TAUBERIAN THEOREMS FOR STRONG AND
ABSOLUTE BOREL-TYPE METHODS
OF SUMMABILITY

BY
D. BORWEIN AND E. SMET

1. Introduction. Suppose throughout that $s$, $a_n$ ($n = 0, 1, 2, \ldots$) are arbitrary complex numbers, that $\alpha > 0$ and $\beta$ is real and that $N$ is a non-negative integer such that $\alpha N + \beta \geq 1$. Let

$$s_n = \sum_{n=0}^{\infty} a_n \quad (n \geq 0),$$

$$s_{-1} = 0,$$

$$s_{n,\beta}(z) = \sum_{n=N}^{\infty} s_n \frac{z^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},
\quad a_{n,\beta}(z) = \sum_{n=N}^{\infty} a_n \frac{z^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},$$

$$S_{n,\beta}(z) = \alpha e^{-z} s_{n,\beta}(z),
\quad A_{n,\beta}(z) = \alpha e^{-z} a_{n,\beta}(z)$$

where $z = x + iy$ is a complex variable and the power $z^\gamma$ is assumed to have its principal value.

Borel-type methods are defined as follows:
(a) Summability: If $S_{n,\beta}(x)$ exists for all $x \geq 0$ and tends to $s$ as $x \to \infty$, we say that $s_n \to s(B, \alpha, \beta)$ or $\sum a_n = s(B, \alpha, \beta)$;
(b) Strong summability with index $p > 0$: If $S_{n,\beta-1}(x)$ exists for all $x \geq 0$ and

$$\int_0^x e^{-t} |S_{n,\beta-1}(t) - s|^p \, dt = o(e^{x}),$$

we say that $s_n \to s[B, \alpha, \beta]_p$;
(c) Absolute summability: If $s_n \to s(B, \alpha, \beta)$ and $S_{n,\beta}(x) \in BV_x[0, \infty)$, we say that $s_n \to s[B, \alpha, \beta]$;
(d) Boundedness: If $S_{n,\beta}(x)$ exists and is bounded on $[0, \infty)$, we say that $s_n = O(1[B, \alpha, \beta])$;
(e) Strong boundedness with index $p > 0$: If $S_{n,\beta-1}(x)$ exists for all $x \geq 0$ and

$$\int_0^x e^{-t} |S_{n,\beta-1}(t)|^p \, dt = o(e^{x}),$$

we say that $s_n = O(1[B, \alpha, \beta]_p)$.

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\(^{(2)}\) $f(x) \in BV_x[0, \infty]$ means that $f(x)$ is of bounded variation with respect to $x$ on $[0, \infty)$. 161
The summability method \((B,1,1)\) is the Borel exponential method \(B\) (see [7]). The \((B,\alpha,\beta)\) method is due to Borwein (see [2]) and the \([B,\alpha,\beta]_p\) and \([B,\alpha,\beta]\) methods are due to Borwein and Shawyer (see [4], [3], respectively). Strong Borel-type summability \([B,\alpha,\beta]\) (see [3]) is the \([B,\alpha,\beta]\) method.

The actual choice of the integer \(N\) in the above definitions is clearly immaterial. We shall therefore tacitly assume whenever a finite number of methods, with \(\alpha\) fixed and \(\beta = \beta_1, \beta_2, \ldots, \beta_n\) are under consideration that \(N\) is such that \(\alpha N + \beta = 1\) (for \(r = 1,2,\ldots, k\)).

The following known result establishes a natural scale for these summability methods. (Theorem A(ii) is [1], (ii), Theorem A(ii) is [3, Theorem 9] when \(p = 1\) and part of [4, Theorem 9(ii)] when \(p \neq 1\). Theorem A(iii) is [8, Lemma 1].)

**Theorem A.** Let \(\beta > \mu\).

(i) If \(s_n \to s(B, \alpha, \mu)\), then \(s_n \to s(B, \alpha, \beta)\).

(ii) If \(p \geq 1\) and \(s_n \to 0(B, \alpha, \mu)\), then \(s_n \to 0(B, \alpha, \beta)\).

