Since \( \phi \| \in \hat{U} \) and \( \| \phi \| \leq 1 \), it follows that
\[
\|\phi\| \leq \|[\hat{g}]\| + \epsilon = \|P(\hat{g})\| + \epsilon,
\]
since \( \epsilon \) is arbitrary it follows that \( \|g\| \leq \|P(\hat{g})\| \). Thus \( \|g\| = \|P(\hat{g})\| \) and the proof is complete.

The authors would like to thank the referee for several helpful suggestions and for pointing out an error in the original version.

REFERENCES


Printed in Great Britain.

Tauberian and other theorems concerning Dirichlet's series with non-negative coefficients

By DAVID BORWEIN

Department of Mathematics, University of Western Ontario, London, Ont. N6A 5B7, Canada

(Received 13 November 1986; revised 9 March 1987)

Abstract

The paper is concerned with properties of the Dirichlet series \( a(x) = \sum_{n=1}^\infty a_n e^{-\lambda_n x} \), where \( \lambda_n \) is a strictly increasing unbounded sequence of real numbers with \( \lambda_n \geq 0 \). One of the main Tauberian results proved is that if \( a_n > 0, a_n \geq 0 \) for \( n = 2, 3, \ldots \), \( a(x) < \infty \) for all \( x > 0 \), \( A_n = a_1 + a_2 + \cdots + a_n \to \infty \), \( A_n A_{n-1} = \sigma(\lambda_{n-1} - \lambda_n) A_n \), \( a_n A_n A_{n-1} \geq -\sigma(\lambda_{n-1} - \lambda_n) A_n \) and \( \sum_{n=1}^\infty a_n e^{-\lambda_n x} = a(x) \) as \( x \to 0^+ \), then \( \sum_{n=1}^\infty a_n e^{-\lambda_n x} \). A new summability method \( D_{\lambda_n} \) based on the Dirichlet series \( a(x) \) is defined and its relationship to the weighted mean method \( M_{\lambda_n} \) investigated.

1. Introduction

Suppose throughout that \( \lambda = (\lambda_n) \) is a strictly increasing unbounded sequence of real numbers with \( \lambda_n \geq 0 \), and that \( a = (a_n) \) is a sequence of real numbers. Let
\[
A_n := a_1 + a_2 + \cdots + a_n, \quad a(x) := \sum_{n=1}^\infty a_n e^{-\lambda_n x}.
\]

The same system of notation will be used with letters other than \( a, A \). Except in \( \S 6 \) and \( \S 7 \), we shall suppose that
\[
a_n > 0 \quad \text{and} \quad a_n \geq 0 \quad \text{for} \quad n = 2, 3, \ldots,
\]
and that the Dirichlet series \( a(x) \) is convergent for all \( x > 0 \). Let \( (a_n) \) be a sequence of real numbers,
\[
t_n := \sum_{k=1}^n a_k e^{-\lambda_n x} \quad \text{and} \quad \sigma(x) := \sum_{n=1}^\infty a_n e^{-\lambda_n x}.
\]

We shall be concerned with the weighted mean summability method \( M_{\lambda_n} \) and the Dirichlet series method \( D_{\lambda_n} \), the latter method being new. These methods are defined as follows:
\[
a_n \to s(M_{\lambda_n}) \quad \text{if} \quad t_n \to s;
\]
\[
a_n \to s(D_{\lambda_n}) \quad \text{if} \quad \sigma(x) \text{ exists for all } x > 0 \text{ and } \sigma(x) \to s \text{ as } x \to 0^+.
\]

When \( \lambda_n \) is the method \( D_{\lambda_n} \) reduces to the power series method \( J_{\lambda_n} \) (as defined in [1] for example). It is familiar that the method \( M_{\lambda_n} \) is regular (i.e., \( s_n \to s(M_{\lambda_n}) \) whenever \( s_n \to s \)) if and only if \( A_n \to \infty \) (see [5], theorems 2 and 12), and, since
The primary purpose of this paper is to prove the following four theorems, the latter three being Tauberian in character:

**Theorem 1.** If \( A_n \to \infty \) and \( s_n \to s(M) \), then \( s_n \to s(D_{i,k}) \).

**Theorem 2.** Let \( s_n \to s(D_{i,k}) \); let \( s_n > -H \) for \( n = 1, 2, \ldots \), where \( H \) is a positive constant, and let \( a(x) \) satisfy

\[
\lim_{x \to a} \frac{a(mx)}{a(x)} = a_m > 0 \quad \text{for} \quad m = 2, 3, \ldots.
\]

Then \( s_n \to s(M) \).

