THE HAUSDORFF MOMENT PROBLEM

BY

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1. Introduction. Suppose throughout that

\[ 0 \leq \lambda_0 < \cdots < \lambda_n, \quad \lambda_n \to \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty, \]

and that \( \{\mu_n\} (n \geq 0) \) is a sequence of real numbers. The (generalized) Hausdorff moment problem is to determine necessary and sufficient conditions for there to be a function \( x \) in some specified class satisfying

\[ \mu_n = \int_0^1 t^{\lambda_n} \, dx(t) \quad \text{for} \quad n = 0, 1, 2, \ldots. \]

Let

\[ D_0 = \beta_0 = 1, \quad D_n = \frac{(1 + \lambda_n) d_n}{(1 + \lambda_0) d_0} = \left( 1 + \frac{1}{\lambda_1} \right) \cdots \left( 1 + \frac{1}{\lambda_n} \right). \]

Define the divided difference \([\mu_k, \ldots, \mu_n]\) inductively by \([\mu_k] = \mu_k\),

\[ [\mu_k, \ldots, \mu_n] = \frac{[\mu_k, \ldots, \mu_{n-1}] - [\mu_{k+1}, \ldots, \mu_n]}{\lambda_n - \lambda_k} \quad \text{for} \quad 0 \leq k < n. \]

For \( 0 \leq k \leq n, \ 0 \leq t \leq 1 \), let

\[ \lambda_{nk} = \lambda_{k+1} \cdots \lambda_n [\mu_k, \ldots, \mu_n], \]

\[ \lambda_{nk}(t) = \lambda_{k+1} \cdots \lambda_n [t^{\lambda_k}, \ldots, t^{\lambda_n}] \]

with the convention that products such as \( \lambda_{k+1} \cdots \lambda_n = 1 \) when \( k = n \). Let

\[ M_{p,n} = \begin{cases} \left( \sum_{k=0}^{n} \frac{|\lambda_{nk}|^p D_n}{d_k} \right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max_{0 \leq k \leq n} |\lambda_{nk}| \frac{D_n}{d_k} & \text{if} \quad p = \infty, \end{cases} \]

\[ M_p = \sup_{n \geq 0} M_{p,n}. \]

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2. Preliminary results. The following simple identities and inequalities are known.

\begin{align*}
(3) \quad & \mu_n = \sum_{k=0}^n \lambda_k \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k-1}} \right) \ldots \left( 1 - \frac{1}{\lambda_1} \right) \quad \text{for } 0 \leq s \leq n, \quad [6, (5)] \\
(4) \quad & 0 \leq \lambda_n(t) = \sum_{k=0}^n \lambda_k(t) \leq 1 \quad \text{for } 0 \leq t \leq 1, \quad 0 \leq s \leq n, \quad [10, Lemma 1] \\
(5) \quad & \int_0^1 \lambda_n(t) \, dt = \frac{D_n}{D_s} \quad \text{for } 0 \leq k \leq n. \quad [6, p. 294]
\end{align*}

We require some lemmas.

**Lemma 1.** If \( M_{k} = \infty \), then

\[ \mu_n = \lim_{s \to \infty} \sum_{k=0}^n \lambda_k \left( \frac{D_n}{D_k} \right)^s \quad \text{for } s = 0, 1, 2, \ldots. \]

**Proof.** Let \( \lambda > 0 \), \( \mu_k = e^{\lambda u_k} \),

\[ \phi_n(\lambda) = \sum_{k=0}^n \lambda_k u_{k+1} \ldots u_n, \]

and let

\[ \phi_n = \sum_{k=0}^n \lambda_k u_{k+1} \ldots u_n, \]

where \( u_k = e^{\lambda u_k} \) for sufficiently large \( n \) and \( \gamma_k \) as \( n \to \infty \).

Let \( 0 < \leq \lambda \). Then, for \( \delta > 0 \), \( |\gamma - \lambda| < \delta \), we have that

\[ |e^{-s} - e^{-\lambda |t|}| \leq \delta |\gamma - \lambda| e^{-s|\lambda - \gamma|} \leq \frac{\delta}{\lambda - \gamma}. \]

Choose a positive integer \( N \) so large that \( |\gamma_n - \lambda| < \epsilon \) for \( n > N \). Then, for \( n > N \), we have that

\[ |\phi_n - \phi_n(\lambda)| \leq M_1 \sum_{k=0}^n |u_{k+1} \ldots u_{n-k-1}| + \frac{\epsilon}{\lambda - \gamma} \sum_{k=0}^n |\lambda_k|. \]

Since \( u_n \to 0 \) and \( u_0 \to 0 \) as \( n \to \infty \), it follows that

\[ \lim_{n \to \infty} |\phi_n - \phi_n(\lambda)| \leq M_1 e^{\frac{\epsilon}{\lambda - \gamma}}, \]

and hence that

\[ \lim_{n \to \infty} (\phi_n - \phi_n(\lambda)) = 0. \]
Note that when \( v_n = 1 - \lambda_n \lambda_n \), then, by (3), the corresponding \( \phi_n = \mu_n \) for \( n \geq 1 \).