(iii) If \(s_n \to 0(B, \alpha, \mu)\), then \(s_n \to 0(B, \alpha, \beta)\).

In [5] we established a number of tauberian theorems for the \((B,\alpha,\beta)\) method. In this paper we investigate all the corresponding results for the \([B,\alpha,\beta]_p\) method with \(p \geq 1\) and either prove them or show, by means of counterexamples, that they are false. We also examine some of the corresponding results for the \([B,\alpha,\beta]\) method.

2. Preliminary results. We first state some known results.

**Lemma 1.**

(i) If \(p \geq 1\) and \(s_n \to 0(B, \alpha, \beta)\), then \(s_n \to 0(B, \alpha, \beta)\).

(ii) If \(s_n \to 0(B, \alpha, \beta)\), then \(s_n \to 0(B, \alpha, \beta)\).

**Lemma 2.**

(i) If \(p \geq 1\) and \(s_n \to 0(B, \alpha, \beta)\), then \(s_n \to 0(B, \alpha, \beta)\).

(ii) If \(p > 0\) and \(s_n \to 0(B, \alpha, \beta)\), then \(s_n \to 0(B, \alpha, \beta + 1)\).

Lemma 10) is included in [3, Theorem 15] when \(p = 1\) and in [4, Theorem 12*] when \(p > 1\). Lemma 10) is included in [3, Theorem 14]. Lemma 20) is [3, Theorem 5] when \(p = 1\) while Lemma 20) follows from [4, Theorem 3*] and Theorem A) when \(p > 1\). Lemma 20) is [4, Theorem 5*].

Wherever it occurs in the following lemmas, we suppose that \(f(x)\) is bounded and Lebesgue measurable on every finite interval \([0,X]\) and we let \(f(x)\) be defined by

\[
\hat{f}(x) = \frac{1}{1/(\beta)} \int_0^1 (x - t)^{-1} f(t) \, dt
\]

where \(\beta > 0\).

**Lemma 3.** If \(\delta > 0\) and \(\gamma > 0\), then

\[
\hat{f}(x) = \frac{1}{1/(\gamma)} \int_0^1 (x - t)^{-1} f(t) \, dt.
\]

**Lemma 4.**

(i) Let \(f(x) = S_{\alpha,\beta}(x)\) and let \(\delta > 0\). Then \(s_n \to 0(B, \alpha, \beta)\).

(ii) \(A_{\alpha,\beta}(x) = S_{\alpha,\beta}(x) - S_{\alpha,\beta}(x) - \alpha e^{-\alpha x} + O(1/(\alpha n + \beta))\).

**Lemma 5.** If \(s_n = 0(B, \alpha, \beta)\), then \(s_n = 0(B, \alpha, \beta + \delta)\) for every \(\delta > 0\).

**Lemma 6.** If \(s_n = 0(B, \alpha, \beta)\), then

(i) \(s_n = 0(B, \alpha, \beta)\),

(ii) \(s_n = 0(B, \alpha, \beta + \delta)\), where \(0 < \delta < 1\), and

(iii) \(s_n = 0(B, \alpha, \beta + \delta)\), where \(\delta > 0\) and \(\delta \neq 1\).

**Proof.** (i) When \(p = 1\) the result is [3, Theorem 4]. Thus we suppose that \(p > 1\) and we let \(1/p + 1/q = 1\). Using Hölder's inequality and Lemma 4), we have that

\[
\left| S_{\alpha,\beta}(x) \right| \leq \int_0^1 e^{-\alpha x} \left| S_{\alpha,\beta}(t) \right| dt 
\]

\[
\leq e^{-\alpha x} \left( \int_0^1 e^{t^2/\beta} \right)^{1/2} \left( \int_0^1 e^{t^2/\beta} \right)^{1/2} 
\]

\[
= e^{-\alpha x} \left( K e^{x^2/\beta} \right)^{1/2} = K x^{1/2} 
\]

for some positive constant \(K\) since \(s_n = 0(1/B, \alpha, \beta)\).