**Theorem 3.** Suppose that \( \lambda_{n+k} \to \lambda_n \) and \( A_n = o(\lambda_{n+k} - \lambda_n) \).

**Theorem 4.** If \( a(x) \) satisfies (1), then

\[
a(x) = x^{-\rho}L \left( \frac{1}{x} \right) \quad \text{for} \quad x > 0,
\]

where \( \rho = -\log_2 a_1 \geq 0 \) and \( L(x) \) is a function (defined for \( x > 0 \)) satisfying

\[
\lim_{x \to a} \frac{L(tx)}{L(x)} = 1 \quad \text{for all} \quad t > 0,
\]

and

\[
A_n \sim \frac{1}{\Gamma(p+1)} \left( \frac{1}{a_m} \right) = \frac{\lambda_n}{\Gamma(p+1)} L(a_m).
\]

With regard to condition (1), it should be noted that the condition

\[
\lim_{x \to a} \frac{a(mx)}{a(x)} = 1
\]

implies that

\[
\lim_{x \to a} \frac{a(mx)}{a(x)} = 1 \quad \text{for} \quad m = 2, 3, \ldots.
\]

The special case \( \lambda_1 = a \) of Theorem 1 is due to Ishiguro [6]; and the same case of Theorem 2 to Borwein and Mert [2], and of Theorem 3 to Borwein [1]. Theorem 4 generalizes a theorem due to Hardy and Littlewood [3], theorem D), which has in place of condition (1) the stronger condition

\[
a(x) \sim A x^{-\gamma} \quad \text{as} \quad x \to 0 +,
\]

with \( A > 0 \) and \( \rho \geq 0 \). Theorem 4 is in fact a corollary of Karamata's Tauberian theorem and a known result about regularly varying functions (see [9], theorem 2.5 and 1.8), and Theorem 2 can be deduced from Theorem 4. Our proofs of Theorems 2 and 4, however, are more direct and more elementary in that they do not involve the extended continuity theorem for Laplace-Stieltjes transforms on which the proof of Karamata's theorem is based. We indicate the scope of Theorem 3 by means of two examples at the end of §4. In §5 we express a slight extension of Theorem 2 in a different form. In §6 we show how to generate Dirichlet series \( a(x) \) that satisfy (1).

In §7 we show that Theorem 4 remains valid when the condition \( a_n > 0 \) is relaxed.

**2. Proof of Theorem 1**

Suppose that \( x > 0 \). Then, for \( 0 < \delta < x \),

\[
0 \leq A_n e^{-L(x)} \leq e^{-\lambda_n} \sum_{k=1}^{n} a_k e^{-\lambda_{n-k}} \leq e^{-\lambda_n} a(x) \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, by hypothesis, \( t_k \to s \), and so

\[
\sum_{k=1}^{n} a_k t_k e^{-\lambda_n} = \sum_{k=1}^{n} (A_k t_k - A_{k-1} t_{k-1}) e^{-\lambda_n} = \sum_{k=1}^{n} A_{k-1} e^{-\lambda_n} t_k e^{-\lambda_n} + \sum_{k=1}^{n} A_k e^{-\lambda_n} t_k e^{-\lambda_n} \to \sum_{k=1}^{n} A_k e^{-\lambda_n} t_k e^{-\lambda_n} \quad \text{as} \quad n \to \infty.
\]

Consequently

\[
a(x) = \frac{1}{a(x)} \sum_{k=1}^{n} A_k t_k e^{-\lambda_n} e^{-\lambda_n + \cdots} = \frac{1}{a(x)} \sum_{k=1}^{n} A_k e^{-\lambda_n} e^{-\lambda_n} t_k e^{-\lambda_n + \cdots}.
\]

