ON CERTAIN SEQUENCES OF PLUS AND MINUS ONES
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1. Suppose throughout that \( c \) is a fixed positive integer, that
\[
a = 1 - c + \sqrt{c^2 + 1},
\]
and that
\[
\sigma_n = (-1)^{\lfloor x \rfloor}, \quad S_n = \sum_{k=1}^{n} \sigma_k, \quad T_n = \sum_{k=1}^{n} S_k \quad \text{for} \quad n = 1, 2, \ldots,
\]
where \([x]\) is defined to be the largest integer not exceeding \( x \). The following expansions of \( a \) and \( a/2 \) as simple continued fractions are easily verified:
\[
a = \{1, 2a, 2a, \ldots \} \quad \text{if} \quad c = 1
\]
\[
a/2 = \left\{0, 1, 2, 2, \ldots \right\} \quad \text{if} \quad c > 1.
\]
In a recent issue of the American Mathematical Monthly [86, 1976, No. 7, p. 573] H. Rudman posed the problem of proving the convergence of the series \( \sum_{n=1}^{\infty} \sigma_n/n \) in the special case \( a = \sqrt{2} \), and asked for an estimate of its sum. To prove convergence we note that, by Abel's partial summation formula,
\[
\sum_{n=1}^{\infty} \frac{\sigma_n}{n^2} = \sum_{n=1}^{\infty} \frac{S_n}{n^2} \sum_{k=1}^{n} \frac{1}{k} + \sum_{n=1}^{\infty} \frac{S_n}{n^2} \sum_{k=n+1}^{\infty} \frac{1}{k}.
\]
Furthermore we have \( S_n = 2a - 1, \) where \( a \) is the number of positive integers \( k \leq n \) for which \([ak]\) is even, or, equivalently, for which the fractional part of \( ak/2 \) is in the interval \((0, 1/2)\). The familiar result that the sequence \( (\sigma_n/2)^2 \) is uniformly distributed modulo 1 when \( a \) is irrational, yields only that \( \sigma_n/n \rightarrow 1/2 \) as \( n \rightarrow \infty \), and hence that \( S_n = o(n) \); but this is insufficient to establish the convergence of \( \sum_{n=1}^{\infty} \sigma_n/n \). A better estimate of \( S_n \) is obtained, however, from a known result on the discreteness of the sequence \( (\sigma_n/2)^2 \) [3, Theorem 3.4, p. 125] which yields
\[
S_n = 2a - 1, \quad \frac{S_n - 1}{n^2} \leq 6 + 2M_n, \quad n \geq 1.
\]

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Let
\[ n_k = \frac{1}{2}(p_k - 1), \quad k \geq 0. \]
It is easily verified that \( n_k \) is an integer, that
\[ a_\beta = 2\alpha, \quad -1/2 < 1 - \alpha < 0, \quad \alpha = (-1)^{n_k} \]
and that, for \( k \geq 0 \),
\[ 2p_k \sqrt{d} = a(1 + \beta)^{2k+1} + \beta(1 - \alpha)^{2k+1}, \]
\[ 2p_k \sqrt{d} = (1 + \beta)^{2k+1} - (1 - \alpha)^{2k+1}, \]
\[ p_k - a_\beta = (1 - \alpha)^{2k+1}, \quad n_k + \alpha = n_{k+1}, \quad n_k + p_{k+1} = n_{k+2}. \]

The first lemma is concerned with some basic identities involving the sequences \((s_\gamma), (S_\gamma)\) and \((T_\gamma)\).

**Lemma 1.** The following identities hold for \( k \geq 1 \).

(a) \( s_{2k} = (-1)^{2k+1} \).
(b) \( s_{2k+1} = (-1)^{2k+2} \) if \( 1 \leq j \leq 2k \).
(c) \( r_{2k} = (-1)^{2k+1} \) if \( 1 \leq j \leq 2k+1 \).
(d) \( r_{2k+1} = (-1)^{2k+2} \) if \( j = 2k+1, \quad 1 \leq i \leq 1 + r \).
(e) \( s_{k+1} = r_k = 0 \) if \( 1 \leq j \leq 2k+1 \).
(f) \( s_{k+1} = r_k = 0 \) if \( 1 \leq j \leq 2k \).
(g) \( r_{2k+1} = (-1)^{2k+1} \) if \( j = 2k+1, \quad 1 \leq i \leq 1 + r \).
(h) \( S_{2k+1} = S_{k+1} = -k \), \( S_{2k+2} = k \).

