EQUIVALENCE OF RIESZ METHODS OF SUMMABILITY

D. BORWEIN AND F. P. CASS

Suppose throughout that \( \lambda = \{ \lambda_n \} \) is an unbounded strictly increasing sequence with \( \lambda_0 = 0 \), that \( p \) is a non-negative integer, that \( 0 < \delta \leqslant 1 \), that \( \kappa \geqslant 0 \), and that \( \sum_{\nu=0}^{\infty} a_{\nu} \) is a series of real numbers.

The series \( \sum_{\nu=0}^{\infty} a_{\nu} \) is said to be summable by the Riesz method \((R, \lambda, \kappa)\) to \( s \) if

\[
\sum_{\lambda_\omega < \omega} \left(1 - \frac{\lambda_\omega}{\omega}\right)^\kappa a_{\nu} \to s \quad \text{as} \quad \omega \to \infty.
\]

A summability method \( Q \) is said to include a method \( P \), and we write \( P \subseteq Q \), if every series summable \( P \) to \( s \) is necessarily summable \( Q \) to \( s \). The methods are said to be equivalent, and we write \( P \sim Q \), if \( P \subseteq Q \) and \( Q \subseteq P \).

It is familiar that

\[
(R, \lambda, \alpha) \subseteq (R, \lambda, \beta) \quad (0 \leqslant \alpha \leqslant \beta).
\] (1)

See [3; Theorem 16].

Our object is to prove the following theorem.

**THEOREM.** If \( \kappa > p \), then a necessary and sufficient condition for \((R, \lambda, \kappa) \sim (R, \lambda, p)\) is

\[
\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.
\] (2)

The case \( p = 0 \) of the above theorem is known; see [4] for full references. The following lemma is essentially part of a theorem we established elsewhere [1; Theorem 5]. A proof is given here for completeness.

**LEMMA.** Suppose that \( a_{n, \nu} \geqslant 0 \) and that

\[ t_n = \sum_{\nu=0}^{\infty} a_{n, \nu} s_{\nu} \]

tends to zero if and only if \( s_n \) tends to zero. Then

\[
\liminf_{n \to \infty} \max_{\nu > 0} a_{n, \nu} > 0.
\]

**Proof.** The hypotheses imply that \( \lim_{n \to \infty} a_{n, \nu} = 0 \) for \( \nu = 0, 1, 2, \ldots \), and that

\[
\sup_{n \geqslant 0} \sum_{\nu=0}^{\infty} a_{n, \nu} < \infty.
\]

Let \( \mu_{\nu} = \max_{n \geqslant 0} a_{n, \nu} \) and assume that \( \liminf_{n \to \infty} \mu_{\nu} = 0 \). There is an increasing sequence of integers \( \{ k_i \} \) such that

\[
\mu_{k_i} < 2^{-i}.
\]

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Define a divergent sequence \( (x_n) \) by setting \( x_n = 1 \) if \( n = k \) and \( x_n = 0 \) otherwise. The corresponding \( t_n \) tends to zero since for each integer \( m \) we have

\[
  t_n = \sum_{k=0}^{\infty} a_n x_k \leq \sum_{k=0}^m a_n x_k + \sum_{k=m+1}^{\infty} \frac{a_n}{2^k}
\]

We now introduce some notation and state a definition of generalized Cesàro summability given by Bosanquet and Russell [2].

Let

\[
h_{n,v}(\lambda) = (a_{n+v} + \lambda) D^v (a_n - \lambda)
\]

where \( D^v \) is a divided difference operator defined inductively by

\[
D^0 b_n = b_n, \quad D^{v+1} b_n = D^v b_n - \frac{D^v b_{n+1}}{\lambda_{n+v+1} - \lambda_n}
\]

and let

\[
C_s^m = \sum_{v=0}^{m} h_{n,v}(\lambda) C_s^v.
\]

The definition of \( C_s^p \) is unambiguous in the case \( \delta = 1 \), since (see [2] or [5])

\[
C_s^p = \sum_{v=0}^{p} (a_{n+p+1} - \lambda) C_s^v.
\]

Let

\[
E_s^p = \frac{C_s^p}{E_s^0},
\]

where \( E_s^p \) is the value of \( C_s^p \) obtained from the series with \( a_0 = 1, a_n = 0 \) \( (n > 0) \), i.e.,

\[
E_s^0 = 1, \quad E_s^{m+1} = \lambda_{m+1} \lambda_{m+2} \ldots \lambda_{n+m+1} \quad (m = 0, 1, \ldots),
\]

\[
E_s^{p+1} = \sum_{v=0}^{p} h_{n,v}(\lambda) E_s^v
\]

The series \( \sum_{v=0}^{p} a_n \) is said to be summable by the generalised Cesàro method \((\Psi_s^*, \lambda_s, n)\) to \( x \) if \( t_n^* \to x \) as \( n \to \infty \).