Since

\[
\sum_{k=1}^{n} A_k e^{-\lambda_n} e^{-\lambda_n + \cdots} = 1,
\]

and, for \( k = 1, 2, \ldots \),

\[
0 < \frac{1}{a(x)} A_k e^{-\lambda_n} e^{-\lambda_n + \cdots} \to 0 \quad \text{as} \quad x \to 0 +,
\]

it follows, by a standard result ([5], theorem 2), that

\[
a(x) \to s \quad \text{as} \quad x \to 0 +.
\]

**3. Proofs of Theorems 2 and 4**

We require the following known result ([9], theorem 1.8):

**Lemma 1.** If \( a(x) \) satisfies (1), then

\[
\lim_{x \to a} \frac{a(tx)}{a(x)} = t^{\rho} \quad \text{for all} \quad t > 0,
\]

where \( \rho = -\log_2 a_1 \geq 0 \), and

\[
a(x) = x^{-\rho} L \left( \frac{1}{x} \right) \quad \text{for} \quad x > 0,
\]

where

\[
\lim_{x \to a} \frac{L(tx)}{L(x)} = 1 \quad \text{for all} \quad t > 0.
\]
Proof of Theorem 2. (Cf. the proof of theorem 1 in [1].) It follows from (4), by Lemma 1, that
\[ \lim_{x \to \infty} \frac{a(mx)}{a(x)} = m^\nu \quad \text{for} \quad m = 1, 2, \ldots . \]
Further, for \( m = 0, 1, \ldots \),
\[ (m+1)^\nu = \int_0^1 (m+1)^\nu \, dx(t), \]
where
\[ x(t) = \frac{1}{\Gamma(\rho)} \int_0^t (-\log u)^{\rho-1} \, du \quad \text{when} \quad \rho > 0, \]
and, when \( \rho = 0 \),
\[ x(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 1, \\ 1 & \text{if} \quad t = 1. \end{cases} \]
Suppose without loss of generality that \( H = 0 \), i.e. that \( a_n > 0 \) for \( n = 1, 2, \ldots \).
Define
\[ \psi(x) = \begin{cases} \frac{1}{a(x)} & \text{for} \quad c \leq x \leq 1, \\ 0 & \text{otherwise}, \end{cases} \]
where \( 0 < c < 1 \), and
\[ \psi(x) = \frac{1}{a(x)} \sum_{n=1}^\infty a_n e^{-2\pi i x a_n} \psi(x - a_n^\nu). \]
Then, for \( m = 0, 1, \ldots \),
\[ \frac{1}{a(x)} \sum_{n=1}^\infty a_n e^{-2\pi i x a_n} \psi(x - a_n^\nu) = \frac{a(x + mx)}{a(x)} \quad \text{as} \quad x \to 0^+; \]
and so, for any polynomial \( p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n \),
\[ \frac{1}{a(x)} \sum_{n=1}^\infty a_n e^{-2\pi i x a_n} \psi(x - a_n^\nu) \to s \sum_{n=1}^\infty p_n(a_n^\nu) \psi(x) \quad \text{as} \quad x \to 0^+. \]
Since \( x \) is continuous at \( \epsilon \) it is readily demonstrated that, given \( \epsilon > 0 \), there are polynomials \( p(t), q(t) \) such that
\[ p(t) \leq \psi(t) \leq q(t) \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and} \quad \int_0^1 (q(t) - p(t)) \, dx(t) < \epsilon. \]
It follows that
\[ \lim_{x \to +0} \psi(x) = \int_0^1 \phi(\epsilon) \, dx(\epsilon) = \int_0^1 (\frac{1}{\Gamma(p+1)} (-\log c)^p e^{(-\log c) \epsilon}). \]
Hence
\[ \psi(\frac{(-\log c)}{\Lambda_n}) = \frac{1}{a(\frac{(-\log c)}{\Lambda_n})} \sum_{n=1}^\infty a_n e_n \psi(\frac{(-\log c) \epsilon}{\Gamma(p+1)}). \]
Taking \( a_k = 1 \) for \( k = 1, 2, \ldots \), we obtain
\[ a_n \left( \frac{(-\log c)}{\Lambda_n} \right) \psi(\frac{(-\log c) \epsilon}{\Gamma(p+1)}). \]
Consequently
\[ t_n = \frac{1}{a_n} \sum_{n=1}^\infty a_n e_n \to s. \]
Note that the case in which (4) holds can be proved somewhat more simply along the lines of the proof of Case 1 of theorem 1 in [1].