Thus
\[
\lim_{n \to \infty} \phi_n(\lambda_n) = \mu_n.
\]

The desired conclusion is now obtained by considering the \( \phi_n \) corresponding to
\[
v_n = \left( 1 + \frac{1}{\lambda_n^2} \right)^{-k}.
\]

**Lemma 2.**

(i) If (1) is satisfied by a function \( \alpha \in BV \), then \( M_k \leq \| \beta \| \).

(ii) If \( 1 < p < \infty \) and (2) is satisfied by a function \( \beta \in L_p \), then \( M_k \leq \| \beta \| \).

**Proof.** Part (i). We have that
\[
\lambda_n = \int_0^1 \lambda_n(t) \, dx(t) \quad \text{for} \quad 0 \leq k \leq n,
\]
and thus, by (4),
\[
\sum_{k=0}^n \lambda_n \leq \int_0^1 |dx(t)| \sum_{k=0}^n \lambda_n(t) \leq \int_0^1 |dx(t)|.
\]
Hence
\[
M_k \leq \int_0^1 |dx(t)|.
\]

Part (ii). We now have that
\[
\lambda_n = \int_0^1 \lambda_n(t) \, \beta(t) \, dt \quad \text{for} \quad 0 \leq k \leq n.
\]

Hence, by (5),
\[
|\lambda_n| \leq \int_0^1 |\lambda_n(t)| |\beta(t)| \, dt \leq \frac{d_k}{D_n} \sup_{0 \leq t \leq 1} |\beta(t)|.
\]

Next, if \( 1 < p < \infty \), then, by Hölder's inequality and (5),
\[
|\lambda_n|^p \leq \int_0^1 |\lambda_n(t)| |\beta(t)|^p \, dt \left( \int_0^1 \lambda_n(t) \, dt \right)^{p-1}
\]
\[
= \left( \frac{d_k}{D_n} \right)^{p-1} \int_0^1 \lambda_n(t) \, |\beta(t)|^p \, dt;
\]
and so, by (4),
\[
\sum_{k=0}^n \lambda_n \left( \frac{d_k}{D_n} \right)^{p-1} \leq \int_0^1 |\beta(t)|^p \, dt \sum_{k=0}^n \lambda_n(t) \leq \int_0^1 |\beta(t)|^p \, dt.
\]

Consequently, if \( 1 < p < \infty \), then \( M_k \leq \| \beta \| \).

**The Hausdorff Moment Problem**

**Lemma 3.** If a normalized function \( x(t) \in BV \) is such that
\[
\int_0^1 t^r \, dx(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,
\]
then \( x(t) = x(0^+) \) for \( 0 < t \leq 1 \). If, in addition, \( \lambda_0 = 0 \), then \( x(0^+) = 0 \).

**Proof.** Suppose first that \( \lambda_0 = 0 \). A known consequence of the hypothesis [11, p. 337] is that
\[
\int_0^1 t^r \, dx(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]
Hence, by a standard result [14, Theorem 6.1], \( x(t) = 0 \) for \( 0 < t \leq 1 \).

Suppose next that \( \lambda_0 > 0 \). Then, by hypothesis,
\[
\int_0^1 t^r \, dx(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,
\]
where \( y(t) = \sum u^n \, dx(u) \). Since \( y \) is normalized [14, Theorem 8b], we have, by the part already proved, that \( y(t) = 0 \) for \( 0 \leq t \leq 1 \). Let \( 0 < < t < 1 \). Then
\[
\int_0^t u^n \, dx(u) = t^n x(t) = t^n x(t) - \epsilon x(t) = \int_t^1 u^n \, dx(u)
\]
and so \( x \) is absolutely continuous in \( [0, 1] \). Therefore \( \int_0^t u^n \, dx(u) \) and consequently \( x(t) = 0 \) a.e. in \( (0, 1) \). It follows that \( x(t) = x(t) \) for \( 0 < t < 1 \), and hence that \( x(t) = x(0^+) \) for \( 0 < t \leq 1 \).

This completes the proof of Lemma 3.

