MATRICES TRANSFORMATION OF WEAKLY MULTIPlicative SEQUENCES OF RANDOM VARIABLES

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1. Introduction

Suppose throughout that \( \{X_n\} (n = 0, 1, \ldots) \) is a sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\), and that \( \{a_{nk}\} (n, k = 0, 1, \ldots) \) is a (summability) matrix satisfying

\[
\sum_{k=0}^{\infty} |a_{nk}| < \infty \quad \text{for} \quad n = 0, 1, 2, \ldots . \tag{1}
\]

Let

\[
b_{n_1 \ldots n_k} = E(X_{i_1} \cdots X_{i_k}) ,
\]

\[
B_{n}(q) = \sum_{0 \leq i_1 < i_2 < \ldots < i_n} |b_{i_1 \ldots i_n}|^q ,
\]

where the summation is extended to all integers \( i_1, i_2, \ldots, i_n \) satisfying \( 0 \leq i_1 < i_2 < \ldots < i_n \). Let

\[
\sigma_n(p) = \left( \sum_{k=0}^{\infty} |a_{nk}|^p \right)^{1/(p-1)} ,
\]

and let

\[
T_n = \sum_{k=0}^{\infty} a_{nk} X_k .
\]

The primary object of this paper is to establish the following two theorems concerning the almost sure convergence to zero of the sequence \( \{T_n\} \).

Theorem 1. Let \( 1 < p \leq 2, \frac{1}{p} + \frac{1}{q} = 1, 0 < M < \infty \), let \( r \) be an even positive integer, and let

\[
EX_n^r \leq M \quad \text{for} \quad n = 0, 1, \ldots , \tag{2}
\]

\[
B_n(q) < \infty , \tag{3}
\]

\[
\sum_{n=0}^{\infty} \sigma_n(p)^{r/q} < \infty . \tag{4}
\]

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Then
\[ E \sum_{n=0}^{\infty} T_n^s < \infty \]
and, in particular, \( T_n \to 0 \) a.s.

**Theorem 2.** Let \( 1 < p \leq \frac{1}{q} + \frac{1}{q} = 1, 0 < M < \infty \), and let
\begin{align*}
|X| &\leq M \text{ a.s. for } n = 0, 1, \ldots, \\
|B_q(a_n)| &\leq M \text{ for } n = 1, 2, \ldots, \\
\sum_{n=0}^{\infty} e^{-\alpha n} &< \infty \text{ for every } \alpha > 0.
\end{align*}

Then
\[ \sum_{n=0}^{\infty} P(|T_n| > \varepsilon) < \infty \text{ for every } \varepsilon > 0 \]
and, in particular, \( T_n \to 0 \) a.s.

In Theorem 2 the conditions on the sequence \( \{X_n\} \) are more restrictive whereas the conditions on the matrix \( \{a_n\} \) are less restrictive than in Theorem 1. It is easily demonstrated (see Hill [4]) that condition (7) is implied by either condition (4) or by
\[ a_n(p) \not\to \infty \text{ for } n = 0, 1, \ldots \text{ and } a_n(p) \log n \to 0 \text{ as } n \to \infty. \]

Evidently condition (7) becomes less restrictive as \( p \) decreases. In §5 it is shown that condition (7) does likewise provide that
\[ \sup_{e^{s+q} \in S} |b_{e^{s+q}}| \sum_{e^{s+q} \in S} |b_{e^{s+q}}| < \infty. \]

The sequence \( \{X_n\} \) is said to be **multiplicative** if \( b_{e^{s+q}} = 0 \) whenever \( 0 \leq i < j < \ldots < l \), in particular, it is multiplicative if it is independent with expectation \( E X_n = 0 \) for \( n = 0, 1, \ldots \). The sequence is said to be weakly multiplicative if it satisfies condition (3) for some pair of positive numbers \( r, q \).

Hill [3, 4] proved Theorems 1 and 2 for the special case in which \( p = q = 2 \), \( \{X_n\} \) is the sequence of Rademacher functions on \( \Omega = [0, 1] \) and \( P \) is Lebesgue measure. Azuma [1] proved Theorem 2 for the special case in which \( p = q = 2 \) and \( \{X_n\} \) is multiplicative. Other results of a similar nature appear in Chapter 4 of Stout’s book [8].