Proof of Theorem 4. The theorem follows immediately from (5) with \( c = 1/e \) and Lemma 1.

4. Proof of Theorem 3
(Cf. the proof of theorem 2 in [1].) Let \( x > 0 \). By Cauchy's mean value theorem,
\[ e^{(x-a)x} - e^{(x-a)x} = \frac{1}{1-e^{(a-a)x}} x = \frac{\lambda_n}{\lambda_{n+1} - \lambda_n} e^{(a-a)x} \quad (0 < \xi < x) \]
\[ = \frac{\lambda_n}{\lambda_{n+1} - \lambda_n} e^{(a-a)x} \quad (0 < \xi < x) \]
\[ < \frac{\lambda_n}{\lambda_{n+1} - \lambda_n} e^{(a-a)x} \quad \text{for} \quad n \geq N, \quad (6) \]
where \( N \) is a sufficiently large positive integer. Since \( a(x) \to \infty \) as \( x \to 0^+ \), it follows from (2) and (6) that
\[ 0 < a(x) - a(2x) = \sum_{n=1}^\infty a_n (e^{(x-a)x} - e^{2(x-a)x}) = \sum_{n=1}^\infty a_n e^{(x-a)x} \to 0 \quad \text{as} \quad x \to 0^+ \]
as in the proof of Theorem 1. Hence
\[ \lim_{x \to 0^+} a(x) = 1, \]
and this implies that
\[ \lim_{x \to 0^+} \frac{a(x)}{x} = 1 \quad \text{for all} \quad t > 0. \quad (7) \]
Now let \( (\gamma_n) \) be a sequence of positive numbers such that
\[
a_n a_n + \gamma_n = H(\lambda_n + \lambda_n - \lambda_n) A_n,
\]
so that, by (3), \( \gamma_n + a_n \geq 0 \). Next, for \( \psi(t) \) defined as in the proof of Theorem 2, we have
\[
\psi(x) = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n (\gamma_n + \gamma_n) e^{-\lambda_n} e^{-\lambda_n x} = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n (e^{-\lambda_n x} - e^{-\lambda_n x})
\]
and hence, by (6),
\[
\psi(x) = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n (\gamma_n + \gamma_n) e^{-\lambda_n} e^{-\lambda_n x} = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n (e^{-\lambda_n x} - e^{-\lambda_n x}).
\]
where \( M \) is a positive constant. Therefore
\[
\limsup_{x \to \infty} \psi(x) = \left(1 + \frac{1}{e}\right) M = \frac{1}{e} + \frac{1}{e} = e + M < \infty.
\]
Similarly
\[
\liminf_{x \to \infty} \psi(x) < -\infty,
\]
and hence \( \phi(x) = O(1) \) as \( x \to \infty \). It follows that
\[
\psi(-\log x) = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n = O(1).
\]
Further, since (5) is a consequence of (7),
\[
A_n \sim a_n \left(-\log x\right),
\]
and therefore
\[
t_n = \frac{1}{A_n} \sum_{n=1}^{\infty} a_n \lambda_n = O(1).
\]
Now let
\[
b_n = A_n (\lambda_{n+1} - \lambda_n)
\]
for \( n = 1, 2, \ldots \), and
\[
\tau(x) = \frac{1}{b(x)} \sum_{n=1}^{\infty} a_n \lambda_n e^{-\lambda_n x}.
\]
Then
\[
x b(x) \geq \sum_{n=1}^{\infty} A_n \left(\lambda_{n+1} - \lambda_n\right) e^{-\lambda_n x} = a(x).