3. The main results. The proofs of both parts of the following theorem are based on proofs in Shohat and Tamarkin's book [13, pp. 99–101] of the case \( n = 0, 1, 2, \ldots \). Hildebrandt [8] originally proved this case of part (i) by a similar method.

**Theorem 1.**

(i) If \( M_k \leq \infty \), then there is a normalized function \( \alpha \in BV \) such that (1) is satisfied and \( \| \beta \| \leq M_k \).

(ii) If \( 1 < p < \infty \) and \( M_k \leq \infty \), then there is a function \( \beta \in L_p \) such that (2) is satisfied and \( \| \beta \| \leq M_k \).

**Proof.** Define \( A \) to be the linear space of functions \( P \) such that
\[
P(t) = \sum_{n=0}^m a_n \lambda_n \quad \text{for} \quad 0 \leq t \leq 1,
\]
where \( m \) is an arbitrary non-negative integer and \( a_0, a_1, \ldots, a_m \) are real.
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constants. Define the moment operator \( \mu \) on \( \Lambda \) by setting

\[
\mu(P) = \sum_{k=0}^{\infty} \lambda_k \Delta_k
\]

when \( P \) is given by (6).

Suppose that \( M_p < \infty \) where \( 1 < p < \infty \). Let \( P \in \Lambda \) and let \( B_n \in \Lambda \) be given by

\[
B_n(t) = \sum_{k=0}^{n} \lambda_k(t) P \left( \frac{\Delta_k}{\Delta_n} \right) \text{ for } 0 \leq t \leq 1.
\]

Then

\[
(7) \quad \mu(B_n) = \sum_{k=0}^{n} \lambda_k P \left( \frac{\Delta_k}{\Delta_n} \right)
\]

and hence, by Lemma 1,

\[
(8) \quad \lim_{n \to \infty} \mu(B_n) = \mu(P).
\]

since, by Hölder's inequality, \( M_1 \leq M_p \).

Part (i). It follows from (7) that

\[
\mu(B_n) = M_n \|P\|_c
\]

and hence, by (8), that

\[
\mu(P) = \sum_{n=0}^{\infty} M_n \|P\|_c.
\]

Thus \( \mu \) is a bounded linear functional on a linear subspace of \( C \). Hence, by the Hahn–Banach theorem [11, Theorem 5.16] and the Riesz representation theorem for bounded linear functionals on \( L^p \) [2, pp. 64, 65], there is a function \( \beta \in L^p \) such that, for every \( P \in \Lambda \),

\[
\mu(P) = \int_0^1 P(t) \beta(t) \, dt \quad \text{and} \quad \|\beta\|_p = M_p.
\]

In particular, taking \( P(t) = t^n \), we get that

\[
\mu_n = \int_0^1 t^n \beta(t) \, dt \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]

This completes the proof of Theorem 1.

Combining Lemma 2 and Theorem 1 we obtain:

**Theorem 2.**

(i) \( M_p < \infty \) if and only if (1) is satisfied by a function \( \alpha \in BV \).

(ii) For \( 1 < p < \infty \), \( M_p < \infty \) if and only if (2) is satisfied by a function \( \beta \in L^p \).

The next two theorems give more precise information about \( M_p \).

**Theorem 3.**

(i) If (1) is satisfied by a normalized function \( \alpha \in BV \), then

\[
\begin{align*}
M_1 &= \int_0^1 |\alpha(t)| \, dt \quad \text{when} \quad \lambda_0 = 0, \\
M_1 &= \int_0^1 |\alpha(t)| \, dt - |\alpha(0^+)| \quad \text{when} \quad \lambda_0 > 0.
\end{align*}
\]

(ii) If \( 1 < p \leq \infty \) and (2) is satisfied by a function \( \beta \in L^p \), then \( M_p = \|\beta\|_p \).

**Proof.** Part (i). By Lemma 2(i), we have that \( M_1 = \frac{1}{p} |\alpha(0)| < \infty \). Hence by Theorem 1(i), there is a normalized function \( \tilde{\alpha} \in BV \) such that \( \mu_n = \frac{1}{p} t^n \tilde{\alpha}(t) \) for \( n = 0, 1, 2, \ldots \) and \( \|\tilde{\alpha}(t)\| = M_1 \).

If \( \lambda_0 = 0 \), then, by Lemma 3, \( \tilde{\alpha}(t) = \alpha(t) \) for \( 0 \leq t \leq 1 \), and hence \( M_1 = \frac{1}{p} |\alpha(0)| \).