(ii) When \(p = 1\) the result is included in [3, Theorem 10]. Thus we again suppose that \(p > 1\) and we let \(1/p + 1/q = 1\). Furthermore, we let \(f(x) = S_{\alpha,\beta}(x)\), \(L = 2/(1/\beta)\), and \(M = 1/(1/(\beta^2))\). Then, using Lemma 4), Hölder's inequality, and part of the proof of (i), we have for \(x \geq 1\) that

\[
\int_0^1 e^{-\alpha x} \left| S_{\alpha,\beta}(t) \right| dt 
\]

\[
\leq L \left( \int_0^1 e^{t^2/\beta} \right)^{1/2} \left( \int_0^1 |f(u)|^q du \right)^{1/2} 
\]

\[
+ L \left( \int_0^1 e^{t^2/\beta} \right)^{1/2} \left( \int_0^1 |f(u)|^q du \right)^{1/2} 
\]

\[
\times \left( \int_0^1 (t - u)^q du \right)^{1/2} 
\]

\[
+ M 
\]
This establishes the desired result.

(iii) If $K \geq 1$, then
\[ \int_0^1 e^{(t-u)^2} |f(u)|^p \, du \leq K \int_0^1 e^{t} \, dt \leq K e^t \]
for some positive constant $K$ by Lemma 6(ii) and Lemma 5.

**Lemma 7.** If
\[ e^{-t} \int_0^t f(t) \, dt = o(1), \]
then
\[ e^{-t} \int_0^t f(t) \, dt = o(1) \]
for every $0 < \delta < 1$.

The proof of Lemma 7 is essentially the same as the proof of [3, Lemma 5].

**Lemma 8.** Let $p \geq 1$. If
\[ e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1) \quad \text{and} \quad e^{-t} \int_0^t f(t) \, dt = o(1), \]
then
(i) $e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1)$ where $0 < \delta < 1$ and
(ii) $e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1)$ where $\delta > 0$ and $\delta \leq 1$.

**Proof.** (i) Let $e > 0$. By hypothesis, there exists a number $Y = 0$ such that
\[ \int_0^Y f(t) \, dt \leq \epsilon e^t \]
and
\[ \int_0^Y f(t) \, dt \leq \epsilon e^t. \]

Now
\[ \lim_{e \to 0} \sup_{x(t) \neq T} e^{-x(t)} \left| \int_0^x f(t) \, dt \right| < w. \]

Thus, the desired result holds.

**Corollary.** For every $0 < \delta < 1$ and
\[ e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1). \]

For $\delta > 0$ and $\delta \leq 1$,
\[ e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1). \]

**Example.** Let $N(e) = \sup_{x(t) \neq T} \left| \int_0^x f(t) \, dt \right| < w.$

\[ \lim_{e \to 0} \sup_{x(t) \neq T} e^{-x(t)} \left| \int_0^x f(t) \, dt \right| < w. \]

and
\[ e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = o(1). \]

But, using the Second Mean Value Theorem,
\[ \lim_{e \to 0} \sup_{x(t) \neq T} e^{-x(t)} \left| \int_0^x f(t) \, dt \right| < w. \]

since
\[ \left| \int_0^x f(t) \, dt \right| \leq 2 \sup_{x(t) \neq T} \left| \int_0^x f(t) \, dt \right| \]
and
\[ \lim_{e \to 0} \sup_{x(t) \neq T} e^{-x(t)} \left| \int_0^x f(t) \, dt \right| = 0. \]

Also, by hypothesis there is a number $K \geq 0$ such that
\[ e^{-t} \int_0^t e^{(t-u)^2} |f(u)|^p \, du = K. \]
for all $x \geq 0$, and therefore, when $p = 1$,

$$
\lim_{\delta \to 0} \sup_{x} I_{\delta}(x) \leq \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} (t-u)^{p-1}|f(u)| \, du
$$

$$
\leq \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} |f(u)| \, du \int_{0}^{x} (t-u)^{p-1} \, dt
$$