\]
Also, given \( \gamma \in (0, 1) \), there is a positive integer \( r \) such that \( \lambda_n > \gamma \lambda_{n+1} \) for \( n > r \), and so
\[
\gamma b(x) \leq \sum_{k=1}^{\infty} A_k \gamma e^{-\lambda_k x} dt + \gamma x e^x B_x = a(x) + \gamma x B_x.
\]
Consequently, by (7),
\[
1 \leq \liminf_{x \to \infty} \frac{\gamma b(x)}{a(x)} \leq \limsup_{x \to \infty} \frac{\gamma b(x)}{a(x)} = \frac{1}{\gamma} + \gamma x B_x = \frac{1}{\gamma} + \gamma x B_x
\]
and therefore
\[
\gamma b(x) \sim o(x) \text{ as } x \to 0^+.
\]
Further, \( \lambda_{n+1} > \gamma \lambda_n \) for \( n = 1, 2, \ldots \), and some positive constant \( K \), and, as in the proof of Theorem 1,
\[
(\sigma(x) + K) o(x) = \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \lambda_n e^{-\lambda_n x} \sim (\lambda_{n+1} + K) o(x) \text{ as } x \to 0^+.
\]
Hence, as in the proof of (9),
\[
x r(x) b(x) + K x b(x) = \sum_{n=1}^{\infty} (\lambda_{n+1} + K) A_n (\lambda_{n+1} - \lambda_n) e^{-\lambda_n x} \sim (\lambda_{n+1} + K) o(x) \text{ as } x \to 0^+.
\]
and so, by (9),
\[
\tau(x) \sim o(x) \text{ as } x \to 0^+.
\]
Since, by (7) and (9),
\[
\lim_{x \to \infty} b(x) = \frac{1}{t} \text{ for all } t > 0,
\]
it follows, by (8) and Theorem 2, that
\[
u_n = \frac{1}{B_n} \sum_{n=1}^{\infty} b_n \lambda_n = o(x).
\]
Further, by (2), (3) and (8), we have that, for \( n > 1 \),
\[
t_n - t_{n-1} = \frac{1}{A_n} \frac{\lambda_n}{\lambda_n - \lambda_{n+1}} \frac{A_n}{A_n - \lambda_n} \geq -\lambda_n \frac{\lambda_n - \lambda_{n+1}}{\lambda_n}
\]
for some positive constant \( \mu \). Thus, since \( \lambda_{n+1} \sim \lambda_n \) and \( \lambda_n \to \infty \),
\[
t_n - t_{n-1} \geq -\mu \sum_{n=1}^{\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n} \sim -\mu \log \lambda_n \text{ when } m > N \to \infty
\]
(see (7), p. 292), and so
\[
\liminf_{x \to \infty} (t_m - t_{m-1}) \geq 0 \text{ when } m > n \to \infty \text{ and } \lambda_n \to 1.
\]
Now, by (2),
\[ \lambda_{a_n} A_n - \lambda_{a_{n+1}} A_{n+1} = b_n - \lambda_{a_n} a_n \sim 0, \]
and hence, since \( B_n \geq A_1 \), \( \lambda_{a_{n+1}} A_{n+1} \sim 0 \),
\[ \lambda_{a_{n+1}} A_{n+1} \sim \lambda_{a_n} A_n. \]
It follows that, provided \( \lambda_{a_{n+1}} > (1 + \delta) \lambda_{a_n} \delta > 0 \),
\[ \frac{B_n}{B_{n+1}} \leq 1 + \frac{\sum_{k=1}^{n} \lambda_{a_k} A_k - \lambda_{a_{n+1}} A_{n+1}}{B_{n+1}} \leq 1 + \frac{q \delta \lambda_{a_n}}{B_{n+1}} \leq 1 + q \delta \lambda_{a_n} \quad \text{as} \quad n \to \infty. \]  
(12)