(i) For \( i + j = 1, \quad 0 \leq i, j \leq k \), \( S_{k+i} = 0 \) if \( k \) is even,
\[ 1 \] otherwise.

(j) For \( 1 \leq r \leq 2k+1 \), \( S_{k+i} = \begin{cases} 1 & \text{if } k \text{ is even, } r \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \)

(k) For \( j = r_k, \quad 0 \leq i < q_k, \quad 1 \leq r \leq 2k \), \( S_{k+i} = 1 \), \( S_{k+i} = -k \) if \( k \) is even and \( r \) is odd.

(l) For \( j = r_k, \quad 0 \leq i < q_k, \quad 1 \leq r \leq 2k \), \( S_{k+i} = 1 \), \( S_{k+i} = -k \) if \( k \) is even and \( r \) is odd.

(m) For \( j = r_k, \quad 0 \leq i < q_k, \quad 1 \leq r \leq 2k \), \( T_{k+i} = T_{k+i} = 0 \) if \( k \) is odd and \( r \) is odd.

(n) For \( 1 \leq r \leq 2k+1 \), \( T_{k+i} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ -r_{k+i} / 2 & \text{if } r \text{ is odd} \end{cases} \)
\[ (r - 1)q_{k+i} / 2 - 1 \text{ if } r \text{ is odd} \]
\[ k \text{ is odd} \]

\[ k \text{ is even} \]

**Proof.** (a) We have \( \beta \beta = 2q_\beta + (1 - \alpha)^{2k+1} \) and thus, since \( 0 < \beta(1 - \alpha)^{2k+1} < 1 \) and \( -1 < 1 - \alpha < 0 \),
\[ [\beta] \beta = \beta 2q_\beta, \quad \text{if } \beta \text{ is odd} \]
and this implies that \( e_{2k} = (-1)^{n_k+1} = (-1)^{2k+1} \).
(b) For \( 1 \leq j \leq 2k \), we have \( [\alpha] = [j/\sqrt{d}] + j \) which implies that \( e_j = (-1)^{j-1} \).
(c) Starting from the identity \( \alpha p_{2k} = p_{2k} - j(1 - \alpha)^{2k+1} \) we get, since \( 0 < j(1 - \alpha)^{2k+1} \leq (2k+1)/\sqrt{d} < 1 \), that
\[ \alpha p_{2k} = \begin{cases} 1 \quad \text{if } k \text{ is odd} \\ 0 \quad \text{if } k \text{ is even} \end{cases} \]
Further \( p_{2k} \) is odd and hence \( e_{2k} = (-1)^{2k+1} \).
(d) For \( j = p_k + 1, \quad 1 \leq p_k < q_k \), we have
\[ \beta = \alpha p_k + \beta(1 - a)^{2k+1} + a^2 + 2(1 - a) \]
\[ = 2q_k + 2(1 - a)^{2k+1} + a + \delta, \quad a = [\alpha] \]
Since \( i < q_k \), it follows that \( \delta = q_k - a \geq [\alpha] - p_k - [\alpha] = (a - 1)^k \) by standard theory. (See e.g. p.167, Theorem 7.1S.) Further \( 0 < \beta(1 - a) < 2\sqrt{d}/(1 - \alpha) < 1 \) and so \( \delta > (\alpha - 1)^{2k+1} \). Likewise we obtain \( 1 - \delta = 1 + a - a > (\alpha - 1)^{2k+1} \) and so
\[ [\delta] = 2q_k + 2(1 - a)^{2k+1} + [\alpha] \]
from which it follows that \( e_k = e_k \).
(e) Let \( i + j = p_k, \quad 1 \leq i < p_k/2 \). Since \( 1 < \alpha < 3/2 \), we have
\[ \frac{p_k - 1}{2} \leq \frac{p_k + 1}{2} \leq q_k \leq q_k - (1 - \alpha)^{2k+1} + a - 1 \]
and by the same argument as in the proof of (d) it follows that \( [\beta] = 2q_k - 2(1 - a)^{2k+1} + 1 - [\alpha] \) and thus that \( e_k = e_k \).
(f) For \( j = q_k - 1, \quad 1 \leq i < q_k \), we have \( a_\gamma = p_k - (1 - a)^{2k+1} + a - 1 \), where \( a = [\alpha] \) and as above \( \delta > (\alpha - 1)^{2k+1} \) and \( 1 - \delta > (\alpha - 1)^{2k+1} \). Hence \( [\alpha] = p_k - 1 - [\alpha] \), and so, since \( p_k - 1 \) is even, \( e_k = e_k \).
(g) For \( j = p_k + 1, \quad 1 \leq i < q_k, \quad 1 \leq r \leq 2k \), we have
\[ a_\gamma = p_k + r(1 - a)^{2k+1} + a + 1 \]
where \( a = [\alpha] \) and as above \( \delta \geq (\alpha - 1)^{2k+1} + r(1 - a)^{2k+1} \) and \( 1 - \delta > (\alpha - 1)^{2k+1} \). Thus \( [\alpha] = p_k + [\alpha] \) and so \( e_k = (-1)^{n_k+1} \) since \( p_k \) has the same parity as \( r \).
(1) By (c) we have, for \( i + 1 + j = p_n, \ 0 \leq i < j, \) that
\[
S_j = S_i + \sum_{k=i+1}^{j} x_k - S_n,
\]
since \( j - 1 \) is even.

