - M_2 \quad \text{whenever} \quad x \geq y \geq 0,$$

and such that, for all $x \geq 0$,

$$F(x) = \int_0^x h(x) f(t) dt$$ exists as a Lebesgue integral. Then, $f(x)$ is bounded on $[0, \infty)$ whenever $F(x)$ is bounded on $[0, \infty)$.

Proof. Suppose that $M_2 = \sup_{x \geq 0} |F(x)| < \infty$.

Choose $X$ such that

$$L = \int_0^X h(u) du > 0.$$
Now
\[ f(x) = \int_0^x h(x-t) \, dt = \int_0^x (h(x-t) - f(t)) \, dt + F(x) \]
\[ \geq \int_0^x h(x-t) \left[ -M_3(x-t) - M_4 \right] \, dt + F(x) \]
\[ \geq -M_4 \int_0^x h(x-t) \, dt - M_3 \int_0^x h(x-t) \, dt - M_4 \]
\[ = -M_4, \]

say, and hence \( f(x) \geq -M_4 \) for \( x \geq 0 \). But \( f(x) \geq -M_4x + f(0) \) for \( 0 \leq x \leq X \).

Hence there exists a positive number \( M_4 \) such that \( f(x) \geq -M_4 \) for all \( x \geq 0 \).

If \( x \geq X \), then
\[ M_4 \geq f(x) \]
\[ = \int_0^{x-X} h(x-t)\, dt + \int_{x-X}^x h(x-t)f(t) \, dt \]
\[ \geq -M_4 \int_0^{x-X} h(x-t) \, dt + \int_{x-X}^x h(x-t)[f(x-X) - M_4(t-x+X)] \, dt \]
\[ \geq M_4 + f(x-X) \int_0^{x-X} h(x-t) \, dt \]

where
\[ M_4 = -M_4 \int_0^{x-X} h(x-t) \, dt + M_3 \int_0^{x-X} h(x-t) \, dt. \]

It follows that \( M_4 = M_3 \int_0^{x-X} h(x-t) \, dt \) for \( x \geq 0 \).

Thus \( f(x) \) is bounded on \([0, \infty)\).

**Theorem 10.** Let \( f(t) \) be a real-valued non-decreasing function defined on \([0, \infty)\) and, for \( \delta > 0 \), let
\[ F(x) = \int_0^x e^{-\delta t} f(t) \, dt, \quad x \geq 0. \]

Then \( e^{-\delta} f(x) \) is bounded on \([0, \infty)\) whenever \( F(x) \) is bounded on \([0, \infty)\).

**Proof.** Suppose \( M = \sup_{t \geq 0} |f(t)| < \infty \). Since \( f(t) \) is non-decreasing, we have, for all \( x \geq 0 \),
\[ \delta f(x) \geq \delta t f(x) \delta \int_0^x e^{-\delta t} f(t) \, dt \]
\[ = \delta e^{-\delta x} \int_0^x (x-\delta t)^{\delta-1} t f(t) \, dt + \delta e^{-\delta x} \int_0^x (x-\delta t)^{\delta-1} t f(t) \, dt \]
\[ \geq \delta e^{-\delta x} \int_0^x (x-\delta t)^{\delta-1} t f(t) \, dt + \delta e^{-\delta x} \int_0^x (x-\delta t)^{\delta-1} t f(t) \, dt \]
\[ = f(0)(x+1)^{-\delta} e^{-\delta x} - e^{-\delta x} f(x). \]

It follows that \( e^{-\delta} f(x) \) is bounded on \([0, \infty)\).
on $[0, \infty)$. Hence, by Theorem 2, $S_{\alpha, \beta}(x) \to x$ as $x \to \infty$. Thus, in view of Lemma 1(i), it follows, by Theorem 8 (with $G(x) = S_{\alpha, \beta}(x)$, $f(x) = S_{\alpha, \beta}(x)$), that $S_{\alpha, \beta}(x) \to x$ as $x \to \infty$.

Proof of Theorem 4. In view of Lemma 1(i), we can assume without loss of generality that $\mu = \alpha + \beta$ where $\beta > 0$. The result then follows by Theorem 10 with

$$F(x) = S_{\alpha, \beta}(x) + \alpha x \sum_{n=0}^{\infty} K \frac{x^{n+\beta-1}}{(\sin x)^{\alpha + \beta}}$$

$$e^{-\alpha f(x)} = S_{\alpha, \beta}(x) + \alpha x \sum_{n=0}^{\infty} K \frac{x^{n+\beta-1}}{(\sin x)^{\alpha + \beta}}$$

Proof of Theorem 5. Let $k$ be a positive integer such that $k < \beta$. Then, by Theorem 4, $s_{k} = O(1)$ if $\beta = 1$ and hence, by Theorem 2, $s_{k} = r_{k}(B, \alpha, \beta)$.