Suppose now without loss of generality that \( s = 0 \), i.e., \( \lambda_{a_n} \to 0 \). It follows from (11) that, given \( \epsilon > 0 \), there are positive numbers \( p, \delta \) such that \( t_n - t_{n+1} > -\epsilon \) when \( m > n > p \) and \( \lambda_{a_n} < (1 + 2\delta) \lambda_{a_{n+1}} \). Consequently if \( m, n \) satisfy these conditions we have, by (10), that
\[ (t_n - t_{n+1}) \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} t_k b_k - \sum_{k=1}^{n} u_k b_k \leq (t_n - t_{n+1}) \sum_{k=1}^{n} b_k, \]
and hence that
\[ t_n - t_{n+1} \leq \frac{\sum_{k=1}^{n} b_k}{B_n - B_{n+1}} \leq \frac{\sum_{k=1}^{n} b_k}{B_n - B_{n+1}} = \frac{1}{(B_n/B_{n+1}) - 1}. \]  
(13)

Letting \( m, n \to \infty \) subject to \( 1 + \delta < \lambda_{a_{n+1}} A_{n+1} < 1 + 2\delta \), it follows from (12) that
\[ \frac{1}{(B_n/B_{n+1}) - 1} = O(1), \]
and hence from (13) that
\[ \limsup_{n \to \infty} t_n \leq \epsilon \quad \text{and} \quad \liminf_{n \to \infty} t_n \geq -\epsilon. \]

Therefore \( t_n \to 0 \).

Example 1. Since \( \lambda_{a_n} = n, \lambda_{a_n} = 1/n \) satisfies the conditions of Theorem 3, we get as a corollary of that theorem a result due to Kouchanovskii [8], namely if
\[ \frac{1}{\log(1-y)} \sum_{n=1}^{\infty} \frac{y^n}{n} = y - 1 \quad \text{and} \quad y_n \geq -H \log n \quad \text{for} \quad n \geq 1, \]
then
\[ \frac{1}{\log n} \sum_{k=1}^{n} y_k \to y. \]

Example 2. Let \( \lambda_{a_n} = \log n, a_n = 1, a_{n+1} = 1/(n \log n) \) \( (n \geq 2) \). These sequences satisfy the conditions of Theorem 3. Further, for \( x > 0 \),
\[ a^n(x) = - \sum_{n=1}^{\infty} \frac{1}{\log nx} = 1 - (1 + x) \sim \frac{1}{x} \quad \text{as} \quad x \to 0^{+}, \]
by a known property of the Riemann zeta function or by Lemma 2 (below) with \( \lambda_{a_n} = \log n \). Consequently, as \( x \to 0^{+} \),
\[ a^n(x) = a(1) - \int_{x}^{1} \frac{a'(t)}{t} \, dt \sim \frac{1}{x} \quad \text{as} \quad x \to 0^{+}, \]
\[ a^n(x) = a(1) - \int_{x}^{1} a'(t) \, dt \sim \frac{1}{x} \quad \text{as} \quad x \to 0^{+}. \]

6. Auxiliary results

The following two lemmas show how to generate Dirichlet series \( a(z) \) that satisfy (1). The conditions in the lemmas on \( L(z) \) are satisfied when \( L(z) \) is (for large \( z \)) a logarithmico-exponential function (see [4]) in the range
\[ x < L(z) < e^x. \]
Examples of such functions are given by
\[ L(x) = (\log x)^{\kappa} (\log \log x)^{\nu}, \]
where \( \kappa, \nu \) are real numbers.

**Lemma 2.** Suppose that \( \rho > \delta > 0, \epsilon > 0, \epsilon > 0 \) and that \( L(x) \) is a positive continuous function on \((0, \infty)\) such that \( x^{\rho} L(x) \) is increasing and \( x^{\rho} L(x) \) is decreasing on \((\epsilon, \infty)\), and
\[
\lim_{x \to \infty} \frac{L(t)}{L(x)} = 1 \quad \text{for all} \quad t > 0.
\]
Suppose also that \( \lambda_{n+1} \sim \lambda_n \) and
\[
a_n \sim (\lambda_{n+1} - \lambda_n) L_{n+1} L_{n+1}. \]
Then
\[
a(x) \sim \frac{1}{\Gamma(p+1)} \frac{1}{\lambda_n} \sim \frac{\lambda_{n+1}}{\rho} L_{n+1},
\]
and
\[
A_n \sim \frac{1}{\Gamma(p+1)} \frac{1}{\lambda_n} \sim \frac{\lambda_{n+1}}{\rho} L_{n+1}. \]