(1) By (b), \( S_{n+1} = 0 \) and so, by (a), \( S_k = s_k = (-1)^{k+1}. \) Next, since \( q_i + q_{i+1} = s_{i+1} \) and \( q_i < q_{i+1}, \) we have, by (d), that
\[
S_{n+i} - S_{n+i+1} = \sum_{k=i+1}^{n+i} x_k = S_n
\]
and so \( S_{n+i} - S_n = S_{n+i+1} = (-1)^i \) for \( i \geq 0. \) Hence
\[
S_{n+i} = \sum_{k=1}^{i} (S_{n+k} - S_{n+k-1}) = k
\]
and
\[
S_{n+i+1} = \sum_{k=1}^{i+1} (S_{n+k} - S_{n+k-1}) + S_n = -(-k - 1) - 1 = -k
\]
since \( n_i = 1 \) and \( S_1 = -1. \)

(1) By (i) we have, for \( i + 1 + j = q_k, \ 0 \leq i \leq j, \) that \( S_i = S_k - S_j; \) and hence that \( S_i + S_j = S_k - S_j - S_k + (-1)^j \) by (c). Next, since \( q_i + q_{i+1} = n_{i+1} \) and \( q_i < q_{i+1} \) it follows, by (g) with \( r - 1, \) that \( S_{n+i} - S_{n+i+1} = -S_n \) and so, by (i),
\[
S_i = S_k + S_{n+i+1} = 0 \text{ when } k \text{ is odd};
\]
Hence
\[
S_i + (-1)^j = 0 \text{ when } k \text{ is even}\]
and this completes the proof.

(1) Let \( j = r_n + i, \ 0 \leq i < q_r, \ 1 \leq r \leq 2r. \) Applying (a) we get \( S_j - S_{r_n} = (-1)^{r+1} \) for \( i < q_r \) and hence
\[
S_k = S_{r_n+i+1} = (-1)^{r+i} S_{r_n+i+1} = (-1)^{r+i} S_{k+i} + (-1)^r q_r
\]
\[
= (-1)^{r+i} S_{k+i} + (-1)^r q_r, \text{ by (c)}.
\]
Consequently
\[
S_{r_n+i} = \sum_{k=1}^{i} (S_{r_n+k} - S_k) + S_k = S_k \sum_{k=1}^{i} (-1)^k + S_k
\]
\[\begin{cases}
10 & \text{when } r \text{ is odd}, \\
1 & \text{when } r \text{ is even}.
\end{cases}\]
Our next lemma shows that $n_{k-1}$ is the first value of $n$ for which $S_n$ attains the value $-k$.