$$
\leq K \epsilon \frac{p}{\delta},
$$

while, when $p > 1$,

$$
\lim_{\delta \to 0} \sup_{x} I_{\delta}(x) \leq \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} e^{x-t} \int_{0}^{x} (t-u)^{p-1}|f(u)|^{p} \, du
$$

$$
\times \left( \int_{0}^{x} (t-u)^{p-3} \, dt \right)^{p-1} \, dt
$$

$$
= \left( \frac{p}{\delta} \right)^{p-1} \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} e^{x-t} \int_{0}^{x} (t-u)^{p-1}|f(u)|^{p} \, du
$$

$$
\leq \left( \frac{p}{\delta} \right)^{p-1} \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} \int_{0}^{x} (t-u)^{p-1}v^{p-1} \, dt
$$

$$
\leq \left( \frac{p}{\delta} \right)^{p} \lim_{\delta \to 0} \sup_{x} e^{-x} \int_{0}^{x} e^{x-t} \int_{0}^{x} (t-u)^{p-1}|f(u)|^{p} \, du \leq K \epsilon \frac{p}{\delta}.
$$

Thus for $p \geq 1$ we have that

$$
\lim_{\delta \to 0} e^{-x} \int_{0}^{x} e^{x-t} \int_{0}^{x} (t-u)^{p-1}|f(u)|^{p} \, du = 0
$$

from which it follows that

$$
\lim_{\delta \to 0} e^{-x} \int_{0}^{x} e^{x-t} \int_{0}^{x} (t-u)^{p-1}|f(u)|^{p} \, dt = 0
$$

since $\epsilon$ is arbitrary. This establishes the desired result.

(ii) Since $e^{-x}f_{i}(x) = o(1)$ by hypothesis, we have, when $\delta = 1 + \mu$ where $\mu > 0$, that

$$
e^{-x} f_{i}(x) = e^{-x} \int_{0}^{x} f_{i}(t) \, dt = o(1),
$$

using Lemma 3 and Lemma 7. Hence, for $\delta = 1$,

$$
e^{-x} \int_{0}^{x} e^{x-t} \left\{ f_{i}(t) \right\}^{p} \, dt = e^{-x} \int_{0}^{x} e^{x-t} \left\{ e^{-x} f_{i}(t) \right\}^{p} \, dt
$$

$$
= e^{-x} \int_{0}^{x} e^{x-t} o(1) \, dt
$$

$$
= o(1).
$$

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If $b$ is a real number, we let

$$H_{b} = \{ z \mid \Re z \geq b \}.
$$

A function $g(z)$ is said to be of exponential type in $H_{b}$ if $g(z)$ is analytic in $H_{b}$ and if there are positive numbers $A, a$ such that $|g(z)| = Ae^{\alpha|z|}$ for all $z$ in $H_{b}$.

**Lemma 9.** If $g(z)$ is of exponential type in $H_{b}$ and if

$$
\int_{0}^{\infty} \left| g(x) \right|^{p} \, dx < \infty (p > 0),
$$

then

$$
\int_{0}^{\infty} \left| \chi(x) \right|^{p} \, dx < \infty.
$$

**Lemma 9** is due to Gaiser [6, Theorem 2].

**Lemma 10.** If $g(z)$ is of exponential type in $H_{b}$ and $g(z) \in BV_{L}(b, w)$, then $\hat{g}^{(k)}(x) \in BV_{L}(b, w)$ for every non-negative integer $k$.

**Proof.** Suppose that $g^{(k)}(x) \in BV_{L}(b, w)$ where $k$ is a non-negative integer. Then

$$
\int_{0}^{\infty} \left| g^{(k)}(x) \right|^{p} \, dx < \infty
$$

and

$$
\left| g^{(k+1)}(x + b + 1) \right| \leq \frac{B_{k+1}}{2w} \int_{0}^{\infty} \left| g(x + b + c^{w}) \right| \, dc
$$

$$
\leq \frac{B_{k+1}}{2w} \int_{0}^{\infty} \left| g(x + b + c^{w}) \right| \, dc < \infty
$$

for all $z$ in $H_{0}$ where $A, a$ are positive constants. Hence, by Lemma 9,

$$
\int_{0}^{\infty} \left| g^{(k+1)}(x + b + 1) \right| \, dx = \int_{0}^{\infty} \left| g^{(k+2)}(x) \right| \, dx < \infty
$$

i.e. $g^{(k+1)}(x) \in BV_{L}(b + 1, \infty)$. Since $g^{(k+1)}(x) \in BV_{L}(b + 1, \infty)$, therefore $g^{(k+1)}(x) \in BV_{L}(b, \infty)$. The desired result now follows by induction.