**Proof:** Suppose without loss of generality that \( \lambda_1 > \lambda_0 = 0 \), and let
\[
b_n := (\lambda_{n+1} - \lambda_n) L_{n+1} L_{n+1} \quad \text{for} \quad n = 1, 2, \ldots.
\]
Given \( \gamma > 1 \), there is a positive integer \( N \) such that \( \lambda_{n+1} > c \) and \( \lambda_n < c \lambda_{n+1} \) for \( n > N \). Hence, for \( n > N \) and \( n+1 \leq \iota \leq n+1 \), we have \( n+1 < \iota \leq n+1 \) and so
\[
t \rho t^{\rho-1} (y t^{\rho})^{\gamma-1} e^{-\gamma t} \leq \lambda_{n+1} L_{n+1} L_{n+1} e^{-\lambda_n x} \leq (\gamma t)(t^{\rho}) (y t^{\rho})^{\gamma-1} e^{-\gamma t} e^{-\lambda_n x}
\]
for \( x > 0 \). It follows that
\[
y^{\gamma-1} t^{\rho} \int_{x=a}^{\infty} (y t^{\rho})^{\gamma-1} e^{-\gamma t} \, dt \leq \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) L_{n+1} L_{n+1} e^{-\lambda_n x}
\]
and hence that
\[
y^{\gamma-1} \int_{x=a}^{\infty} y^{\gamma} x^{\rho} e^{-\gamma x} \, dx \leq \frac{\gamma}{\Gamma(p+1)} \frac{1}{\lambda_n} \sum_{n=1}^{\infty} b_n e^{-\lambda_n x}.
\]
Observe next that, for \( y \geq e \) and \( t \geq c/y \),
\[
\frac{L(t)}{L(y)} = \left( \frac{\gamma t^{\rho}}{\rho} \right)^{\gamma} L(t), \quad t \leq c \leq t^{-1} \quad \text{when} \quad t \leq 1,
\]
and
\[
\frac{L(t)}{L(y)} = \left( \frac{\gamma t^{\rho}}{\rho} \right)^{\gamma} L(t), \quad t \leq c \quad \text{when} \quad t > 1.
\]

Therefore (1) is satisfied with \( b(x) \) in place of \( a(x) \) and so, by Theorem 4,
\[
B_n \sim \frac{1}{\Gamma(p+1)} \frac{1}{\lambda_n} \frac{\lambda_{n+1}}{\rho} L_{n+1} L_{n+1}.
\]

This then completes the proof of the theorem.
Finally, since $B_n \to \infty$, the methods $M_\lambda$ and $D_{\lambda_i}$ are regular; and hence, since $a_n \sim b_n$, we have
\[ a(x) \sim b(x) = \Gamma(p)x^{L_n(1)} \quad \text{as } x \to 0^+, \]
and
\[ A_n \sim B_n \sim \frac{1}{\Gamma(p+1)} a_n \sim \frac{\lambda_n}{p} L(\lambda_n). \]

**Lemma 3.** If the function $L_n(x)$ and the sequence $(\lambda_n)$ satisfy the conditions of Lemma 2 and
\[ A_n \sim B_n \sim a_n, \]
then
\[ a(x) \sim \Gamma(p)x^{L_n(1)} \quad \text{as } x \to 0^+. \]

**Proof.** Let $x > 0$. Suppose without loss of generality that $\lambda_1 > 0$, and let
\[ B_n = \sum_{k=1}^n b_k = \lambda_n L(\lambda_n) \quad \text{for } n = 1, 2, \ldots. \]

Then $B_n \sim a_n = o(1)$ as $n \to \infty$, and hence, as in the proof of Theorem 1,
\[ b(x) = \sum_{n=1}^\infty B_n x^{\lambda_n \psi(x)} = \lambda_n L(\lambda_n) x^{\lambda_n \psi(x)}, \]

Likewise,
\[ a(x) = \sum_{n=1}^\infty A_n x^{\lambda_n \psi(x)}. \]

and, since $\rho A_n \sim B_n$, it follows, again as in the proof of Theorem 1, that $\rho a(x) \sim b(x)$ as $x \to 0^+$. Thus it suffices to prove that
\[ b(x) \sim \Gamma(p+1)x^{L_n(1)} \quad \text{as } x \to 0^+. \]