**Lemma 2.** If $k \geq 1$ and $n < n_{k-1}$, then $|S_n| < k$.

**Proof.** We proceed by induction with respect to $k$. The proposition that $|S_1| < k - 1$ for $n < n_{k-1}$ holds for $k = 2$. Assume it to be true for a given $k \geq 2$. Suppose first that $k \geq 3$. We proceed from the induction hypothesis as follows. Since $q_{n-1} \leq n_{k-1}$, we have, for $j = r_{q_{n-1}} + i$, $0 \leq i < q_{n-1}$, $1 \leq r \leq 2n$, by (0), that

\[ S_j = \begin{cases} S_i & \text{if $r$ is even} \\ -S_i - 1 & \text{if $r$ is odd} \end{cases} \]

Also, by (0), $S_i = -1$ for $i = (2c + 1)q_{n-1}$. Thus

\[ -k < S_i < k - 1 \quad \text{for} \quad i \leq (2c + 1)q_{n-1}. \]

Further, $q_{n-1} < (2c + 1)q_{n-1}$ and $n_{k-1} - q_{n-1} = n_{k-2}$ and so, by (1), we have, for $j = q_{n-1} + i$, $0 \leq i < n_{k-1}$, that

\[ |S_j| = |S_i| < k - 1, \]

since $n_{k-1} < q_{n-1}$. Therefore

\[ -k < S_i < k - 1 \quad \text{for} \quad i < n_{k-1}. \]

But, by (b), the final inequalities also hold for $k = 2$, since $n_1 = 2c + 1$. In what follows we suppose $k \geq 2$. By (1) again, we have, for $j = r_{q_{n-1}} + i$, $0 \leq i < n_{k-1}$, that $S_j = S_i$ and so, since $q_{n-1} < n_{k-1}$,

\[ -k < S_i < k - 1 \quad \text{for} \quad i < (2c + 1)q_{n-1}. \]

Finally, the relations $q_{n-1} < (2c + 1)q_{n-1}$ and $n_{k-1} - q_{n-1} = n_{k-2}$ imply, by (1), that

\[ -S_i = S_i + 1 \quad \text{for} \quad j = q_{n-1} + i, 0 \leq i < n_{k-1}. \]

Hence

\[ -k < S_i < k - 1 \quad \text{for} \quad q_{n-1} < j < n_{k-1}. \]

Since $q_{n-1} < (2c + 1)q_{n-1}$, we have established that

\[ |S_j| < k \quad \text{for} \quad i < n_{k-1}. \]

This completes the proof.

Similar considerations show that $n_{k-1}$ is the first value of $n$ for which $S_n$ attains the value $k$.

**Theorem 1.** If $n \geq 1$, then

\[ |S_n| < \frac{2}{\log (1 + \beta)} \frac{(2cn + 1)\sqrt{d}}{2}. \]
Also,

\[ T_j = T_{j\alpha} - T_i - i \geq \frac{(r - 1)\alpha}{2} - 1 + \frac{i + 1}{2} = \frac{i}{2} \]

Case 3: \( r \) and \( \alpha \) are odd. Then

\[ T_j + j/2 = T_{j\alpha} - T_i + j/2 > -q_j/2 + r_q/2 \geq 0 \]

and

\[ T_i = T_{j\alpha} - T_i \leq -q_j/2 + (i + 1)/2 \leq -q_j/2 + q_j/2 = 0. \]

Hence we have in every case that \( 0 \geq -T_j \leq (j + 1)/2 \) for \( j < (2\alpha + 1)q_\alpha \).

Since \( q_\alpha < (2\alpha + 1)q_\alpha \), the proof is complete.

It follows from Lemma 1, (k), (m), and (n) that if \( n = 2q_\alpha + 1 \), then \( T_n = T_{2q_\alpha + 1} = -q_\alpha - 1 = -(\alpha + 1)/2 \), whereas if \( n = 2q_\alpha - 1 \), then \( T_n = T_{2q_\alpha - 1} = T_{n - 1} - 1 = 0 \). This shows that the inequalities in Theorem 2 are sharp.