In §6 it is shown that when \( P \) is Lebesgue measure on \( \Omega = [0, 1] \) then the sequence \( \{\cos(k_n x + s)\} \) on \( \Omega \) satisfies condition (6) (for every \( q > 1 \) provided \( c > 2 \) and \( k_1 > 0 \) for \( n = 0, 1, \ldots \)). The sequence is known to be multiplicative if \( k_n(2n) \) is an integer and \( k_{n+1} > 2k_n > 0 \) for \( n = 0, 1, \ldots \).

In §7 it is shown that the standard Cesàro and Euler summability matrices satisfy condition (4) for certain pairs of positive numbers \( r, p \).

2. Preliminary results

Two lemmas are required.

**Lemma 1.** Let \( 1 < p \leq \frac{1}{q} + \frac{1}{q} = 1, 0 < M < \infty \), and let \( \{a_n\} \) be a sequence of real numbers and let \( \{X_n\} \) satisfy conditions (2) and (3). Then
\[ E \left( \sum_{k_n \in S} a_{k_n} X_{k_n} \right) \leq K \left( \sum_{k_n \in S} |a_{k_n}|^p \right)^{1/p} \text{ for } m = 0, 1, \ldots, \]
where \( K \) is a positive number independent of \( \{a_n\}, \{X_n\} \) and \( m \).

This result is due to Móricz [6]; his proof is based on an inequality established by Gapokin [2]. The case in which \( p = 2 \) of the following lemma is also due to Móricz [7]; our proof is modelled on his.

**Lemma 2.** Let \( 1 < p \leq 2, u > 0 \), let \( \{a_n\} \) be a sequence of real numbers, \( \{X_n\} \) satisfy conditions (5) and (6), and let
\[ a_n = \sum_{k_n \in S} |a_{k_n}|^p, \quad S_n = \sum_{k_n \in S} a_{k_n} X_{k_n}. \]

Then
\[ E e^{\alpha \delta_n} \leq C e^{\alpha \delta_m} \text{ for } m = 0, 1, \ldots, \]
where \( C, \alpha \) are positive numbers independent of \( \{a_n\}, \{X_n\}, u \) and \( \delta_m \).

**Proof.** Let \( B_n = B_q(a) \) and let
\[ \beta > B = \limsup \sup_{n \to \infty} B_n \]
the finiteness of \( B \) being ensured by condition (6).

Because of the convexity of \( e^{\alpha \delta} \) we have, for every real \( s \) and \( -1 \leq x < 1 \), that \( e^{\alpha \delta} \leq \cosh \alpha (1 + x + x^2) \), Thus
\[ E e^{a_m} = E \prod_{k_n \in S} \exp(u M_{k_n} X_{k_n}/M) \leq \prod_{k_n \in S} \cosh(u M_{k_n} - E \prod_{k_n \in S} \left( 1 + \delta_{k_n} X_{k_n} \right)). \]

where \( \delta_{k_n} = \frac{1}{M} \left( 1 - \frac{1}{u M_{k_n} - 1} \right) \cosh u M_{k_n} \).
Next, since \( \cosh t \leq e^{0.5t} \leq e^t \), \( \cosh t \leq e^t \), and \( 1 < \rho \leq 2 \), we have that
\[
\cosh t \leq e^{\rho t} \quad \text{and so} \quad \prod_{k=0}^{n} \cosh u \cdot M_{n0} \leq \prod_{k=0}^{n} \exp (\omega \cdot M_{n0}^{\rho} u) = \exp (\omega \cdot M_{n0}^{\rho} u).
\]
Further, by Hölder's inequality,
\[
E \prod_{k=0}^{n} (1 + \delta \cdot X_k) \leq 1 + \sum_{j=1}^{n} \beta_j \nu_{k,j} \nu_{k,j}^{\rho} \cdot \left( \sum_{k=0}^{n} \beta_j \nu_{k,j}^{\rho} \right)^{1/\rho} \leq 1 + \sum_{j=1}^{n} \beta_j \nu_{k,j}^{\rho} \cdot \left( \sum_{k=0}^{n} \beta_j \nu_{k,j}^{\rho} \right)^{1/\rho}.
\]
Since \( 1 + t \leq e^t \) and \( \tanh t \leq t \) when \( t > 0 \), it follows that
\[
E \prod_{k=0}^{n} (1 + \delta \cdot X_k) \leq \prod_{k=0}^{n} (1 + \beta \nu_{k,j}^{\rho} \delta \cdot \nu_{k,j}^{\rho}) \cdot \left( 1 + \sum_{j=1}^{n} B_j \beta_j \nu_{k,j}^{\rho} \right)^{1/\rho} \leq C \exp \left( \frac{1}{\rho} \beta \nu_{k,j}^{\rho} \cdot \sum_{j=1}^{n} B_j \beta_j \nu_{k,j}^{\rho} \right) \leq C \exp \left( \frac{1}{\rho} \beta \nu_{k,j}^{\rho} \cdot \sum_{j=1}^{n} B_j \beta_j \nu_{k,j}^{\rho} \right)
\]
where \( C = \left( 1 + \sum_{j=1}^{n} B_j \beta_j \right)^{1/\rho} \leq \infty. \)

