Generalized strong summability of infinite series

By David Borwein at London and J.H. Rizvi at Karachi
Generalized strong summability of infinite series

By David Borwein at London, and J. H. Rizvi at Karachi

In this note, we consider the generalized strong Abel-type summability methods \([A_n]\) and \([A_1]_\lambda\), and establish some equivalence and inclusion relations. We also consider the product of Abel-type methods with regular Hausdorff methods.

1. Introduction

We write throughout:

\[
\sigma_n^\lambda = \left( n + \lambda \right) - \frac{(n+1)(n+2)\cdots(n+n)}{n!},
\]

\[
e_0 = 1,
\]

\[
S_n = \sum_{m=0}^{n} u_m,
\]

\[
S_n(y) = (1 + y)^{-1} \sum_{n=0}^{\infty} e_n S_n \left( \frac{y}{1+y} \right)^n,
\]

\[
u_n(y) = (1 + y)^{-1} \sum_{n=0}^{\infty} e_n u_n \left( \frac{y}{1+y} \right)^n,
\]

\[
U_n(y) = \lambda \int_0^1 u_n(t) dt.
\]

We also use \(M\) as a constant, not necessarily having the same value at each occurrence. The Abel-type methods \(A_n\) and \(A_1^\lambda\), introduced in [2] and [3], are defined as follows:

If

\[
(1 - \sigma)^{\lambda+1} \sum_{n=0}^{\infty} e_n S_n x^n
\]

is convergent for all \(x\) in the open interval \((0,1)\) and tends to a finite limit \(l\) as \(x \to 1\) in the open interval \((0,1)\), we say that the sequence \(S_n\) is \(A_1\)-convergent to \(l\) and write \(S_n \to l(A_1)\).

It is evident that \(S_n \to l(A_1)\) if and only if the series defining \(S_n(y)\) is convergent for all \(y > 0\) and \(S_n(y) \to l\) as \(y \to \infty\). For \(\lambda = 0\), we have the ordinary Abel summability \(A\).

2. Definitions of strong summability

Let \(p\) be a positive number. The strong Abel-type summability methods \([A_n]_p\) and \([A_1]_p\), are defined as follows ([5] and [8]):

**Strong Abel-type Summability \([A_n]_p\):** If

\[
\int_0^1 |S_n(t) - l|^p dt = o(y)
\]

as \(y \to \infty\), we say that the sequence \(S_n\) is strongly \(A_1\)-convergent with index \(p\) or \([A_n]_p\)-convergent to \(l\) and write \(S_n \to [l]_{[A_n]_p}\).

**Strong Abel-type Summability \([A_1]_p\):** If

\[
\int_0^1 |U_n(t) - l|^p dt = o(y)
\]

as \(y \to \infty\), we say that the sequence \(S_n\) is strongly \(A_1^\lambda\)-convergent with index \(p\) or \([A_1]_p\)-convergent to \(l\) and write \(S_n \to [l]_{[A_1]_p}\).

**Strong Product Summability \([A_n]_{H_p}\):**

If \(H_n \to [l]_{[A_n]_{H_p}}\), we say that the sequence \(S_n\) is \([A_n]_{H_p}\)-convergent to \(l\) and write \(S_n \to [l]_{[A_n]_{H_p}}\).

3. Main results

We prove the following theorems:

**Theorem 1:** If \(0 < \lambda < p\), and \(S_n \to [l]_{[A_n]_p}\), then \(S_n \to [l]_{[A_1]_p}\).

**Theorem 2:** If \(\lambda > 0\), \(p = 1\), and \(S_n \to [l]_{[A_n]_p}\), then \(S_n \to [l]_{[A_1]_p}\).

**Theorem 3:** If \(\lambda > 0\), and \(S_n \to [l]_{[A_n]_p}\), then \(S_n \to [l]_{[A_1]_{H_p}}\) for every \(p > 0\).