3. In this section we show how the preceding estimates can be used to determine the sum \( \sigma \) of the series \( \sum_{k=1}^{\infty} \epsilon_k/n \). In addition we contrast the behaviour of the series \( \sum_{k=1}^{\infty} \epsilon_k \) with that of \( \sum_{k=1}^{\infty} (-1)^k \) with regard to summability by certain standard methods.

The problem of estimating the sum of the series \( \sum_{k=1}^{\infty} \epsilon_k/n \) reduces to knowing how close its \( n \)-th partial sum \( \sigma_n \) is to \( \sigma \). Applications of Abel's partial summation formula yield

\[ \sigma = \sigma_n + \frac{S_n}{n + 1} + \sum_{k=1}^{n} \frac{T_k}{k(k + 1)} = \sigma_n, \]

say, and

\[ \sigma = \sigma_n + \frac{S_n}{n + 1} + \frac{T_n}{n(n + 1)} = 2 \sum_{k=1}^{n} \frac{T_k}{k(k + 1)(k + 2)} = \tau_n \]

say. It follows from (2) that

\[ |\sigma_n| < \frac{6 + 2M_f(1 + \log n)}{n}, \]

and from (4) that

\[ 0 < \tau_n < \frac{1}{2(n + 1) + 1}. \]

Consider now the special case \( \sigma = \sqrt{2} \) (i.e., \( \alpha = 1 \)). We find that \( M_f < 3.9 \). For \( \alpha = q_\alpha = 1.0994428 \), we have, by Lemma 1 (k) and (m), that \( S_n = 0 \) and \( T_n = -n/2 \); and a computer yielded \( \sigma_n = -0.5154184531 \). Using the above

estimate for \( \sigma_n \) we get

\[ -0.515428 < \sigma < -0.515409, \]

and using the estimate for \( \tau_n \) we get

\[ -0.5154186 < \sigma < -0.5154184. \]

It is familiar that the series \( \sum_{k=1}^{\infty} (-1)^k \) is summable to \(-1/2\) by the Cesàro method \( C_1 \) and consequently by the Abel method \( A \). It is also summable to \(-1/2\) by the Borel method \( B \). We shall show, on the other hand, that the series \( \sum_{k=1}^{\infty} \epsilon_k \) is not summable by any of the above standard methods. Let \( U_n = \sum_{k=1}^{n} T_k \). Then, by Lemma 1 (m), we have \( T_j = T_{j\alpha} + T_i \) for \( j = 2q_j + i \), \( 1 \leq i < q_j \), and so

\[ U_{2q_j + i} - U_{2q_j} - U_{q_j - 1} = (q_j - 1)T_{2q_j}. \]

It follows, by Lemma 1 (n), that

\[ U_{2q_j} - U_{2q_j - 1} = \frac{1}{n} \quad \text{when} \quad n = q_j - 1 \]

and

\[ U_{2q_j + i} - U_{2q_j} - U_{q_j - 1} = \frac{1}{n} \quad \text{when} \quad n = q_j. \]

If we now suppose that \( U_n/n \) tends to a finite limit \( l \) as \( n \to \infty \), we get the contradictory conclusions that \( U_{2q_j} - U_{2q_j - 1} = 0 \) and \( U_{2q_j + i} - U_{2q_j} - U_{q_j - 1} = -1 \). Hence the sequence \( (U_n/n) \) is not convergent and, equivalently, the sequence \( (T_j/n) \) is not limitable \( C_1 \). Now it is known (see e.g. (1, p. 214)) that if \( \sum_{k=1}^{\infty} \epsilon_k \) is summable \( A \), then \( (T_j/n) \) is limitable \( A \) and hence, by a familiar tauberian theorem, that \( (T_j/n) \) is limitable \( C_1 \) since \( T_j/n \leq 0 \) (2, p. 154, Theorem 95). Thus \( \sum_{k=1}^{\infty} \epsilon_k \) is not summable \( A \) and, a fortiori, not summable \( C_1 \). Another familiar tauberian theorem (2, p. 210, Theorem 147) now shows that the series in question cannot be summable \( B \) for if it were, the order relation \( \epsilon_n = O(1) \) would imply it to be summable \( C_1 \) for every \( \sigma > 1 \).

References


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