Collecting inequalities we arrive at the desired result, namely
\[
E \exp(\epsilon \cdot X_k) \leq C \exp \left( \frac{1}{\rho} \beta \nu_{k,j}^{\rho} \cdot \sum_{j=1}^{n} \beta_j \nu_{k,j}^{\rho} \right) \exp (\omega \cdot M_{n0}^{\rho} u) = C \exp (\epsilon \cdot u).
\]

where \( \epsilon = M^{\rho} \cdot \frac{\beta \nu_{k,j}^{\rho}}{\rho} \).

3. Proof of Theorem 1

Let
\[
T_{n0} = \sum_{k=0}^{n} a_{n0} X_k.
\]

By Hölder's inequality and conditions (1) and (2), we have that
\[
E \left( \sum_{k=0}^{n} |a_{n0} X_k| \right)^{\rho} \leq E \sum_{k=0}^{n} |a_{n0} X_k| \left( \sum_{k=0}^{n} |a_{n0}| \right)^{\rho - 1} \leq M \left( \sum_{k=0}^{n} |a_{n0}| \right) \leq \infty.
\]

It follows that, for \( n = 0, 1, \ldots \), \( \sum_{k=0}^{n} a_{n0} X_k \) converges a.s. and so
\[
\lim_{n \to \infty} T_{n0} = T_{a} \text{ a.s.}
\]

Hence, by Fatou's Lemma and Lemma 1,
\[
ET_{a} = E \liminf_{n \to \infty} T_{n0} \leq \liminf_{n \to \infty} ET_{n0}
\]
\[
\leq \liminf_{n \to \infty} K \left( \sum_{k=0}^{n} |a_{n0}| \right)^{\rho} \leq K \sigma_{\rho}^{\rho}.
\]

Consequently, by condition (4),
\[
E \sum_{k=0}^{n} T_{a} \leq K \sum_{k=0}^{n} \sigma_{\rho}^{\rho} < \infty,
\]
as desired.

4. Proof of Theorem 2

Let \( u, c \) be positive numbers. Then, for \( T_{n0} \) given by (9), we have, by Lemma 2, that
\[
E \exp(\epsilon \cdot u) \leq C \exp (cu \cdot \sigma_{n0}^{\rho - 1}),
\]
where \( C, c \) are positive constants and \( \sigma_{n0} = \sigma_{n0}(p) \). In view of conditions (1) and (5),
\[
\lim_{n \to \infty} T_{n0} = T_{a} \text{ a.s.}
\]
and hence, by Fatou's Lemma,
\[
E \exp(\epsilon \cdot u) \leq \liminf_{n \to \infty} E \exp(\epsilon \cdot u) \leq C \exp (cu \cdot \sigma_{n0}^{\rho - 1}).
\]

Consequently, by Chebyshev's inequality,
\[
P(T_{a} > \epsilon) = P(T_{n0} > \epsilon) \leq e^{-\epsilon^2} \exp (cu \cdot \sigma_{n0}^{\rho - 1}) \leq 2C \exp (cu \cdot \sigma_{n0}^{\rho - 1 - 1}).
\]

It follows, on taking \( u = (c/\rho \sigma_{n0}^{\rho - 1})^2 \), that
\[
P(T_{a} > \epsilon) \leq 2C \exp \left( \frac{-\epsilon^2}{\rho (c/\rho \sigma_{n0}^{\rho - 1})^2} \right).
\]

It follows that for \( n = 0, 1, \ldots \), \( \sum_{k=0}^{n} a_{n0} X_k \) converges a.s. and so
and hence, by condition (7), that

$$\sum_{x \neq 0} P(T_x > \varepsilon) < \infty.$$  

as desired. Since $\varepsilon$ is an arbitrary positive number this implies that $T_x \to 0$ a.s. by a corollary of the Borel-Cantelli Lemma (see [8, Theorem 2.1.1.3]).