The next theorem gives necessary and sufficient conditions for \([A_n]_{H_p}\)-convergence of the sequence \(S_n\).
Theorem 4. For $\lambda > 0$, $p > 1$, necessary and sufficient conditions for the $[A_n]_p$-convergence of the sequence $(S_n)$ to $l$ are:

$$S_n \to I(A_l)$$

and

$$\int_0^1 \left| \frac{d}{dt} U(t) \right|^p dt = o(y) \quad \text{as} \quad y \to \infty.$$  

The following two theorems give relationships between the $[A_n]_p$ and $[A_n]_q$ methods.

Theorem 5. If $\lambda > 0$, $p > 1$, then $S_n \to I(A_l)$ if and only if $S_n \to I(A_{1/p})$ and $n^{1/b} \to 0[A_n]_p$.

Theorem 6. If $\lambda > 0$, $p > 1$, then $S_n \to I(A_{1/p})$ if and only if $S_n \to I(A_{1/q})$.

Finally, we have the following theorem about the product method $[A_n,H_p]$.

Theorem 7. If $\lambda > -1$, $p > 1$, $H_p$ is a regular Hausdorff method, and $S_n \to I(A_{1/p})$, then $S_n \to I(A_{1/p})$.

The corresponding results for ordinary summability are established in [2] and [3], for absolute summability in [4] and for strong summability (i.e. the case $p = 1$) in [5].

4. Preliminary results

We require the following results.

Lemma 1. If $\lambda > \mu > -1$, $y > 0$ and $\sum_{n=0}^\infty a_n c_n S_n \left( \frac{t}{1 + t} \right)^\mu$ is convergent for all $t > 0$, then

$$S_y(t) = \left( \frac{\Gamma(\mu + 1)}{\Gamma(\lambda + \mu) \Gamma(\mu + 1 - \mu)} \right)^{1/\mu} \int_0^1 (y - 1)^{1/\mu - 1} S_n(t) dt.$$  

This lemma is proved in [2] (Lemma 2 (ii)).

Lemma 2. If $\lambda > -1$, $y > 0$, and $\sum_{n=0}^\infty a_n c_n S_n \left( \frac{t}{1 + t} \right)^\mu$ is convergent for all $t > 0$, then

$$U_y(t) = \lambda (1 + y - t)^{-1} \left( 1 + t \right)^{-1} S_n(t) dt,$$

$$S_y(t) = U_y(t) + S_n(t),$$

$$y U_y(t) = U_{1+y}(t) - U_y(t),$$

$$y \frac{d}{dy} U_{1+y}(t) = -\frac{1}{\lambda + 1} \left( U_{1+y}(t) - U_y(t) \right),$$

$$y \frac{d}{dy} S_y(t) = (\lambda + 1) \left( S_{1+y}(t) - S_y(t) \right) - v_y(t),$$

$$U_y(t) = \lambda y^{1/\mu} \left( t^{1/\mu} U_{1+y}(t) \right) dt.$$  

Some of these relations are established in [3]. For complete proofs, see [9].

Lemma 3. If $\lambda > -1$, $\sum_{n=0}^\infty a_n c_n S_n \left( \frac{t}{1 + t} \right)^\mu$ is convergent for $0 \leq x < 1$ and $h_y$ is defined by (1.1), then

$$h_y(t) = (1 + y)^{-1} \sum_{n=0}^\infty a_n c_n \left( \frac{y}{1 + y} \right)^\mu \int_0^1 S_y(t) dt = \int_0^1 S_y(t) dt.$$  

This lemma is proved in [2] (Lemma 5). See also [1], p. 376.

Lemma 4. For $\lambda > -1$, $p > 1$, necessary and sufficient conditions for the $[A_n]_p$-convergence of the sequence $(S_n)$ to $l$ are that

(i) $S_n \to I(A_l)$

and

(ii) $\int_0^1 \left| \frac{d}{dt} S_n(t) \right|^p dt = o(y) \quad \text{as} \quad y \to \infty.$

This lemma is proved by Misra in [8] (Theorem 4).