3. Tauberian theorems for strong Borel-type summability with index $p \geq 1$.

We first show that the scale in Theorem A(ii) is proper. In [5] we showed that there is a sequence $\{ s_{n} \}$ which tends to a limit $(B, a, \beta - 1)$. Hence, in view of Lemma 2, there is a sequence $\{ s_{n} \}$ which tends to a limit $[B, a, \beta - 1]$, for every $p > 0$ but does not tend to a limit $[B, a, \beta - 1]$, for any $p \geq 1$. 

Theorem 1. Let \( p, r > 1 \). If \( s_n \to s(B, \alpha, \mu) \) and \( a_n \to 0(B, \alpha, \beta) \), then \( s_n \to s(B, \alpha, \beta) \).

Proof. By Lemma 2(i), \( s_n \to s(B, \alpha, \mu) \). The result now follows by [9, Theorem 3] and the note following [9, Theorem 3].

Theorem 2. Let \( p \geq 1 \). If \( s_n \to s(B, \alpha, \beta + \epsilon)_p \) for some \( \epsilon > 0 \) and \( s_n = 0(1)(B, \alpha, \beta) \), then \( s_n \to s(B, \alpha, \beta + \delta) \) for every \( \delta > 0 \).

Proof. We can suppose without loss of generality that \( x = 0 \). Then \( s_n \to 0(B, \alpha, \beta + \delta) \) and \( s_n = 0(1)(B, \alpha, \beta) \) by Lemma 2(ii) and Lemma 6(ii). Hence \( s_n \to 0(B, \alpha, \beta + \delta) \) by [5, Theorem 2] for \( \delta > 0 \). Also \( s_n = 0(1)(B, \alpha, \beta + \delta) \) by Lemma 6(ii) or (iii). Therefore, letting \( f(x) = \alpha_{n, B, 2}(x) \), we have that

\[
\epsilon^{-r} \int_0^1 f(t) dt = s_{n, B, 2}(x) = o(1)
\]

and

\[
\epsilon^{-r} \int_0^1 e^{i \epsilon (\frac{1}{2} - \phi)} |f(t)|^p dt = \epsilon^{-r} \int_0^1 e^{i \epsilon (S_{n, B, 2}(t))} dt = o(1)
\]

using Lemma 4(i), and consequently,

\[
\epsilon^{-r} \int_0^1 e^{i \epsilon (S_{n, B, 2}(t))} dt = \epsilon^{-r} \int_0^1 e^{i \epsilon (\frac{1}{2} - \phi)} |f(t)|^p dt = o(1)
\]

using Lemma 4(i) and Lemma 8, i.e. \( s_n \to 0(B, \alpha, \beta + 2\delta) \). This establishes the desired result.

Theorem 2*. Let \( p \geq 1 \). If \( \sum a_n = s(B, \alpha, \beta + \epsilon)_p \) for some \( \epsilon > 0 \) and \( a_n = 0(1)(B, \alpha, \beta) \), then \( a_n \to 0(B, \alpha, \beta + \delta) \) for every \( \delta > 0 \). The result now follows by Theorem 1.

A real-valued function \( g(x) \) with domain \([0,\infty)\) is slowly decreasing if for every \( \epsilon > 0 \) there exist positive numbers \( X, \delta \) such that \( g(x) - g(y) > \epsilon \) whenever \( x > y \geq X \) and \( x - y < \delta \). The following result is [5, Theorem 3]: If \( s_n \to s(B, \alpha, \beta + \epsilon) \) for some \( \epsilon > 0 \) and \( S_{n, B}(x) \) is slowly decreasing, then \( s_n \to s(B, \alpha, \beta) \). We now show that there is no analogue to this result for the \( \{B, \alpha, \beta\} \) method.