We require the following results established by Bosanquet and Russell [2].

\[
(R_s, \lambda, s) = (\Psi_s^*, \lambda_s, s);
\]

\[
0 < h(\lambda_{n+p+1}, \lambda, s) \leq h(\lambda_{n+p+1}, \lambda, p)
\]

\[
\leq \left( \frac{p+3}{p} \right) \left( \lambda_{n+p+1} - \lambda \right)^p \quad (0 \leq \lambda \leq \infty)
\]

[2, Lemma 2];

\[
E_s^{p+1} \geq \left( \frac{p+3}{p+1} \right) E_s^p \Psi_s^{p+1} \quad [2, \text{Lemma 3}].
\]

**Proof of the Theorem.**

**Sufficiency.** In view of (1), and the fact that \( \lambda_{n+p+1} \) increases with \( p \), it is enough to show that condition (2) implies \((\Psi_s^*, \lambda, s+1) = (\Psi_s^*, \lambda, p)\). By (3), we have that

\[
t_n^* \to x \quad \text{ whenever } \quad t_n^{p+1} \to x \quad \text{ and hence that } \quad (\Psi_s^*, \lambda, s+1) = (\Psi_s^*, \lambda, p).
\]

**Necessity.** Define \( A = (a_n) \) to be the matrix such that

\[
t_n^* = \sum_{v=0}^{\infty} a_n \ U_v^p
\]

so that \( a_n = (\Psi_s^*, \lambda, s) \) for \( 0 \leq n \leq n \), and \( a_n = 0 \) for \( n > n \).

By (5) and (6) we have,

\[
0 < \frac{h(\lambda_{n+p+1}, \lambda)}{h(\lambda_{n+p+1}, \lambda, p)} \leq \left( \frac{p+3}{p} \right) \left( \lambda_{n+p+1} - \lambda \right)^p
\]

and

\[
E_s^{p+1} \geq \left( \frac{p+3}{p} \right) E_s^p \Psi_s^{p+1} \quad \text{and therefore}
\]

\[
0 \leq a_n \ U_v^p \left( 1 - \frac{\lambda}{\lambda_{n+p+1}} \right)^p \quad (0 \leq n \leq p).
\]

Suppose now that

\[
(R_s, \lambda, p+1) = (R_s, \lambda, p).
\]

Then, by (4) and (7), the summability method associated with the matrix \( A \) is equivalent to convergence, and hence, by the lemma,

\[
\liminf_{s=0} \frac{a_n}{s^2} > 0.
\]

It follows from (3) and (10) that

\[
\liminf_{s=0} \left( 1 - \frac{\lambda}{\lambda_{n+p+1}} \right)^p > 0,
\]

and hence that

\[
\liminf_{s=0} \frac{\lambda_{n+p+1}}{\lambda} > 1.
\]

We have thus shown that (9) implies (2) and, in view of (1), the proof is complete.

**Remark.** Let \( p, q \) be integers with \( p > q > 0 \), and let the sequence \( \lambda = (\lambda_n) \) be such that

\[
\liminf_{s=0} \frac{\lambda_{n+p+1}}{\lambda} > 1 \quad \text{and} \quad \liminf_{s=0} \frac{\lambda_{n+q+1}}{\lambda} = 1.
\]
Then, using the above theorem, we find that \((R, \lambda, \alpha) \sim (R, \lambda, p)\) whenever \(\alpha > p\), but \((R, \lambda, \beta) \sim (R, \lambda, q)\) whenever \(\beta > q\). An example of such a sequence is given by
\[
\lambda_n = 2^n + r \text{ for } n = m(p+1) + r \text{ and } 0 \leq r \leq p.
\]

References


The University of Western Ontario,
London, Ontario,
Canada N6A 5B9.