5. Proofs of the main results

Theorem 1. We have, by assumption, that (2.2) holds. Using Hölder’s inequality with indices $\frac{p}{q}$ and $\frac{p}{p - q}$, we obtain

$$\int_0^1 \left| U_{1+y}(t) - l \right|^q dt \leq \left( \int_0^1 \left| U_{1+y}(t) - l \right|^p dt \right)^{\frac{q}{p}} \left( \int_0^1 \left| U_{1+y}(t) - l \right|^{p-q} dt \right)^{\frac{p-q}{p}} = o(y^\lambda) \quad \text{as} \quad y \to \infty.$$

Theorem 2. In view of Theorem 1, we may assume that $S_n \to I(A_{1/p})$. Now, by (4.9), we get that

$$\left| U_y(t) - l \right| \leq \lambda y^{-1} \left( t^{1/\mu} \left| U_{1+y}(t) - l \right| \right),$$

$$\lambda y^{-1} \left( t^{1/\mu} \left| U_{1+y}(t) - l \right| \right) \leq o(1) \quad \text{as} \quad y \to \infty.$$  

Theorem 3. The result is a consequence of the regularity of the $(C,1)$-method.

Theorem 4. Necessity: We need only establish (3.2). By (4.7), we have that

$$\int_0^1 \left| \frac{d}{dt} U_y(t) \right| dt \leq \left( \int_0^1 \left| U_{1+y}(t) - l \right|^p dt + \int_0^1 \left| U_y(t) - l \right|^p dt \right)^{\frac{1}{p}} = o(y) \quad \text{as} \quad y \to \infty.$$  

Sufficiency: Again, by (4.7), it follows that

$$\int_0^1 \left| U_{1+y}(t) - l \right|^p dt \leq M \int_0^1 \left| \frac{d}{dt} U_y(t) \right|^p dt + \frac{1}{\lambda} \int_0^1 \left| U_y(t) - l \right|^p dt = o(y) \quad \text{as} \quad y \to \infty,$$

by Theorem 3 and (3.2).
Theorem 5. (i) Suppose that $S_n \to [A_{\lambda}]$, i.e., \[ \int_0^y |S_n(t) - l|^{p-1} dt = o(y) \] as $y \to \infty$.

In view of (4.3), we have that
\[
\int_0^y |U_{n+1}(t) - l|^{p-1} dt \leq M \left[ \int_0^y (1 + t)^{\alpha-1} dt \right] \left[ \int_0^y |S_n(t) - l|^{p-1} dt \right]^{\frac{p-1}{p}} + \int_0^y (1 + t)^{\alpha-1} dt
\]
\[= M \int_0^y J_1(t) dt + M \int_0^y J_2(t) dt.
\]

Now \[ \int_0^y J_1(t) dt = o(y) \] as $y \to \infty$, since $-p(\lambda + 1) + 1 < 0$.

Further, using H"older's inequality with indices $p$ and $\frac{p}{p-1}$, it follows that
\[ J_2(t) \leq (1 + t)^{\alpha-1} \int_0^y |S_n(t) - l|^{p-1} dt \leq (1 + t)^{\alpha-1} \int_0^y |S_n(t) - l|^{p-1} dt = o(1) \] as $t \to \infty$.

Hence,
\[ \int_0^y J_2(t) dt = o(1) \] as $y \to \infty$.

Consequently, \[ \int_0^y |U_{n+1}(t) - l|^{p-1} dt = o(y) \] as $y \to \infty$. Thus $S_n \to [A_{\lambda}]$. Taking note of Lemma 4 and (4.8), we get that $u_n \to 0 [A_{\lambda}]$.

(ii) Suppose that $S_n \to [A_{\lambda}]$, and $u_n \to 0 [A_{\lambda}]$. We show first that $S_n \to [A_{\lambda}]$.