Let \( s_n \) be the sequence defined by \( \sum s_n(x) = s \sin e^t \) (cf. [7, p. 183]). Then \( S_{n, B}(x) = s \sin e^t \) where \( e^t \) is the natural exponential function. Thus, using Lemma 4(i),

\[
S_{n, B}(x) = \epsilon^{-r} \int_0^1 e^{i \epsilon \sin e^t} dt = \epsilon^{-r} \int_0^1 e^{i \epsilon \cos 1 - \cos e^t} = o(1)
\]

for every \( \epsilon > 0 \).

Proof. By Lemma 2(ii), \( s_n \to s(B, \alpha, \mu) \). Hence, by [5, Theorem 5, 6*, 6**, or 7], \( s_n \to s(B, \alpha, \beta - 1) \). The result now follows by Lemma 2(ii).

4. Tauberian theorems for absolute Borel-type summability. We first show that the scale in Theorem (iii) is proper in the sense that for each \( \beta \) there is a sequence \( \{a_n\} \) which is summable \( \{B, \alpha, \beta\} \) but is not summable \( \{B, \alpha, \beta - 1\} \).

Choose an integer \( m \) such that \( am > 1 \) and let \( P \) be the smallest integer such that \( m^p > N \). Let

\[
\epsilon x^p \sin e^t = \sum_{n=0}^{\infty} a_n s_{p,n}
\]

and let

\[
s_n = \begin{cases} \Gamma(\alpha + \beta) b_n & \text{if } n = mk, \\ 0 & \text{otherwise.} \end{cases}
\]

Then

\[
S_{n, B}(x) = a \epsilon^{m^p - 1} \epsilon^{-r} e^{-\epsilon \sin e^t} = o(1)
\]
and
\[
S_{\alpha}(x) = \alpha (am \beta + \beta - 1) e^{am \beta + \beta} - e^{-x^{m \beta + \beta}} - a \alpha e^{am \beta + \beta} + a \alpha e^{am \beta + \beta} e^{-x^m} + \alpha e^{am \beta + \beta} e^{-x^m} e^{-x^m} \sin e^{-x^{m}}
\]
\[+ \alpha e^{am \beta + \beta} e^{-x^m} \cos e^{-x^{m}}
\]
so that \(S_{\alpha}(x) = 0\) and \(S_{\alpha}(x) \in L(0, \infty)\) since \(am \beta + \beta - 2 \geq 0\) and \(N + \beta - 2 \geq 0\).

where \(f(x) \in L(0, \infty)\) and therefore \(S_{\alpha}(x) \in L(0, \infty)\) since \(a \alpha > 1\). Thus, since
\[
S_{\alpha}(x) = S_{\alpha}(x) + \sum_{j=1}^{\infty} S_{\alpha}(x)
\]
and
\[
S_{\alpha}(x) = \sum_{j=1}^{\infty} S_{\alpha}(x)
\]
we have that
\[
x_0 \rightarrow 0 [B, a, \beta - 1] \quad \text{but} \quad x_0 \nrightarrow 0 [B, a, \beta - 1].
\]

**Theorem 4.** If \(x_0 \rightarrow s [B, a, \mu] \) and \(a_0 \rightarrow \beta [B, a, \beta] \), then \(x_0 \rightarrow s [B, a, \beta] \).

**Proof.** By [5, Theorem 1], \(x_0 \rightarrow s [B, a, \beta] \). Thus it remains only to show that \(S_{\alpha}(x) \in BV(0, \infty) \). Let \(k \) be a positive integer. Then, in view of Theorem A(iii), \(A_{\alpha}(x) \) is \(BV(0, \infty) \). Moreover, by Lemma 4(ii),
\[
S_{\alpha}(x) = A_{\alpha}(x) + \sum_{j=1}^{\infty} S_{\alpha}(x) + \alpha \sum_{j=1}^{\infty} \frac{e^{am \beta + \beta} - e^{-x^m}}{e^{am \beta + \beta + \beta} - e^{-x^m}}
\]
Therefore \(S_{\alpha}(x) \in BV(0, \infty) \) if \(S_{\alpha}(x) \in BV(0, \infty) \). Since, in view of Theorem A(iii), \(S_{\alpha}(x) \in BV(0, \infty) \) when \(\beta + k > 0 \), it readily follows that \(S_{\alpha}(x) \in BV(0, \infty) \).