Proof of Theorem 6. Let $k$ be a positive integer such that $k > \beta$. Since $S_{\alpha, \beta}(x) = S_{\alpha, \beta}(x) + \sum_{n=0}^{\infty} \frac{d^{n}}{dx^{n}} S_{\alpha, \beta}(x)$, it is readily seen that

$$S_{\alpha, \beta}(x) = S_{\alpha, \beta}(x) + \sum_{n=0}^{\infty} \frac{d^{n}}{dx^{n}} S_{\alpha, \beta}(x)$$

where $c_{0}, c_{1}, \ldots, c_{k}$ are integers. Since $d^{n} S_{\alpha, \beta}(x)/dx^{n} \to 0$ as $x \to \infty$ (for $j=1, 2, \ldots, k$) by Theorem 11, we have that $S_{\alpha, \beta}(x) \to x$ as $x \to \infty$. Hence $S_{\alpha, \beta}(x) \to x$ as $x \to \infty$ by result (A).

The proofs of Theorems 2*, 3*, 5*, 6* follow the same basic pattern which we illustrate by one example.

Proof of Theorem 2*. By Lemma 2, $s_{k} = O(1)$ if $\beta = 1$ and hence, by Theorem 2, $s_{k} = r_{k}(B, \alpha, \beta)$ for any $\beta > 0$. The desired conclusion follows by Theorem 1.

Proof of Theorem 7. Since $|a_{n}| \leq Kx^{n}$ for all $n \geq 0$, we have that

$$|A_{n, x}(x)| \leq A e^{Kx^{n}}$$

for some positive constant $A$. The desired result follows by Theorem 6*.

5. Final remarks

1. Theorem 2 is false for $\beta = 0$. This is shown by the following example [cf. 4, p. 181]. Let $s_{k}$ be the sequence such that $S_{\alpha, 0}(x) = e^{x}$ if $x \leq 0$ and $S_{\alpha, 0}(x) = e^{-x}$ if $x > 0$. Hence $x_{n} = 0$ if $x_{n} = 1$ and $s_{n}$ does not tend to a limit $(B, 1, 1)$.

2. There exists a sequence $\{s_{n}\}$ which tends to a limit $(B, \beta, \beta)$ but does not tend to a limit $(B, \beta, \beta - 1)$. Choose an integer $m$ such that $m > 1$. Let $P$ be the smallest integer such that $mP \geq N$. Let $x^{n} = e^{-n} = \sum_{n=0}^{\infty} s_{n}$ and let

$$s_{n} = \begin{cases} \lceil (\alpha + \beta) \rceil b_{n} & \text{if } n = m \alpha b_{n} + \frac{\alpha}{b_{n}} \varepsilon \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. The usual theorems proved in this paper are not "empty".

4. Corresponding to the Borwein theorem "exponential" method $(B, \alpha, \beta)$ is an "integral" method $(B', \alpha, \beta)$ defined as follows (see [1]): $\sum_{n} a_{n} = r_{B}(B', \alpha, \beta)$ if $S_{\alpha, \beta}(x) \to x$ for all $x \geq 0$ and limits $x_{n} \to x$ as $n \to \infty$ where $x_{n} = 0$.

The following result is due to Borwein [1, Theorem 2]: $\sum_{n} a_{n} = r_{B}(B, \alpha, \beta)$ if only if $\sum_{n} a_{n} = r_{B}(B', \alpha, \beta)$.

5. The usual theorems proved in this paper suggest that analogous theorems hold for the method $(B', \alpha, \beta)$.

6. Let $p(x) = \sum_{n=0}^{\infty} p_{n} x^{n}$ be an integral function such that $p_{n} > 0$ for all $n$. Associated with $p(x)$ is an integral function method of summability $P$ defined as follows: $\lim_{n \to \infty} p_{n} x^{n} = o(x)$ as $x \to \infty$.

The following result is due to Borwein (see [2]): If $h(x)$ is analytic in $H$, $h(x)$ is real for $x \geq b$ and, when $x \geq b$ and $|x| \leq b$, $h(x) = e^{x^{2}} e^{1/C(x+1/|x|)}$ where $C, a$ are positive and $\beta, \gamma$ are real, then the method associated with the integral function $p(x) = \sum_{n=0}^{\infty} p_{n} x^{n}$ is equivalent to $(B, \alpha, \beta+1/2)$.

There should therefore be a usual theorems of the sort proved in this paper for a wide class of integral function methods.

REFERENCES