Now, by (4.4) and (4.5), we get that
\[ S_n(t) = l \left( 1 + \frac{1}{t} \right) (U_0(t) - \frac{1}{t} U_0(t) - \frac{1}{t} U_0(t) - \frac{1}{t} U_0(t)).
\]

Thus, for $y > 1$, we have that
\[ \int_0^y |S_n(t) - l|^{p-1} dt \leq M \int_0^y |U_0(t) - l|^{p-1} dt + \int_0^y |U_0(t) - l|^{p-1} dt = o(y) \] as $y \to \infty$.

It follows that $S_n \to [A_{\lambda}]$. To complete the proof, we note that by (4.8),
\[ \int_0^y |S_n(t) - l|^{p-1} dt \leq M \left[ \int_0^y |U_0(t) - l|^{p-1} dt + \int_0^y |S_n(t) - l|^{p-1} dt \right] = o(y) \] as $y \to \infty$.

By the above, we have already proved that $S_n \to [A_{\lambda}]$ whenever $S_n \to [A_{\lambda}]$. Now, suppose that $S_n \to [A_{\lambda}]$, i.e., \[ \int_0^y |S_n(t) - l|^{p-1} dt = o(y) \] as $y \to \infty$. In view of (4.2) and (4.5), we have that
\[ U_{n+1}(t) - l = S_n(t) - l - (1 + t)^{-\frac{1}{p}} |S_n(t) - l| \]
\[+ \lambda(1 + t)^{\frac{p-1}{p}} \left( (1 + t)^{\frac{p-1}{p}} - 1 \right) dt - (1 + t)^{-\frac{1}{p}} dt.
\]

so that
\[ |U_{n+1}(t) - l|^p \leq M \left[ |S_n(t) - l|^p + \lambda(1 + t)^{\frac{p-1}{p}} \left( (1 + t)^{\frac{p-1}{p}} - 1 \right) dt - (1 + t)^{-\frac{1}{p}} dt \right] + (1 + t)^{-\frac{p-1}{p}} dt
\]
\[= \int_0^y (1 + t)^{\alpha-1} dt,
\]

By assumption \[ \int_0^y I_1(t) dt = o(y) \] as $y \to \infty$; and since $-p(\lambda + 1) + 1 < 0$, we also have that \[ \int_0^y I_2(t) dt = o(y) \] as $y \to \infty$. Thus we are left with $I_3(t)$. By H"older's inequality, it follows that
\[ I_3(t) \leq M \int_0^y (1 + t)^{\alpha-1} dt \int_0^y |S_n(t) - l|^p dt + (1 + t)^{\alpha-1} dt \]
\[\leq M \int_0^y (1 + t)^{-\frac{1}{p}} dt + \int_0^y (1 + t)^{\alpha-1} dt \int_0^y |S_n(t) - l|^p dt = o(1) \] as $t \to \infty$,

and so \[ \frac{1}{y} \int_0^y I_3(t) dt = o(1) \] as $y \to \infty$. This completes the proof of the theorem.

Theorem 7. Suppose that $S_n \to [A_{\lambda}]$. It follows from the regularity of $H_\lambda$ and (4.40) that $b_{n+1}(t) - l = \int_0^y |S_{n+1}(t)| dt$ using H"older's inequality for Steiltjes integrals (7). Theorem 210), we get
\[ \int_0^y |b_{n+1}(t) - l|^{p-1} dt \leq M \int_0^y |S_{n+1}(t)|^{p-1} dt \]
\[= M \int_0^y |f(t)| dt \] where
\[ f(t) = \int_0^y |S_{n+1}(t)|^{p-1} dt = o(1) \] as $t \to \infty$.

The theorem follows now from a standard argument (cf. [6], proof of Theorem 217), since $\chi(t)$ is continuous at 0.

References

Department of Mathematics, University of Western Ontario, London, Ontario, Canada.

Department of Mathematics, University of Karachi, Karachi 32, Pakistan.