Suppose that \( \lambda_0 > 0 \), and let \( \gamma(0) = 0 \), \( \gamma(t) = \alpha(t) - \alpha(0^+) \) for \( 0 < t \leq 1 \). Then \( \mu_n = \frac{1}{p} t^n \gamma(t) \) for \( n = 0, 1, 2, \ldots \) and hence, by Lemma 2(i), \( M_1 = \frac{1}{p} \|\gamma(t)\|_1 \) is finite. Further, by Lemma 3, \( \gamma(t) = \tilde{\alpha}(t) - \tilde{\alpha}(0^+) \) for \( 0 < t \leq 1 \), and so, since \( \gamma(0) = 0 \), we have that \( M_1 = \frac{1}{p} \|\gamma(t)\|_1 = \frac{1}{p} \|\tilde{\alpha}(t) - \tilde{\alpha}(0^+)\|_1 \).

Hence \( M_1 = \frac{1}{p} \|\tilde{\alpha}(t) - \tilde{\alpha}(0^+)\|_1 \).

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the term multiplying \( M_p \) in the inequality tends to \( \|\beta\|_p \|P(t)\|_p \). In view of (8), it follows that

\[
\|\mu(P)\|_p \leq M_p \|P\|_p.
\]

Thus \( \mu \) is a bounded linear functional on a linear subspace of \( L^p \). Hence, by the

Hahn–Banach theorem and the Riesz representation theorem for bounded linear functionals on \( L^p \) [2, pp. 64, 65], there is a function \( \beta \in L^p \) such that, for every \( P \in \Lambda \),

\[
\mu(P) = \int_0^1 P(t) \beta(t) \, dt \quad \text{and} \quad \|\beta\|_p = M_p.
\]

In particular, taking \( P(t) = t^n \), we get that

\[
\mu_n = \int_0^1 t^n \beta(t) \, dt \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]

This completes the proof of Theorem 1.
Part (ii). By Lemma 2(ii), we have that \( M_n \leq \|f\|_p \leq M_p \). Hence, by Theorem 1(ii), there is a function \( \beta \) on \( \mathbb{R} \) such that 
\( \mu_n = \frac{1}{T} \int_T \beta(t) \, dt \) for \( n = 0, 1, 2, \ldots \) and \( \|f\|_p \leq M_p \). By Lemma 3, \( \beta(0) \) does not depend on \( \beta(t) \) for \( t \leq 1 \), and \( \beta(1) = \beta(0) \) a.e. in \( (0, 1) \). It follows that \( M_n = \|\beta\|_p = \|\beta(1)\|_p \leq M_p \) so that \( M_p = \|f\|_p \).

This completes the proof of Theorem 3.

Theorem 4. If \( 1 \leq p < \infty \), then \( M_n \leq M_{p_n+1} \) for \( n = 0 \) and \( \lim_{n \to \infty} M_{p_n} = M_p \).

Proof. Let \( 0 \leq k \leq n \). Then

\[
\lambda_{n+1,k} = \lambda_{n+1} - \lambda_k \frac{\lambda_{n+2,k} - \lambda_{n+1,k}}{\lambda_{n+2} - \lambda_{n+1}} - \lambda_k
\]

and hence

\[
(8) \quad \lambda_{n+1,k} = \lambda_{n+1} - \lambda_k \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+2} - \lambda_{n+1}} - \lambda_k
\]

It follows that

\[
M_{p_n} \geq M_{p_{n+1}} \left( 1 + \frac{1}{\lambda_{n+1}} \right) D_{n+1} = M_{p_{n+1}}
\]

Finally, for \( 1 \leq p < \infty \), application of Hölder's inequality to (8) yields that

\[
D_{n+1}^{-1} \lambda_{n+1} \delta_{n+1}^{-2} \leq \left( \frac{1}{\lambda_{n+1}} \left| \lambda_{n+1} \delta_{n+1} \right| \right) \left( \lambda_{n+1} \delta_{n+1} \right) D_{n+1}^{-1}
\]

since

\[
1 \leq \lambda_{n+1} \delta_{n+1} \delta_{n+1}^{-1} \delta_{n+1} = 1 + \frac{1}{\lambda_{n+1}} \frac{\delta_{n+1}^{-1}}{\lambda_{n+1}}
\]

Summing the above inequality for \( k = 0, 1, \ldots, n \), we get that

\[
M_{p_n} \leq M_{p_{n+1}} \left( 1 + \frac{1}{\lambda_{n+1}} \right) \delta_{n+1}^{-1} D_{n+1}^{-1} \leq M_{p_{n+1}}
\]

This completes the proof of Theorem 4.

References