5. Variation in strength of condition (\(\mathcal{P}\))

We shall prove the following.

**Proposition 1.** Let $v > w > 1$, and let

$$\sup_{e \in A \cap [0, 1]} |a_e| \leq M < \infty.$$  

Then $\alpha_e(w) \leq \alpha_e(M)^w$ for $n = 0, 1, \ldots$, and $\delta = \frac{1}{w - 1} - \frac{1}{v - 1}$, hence condition (\(\mathcal{P}\)) holds with $p = w$ whenever it holds with $p = v$.

**Proof.** Let $\mu = \frac{v - 1}{w - 1} + \frac{1}{\mu} = 1$. Then, by Hölder’s inequality,

$$\sigma_n(w)^{-\frac{1}{w}} = \sum_{x \neq 0} |a_x|^{\frac{1}{w}} |a_x| \leq \left( \sum_{x \neq 0} |a_x|^{\frac{1}{w}} \right)^\frac{1}{w} \left( \sum_{x \neq 0} |a_x|^\mu \right)^\frac{1}{\mu} \leq \sigma_n(\mu)^{-\frac{1}{\mu}},$$

and the desired inequality follows.

6. A weakly multiplicative sequence

Let $P$ be Lebesgue measure in $\Omega = [0, 1]$, let $s$ be any real number and let

$$X_n = \cos(k_n x + a) \quad \text{for} \quad x \in \Omega, \quad n = 0, 1, \ldots.$$  

We shall prove the following.

**Proposition 2.** Let $v > 2$ and let $k_{n+1} \geq k_n > 0$ for $n = 0, 1, \ldots$. Then the sequence $\{X_n\}$ satisfies condition (6) for every $q \geq 1$.

**Proof.** By induction we have that

$$k_{n+1} \geq k_n + k_{n-1} + \ldots + k_0 + k_0(c-1)^n \quad \text{for} \quad n = 0, 1, \ldots.$$
Hence, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sigma_d(p)^{q-1} = \frac{1}{n^q} \sum_{k=0}^{n-1} \left( \frac{k+n-1}{n} \right)^q = O(n^{e-1}) = O(n^{-r})$$ as $n \to \infty$.

since $p(a-1) > -1$. Therefore $\sigma_d(p)^q = O(n^{-r})$ as $n \to \infty$, and so condition (4) is satisfied.

It follows, by Theorem 1, that

$$T_n = \sum_{k=0}^{n-1} \sigma_k X_k \to 0 \text{ a.s.}$$

that is $X_n \to 0(C_3)$ a.s.

(b) The Euler matrix $S_n(0 < n < 1)$. This is the triangular matrix $[a_k]$ given by

$$a_k = \begin{cases} 0 & k \leq n, \\ \frac{k}{n}(1-\frac{k}{n})^{-1} & k > n. \end{cases}$$

We shall prove the following result.

**Theorem 5.** Let $r > 2q \geq 4$ where $r$ is an even integer, and let $\{X_k\}$ satisfy conditions (2) and (3). Then $X_n \to 0(\mathbb{E}_r)$ a.s.

**Proof.** It is known [9, p. 57] that $n^{r-2}a_k \leq M_k$ for $0 < k < n$, where $M_k$ is a positive number independent of $k$ and $n$. Hence, for $\frac{1}{p} + \frac{1}{q} = 1, n \geq 1$,

$$\sigma_d(p)^{q-1} = \sum_{k=0}^{n-1} |a_k|^{q} \leq M_k n^{-e-\frac{1}{2}} \sum_{k=0}^{n-1} |a_k| = M_k n^{-e-\frac{1}{2}},$$

and so $\sigma_d(p)^q = O(n^{-r})$ as $n \to \infty$. Condition (4) is thus satisfied and consequently, by Theorem 1, $X_n \to 0(\mathbb{E}_r)$ a.s.

References


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