If \(x_0 \) is the sequence described in the paragraph preceding Theorem 3, then, using Lemma 4(i),
\[
S_{\alpha}(x) = e^{-x^{m}} \int_{x_0}^{x} (\cos 1 - \cos e^{-x}) \, dt
\]
and thus it is readily seen that \(x_0 \rightarrow 0 [B, 1, 3] \) and \(x_0 \nrightarrow 0 [B, 1, 2] \). Hence there is also no immediate absolute summability analogue to [5, Theorem 3].

Our final results are extensions of a result due to Gaier (see [6]).

**Theorem 5.** If \(x_0 \rightarrow s [B, a, \mu] \) and \(S_{\alpha}(x) \) is of exponential type in \(H_2 \) for some \(B > 0 \), then \(x_0 \rightarrow s [B, a, \beta] \).

**Proof.** Let \(k \) be a positive integer such that \(\mu - k = \beta \). By [5, Theorem 6] we have that \(x_0 \rightarrow s [B, a, \mu - k] \). Furthermore, since \(S_{\alpha}(x) = S_{\alpha}(x) + S_{\alpha}(x) \), it is readily seen that
\[
S_{\alpha}(x) = S_{\alpha}(x) + \sum_{j=1}^{\infty} S_{\alpha}(x)
\]
Since \(S_{\alpha}(x) \) is of exponential type in \(H_2 \) and since \(S_{\alpha}(x) \in BV(0, \infty) \) by hypothesis, we have, by Lemma 10, that \(S_{\alpha}(x) = BV(0, \infty) \) for \(j = 1, \ldots, k \); also, since we choose \(N \) so that \(\alpha \beta + \alpha k - k > 1 \), we have that \(S_{\alpha}(x) = BV(0, \infty) \) for \(j = 1, \ldots, k \). Therefore, \(S_{\alpha}(x) = BV(0, \infty) \) for \(j = 1, \ldots, k \) and, consequently, \(S_{\alpha}(x) \in BV(0, \infty) \). Hence \(x_0 \rightarrow s [B, a, \beta] \). By Theorem A(iii), \(x_0 \rightarrow s [B, a, \beta] \).

**Theorem 6.** If \(x_0 \rightarrow s [B, a, \mu] \) and \(S_{\alpha}(x) \) is of exponential type in \(H_2 \) for some \(B > 0 \), then \(x_0 \rightarrow s [B, a, \beta] \).

**Proof.** By Lemma 1(ii), \(a_0 \rightarrow \beta [B, a, \beta] \) and thus, by Theorem 5, \(a_0 \rightarrow \beta [B, a, \beta] \). The result now follows by Theorem 4.

**Theorem 7.** If \(x_0 \rightarrow s [B, a, \mu] \) and \(S_{\alpha}(x) \) is of exponential type in \(H_2 \) for some \(B > 0 \), then \(x_0 \rightarrow s [B, a, \beta] \).

**Proof.** Since \(S_{\alpha}(x) \) is of exponential type in \(K \) for all \(x > 0 \) where \(K \) is a positive constant, then \(x_0 \rightarrow s [B, a, \beta] \).

**Proof.** Since \(S_{\alpha}(x) \) is of exponential type in \(K \) for all \(x > 0 \) where \(K \) is a positive constant, then \(x_0 \rightarrow s [B, a, \beta] \).

**References**

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D. BORWEIN AND E. SMET


DEPARTMENT OF MATHEMATICS,
THE UNIVERSITY OF WESTERN ONTARIO,
LONDON, ONTARIO,
CANADA N6A 5B9