Given $\gamma > 1$, there is a positive integer $N$ such that $\lambda_n > \gamma$ and $\lambda_{n+1} < \gamma \lambda_n$ for $n \geq N$. Hence, for $n \geq N$ and $k < n$, we have $t \sim \gamma \lambda_n$, and so
\[ \gamma^{-\sigma} L_{\sigma}(t) = \gamma^{-\sigma} \gamma^{-\sigma} L(t) \leq \gamma^{-\lambda_n \psi(x)} L(\lambda_n) = \lambda_n \psi(x) \sim \rho L(t). \]

It follows that, for $n \geq N$,
\[ \gamma^{\sigma} L(t) \sim \gamma^{-\lambda_n \psi(x)} L(t) \sim \gamma^{-\lambda_n \psi(x)} L(t) \sim \rho L(t). \]

and hence that
\[ \gamma^{-\sigma} \int_{x_n}^{x_{n+1}} y^\frac{1}{\gamma} \frac{L_n(1)}{L(1)} \, dy \leq b_n x^{\lambda_n \psi(x)} \int_{x_n}^{x_{n+1}} L_n(1) e^{L_n(1)} \, dy, \]

where $\gamma = \log \lambda_n$ and $L(t)$ is a function (defined for $x > 0$) satisfying
\[ \lim_{x \to 0^+} \frac{L(t)}{L(x)} = 1 \quad \text{for all } t > 0, \]

and
\[ A_n \sim \frac{1}{\Gamma(p+1)} a_n \sim \frac{\lambda_n}{\Gamma(p+1)} L(\lambda_n). \]
Proof. Let
\[ c(x) := a(x) + Hb(x). \]

Then, for \( m = 1, 2, \)
\[ \frac{c(mx)}{c(x)} = \frac{a(mx)}{a(x)} + H \left( \frac{b(mx)}{b(x)} \right) \frac{a(x)}{c(x)} \alpha_n \text{ as } x \to 0^+. \]

since \( b(x) = O(c(x)) \) as \( x \to 0^+ \). Further, \( c_n \geq 0 \) for \( n = 1, 2, \ldots \). It follows, by Lemma 1, that, for all \( t > 0 \),
\[ \lim_{x \to 0^+} \frac{b(tx)}{b(x)} = \lim_{x \to 0^+} \frac{c(tx)}{c(x)} = t^\beta, \]

where \( \beta = -\log_2 c_n \), and so
\[ \frac{a(tx)}{a(x)} \cdot \frac{c(tx)}{c(x)} \cdot \frac{b(tx)}{b(x)} = t^\beta \text{ as } x \to 0^+. \]

since \( b(x) = O(a(x)) \) as \( x \to 0^+ \). If we now define
\[ L(x) := x^{-\alpha} \left( \frac{1}{x} \right), \]

we see that
\[ \frac{L(tx)}{L(x)} = t^\beta \cdot a \left( \frac{1}{x} \right) \cdot t^\beta = 1 \text{ as } x \to \infty. \]

Finally, we have, by Theorem 4, that
\[ B_n \sim \frac{1}{\Gamma(\rho+1)} b \left( \frac{1}{x_n} \right) \text{ and } C_n \sim \frac{1}{\Gamma(\rho+1)} \left( \frac{1}{x_n} \right). \]

and so
\[ \frac{A_n}{a(1/\lambda_n)} \sim C_n \left( \frac{B_n}{c(1/\lambda_n)} - \frac{C_n}{c(1/\lambda_n)} \right) \frac{b(1/\lambda_n)}{c(1/\lambda_n)} \cdot a(1/\lambda_n) \sim \frac{1}{\Gamma(\rho+1)} \text{ as } x \to 0^+. \]

The following corollary to Theorem 6 generalizes Hardy and Littlewood's theorem E in [3].

**Corollary.** Suppose that \( \rho > 1 > 0, c > 0, \alpha > 0 \) and that \( K(x) \) is a positive continuous function on \((0, \infty)\) such that \( x^\delta K(x) \) is increasing and \( x^{-\alpha} K(x) \) is decreasing on \((0, \infty)\), and
\[ \lim_{x \to \infty} K(x) = 0 \text{ for all } t > 0. \]

Suppose that \( \lambda_{n+1} \sim \lambda_n \) and
\[ a_n \geq -H(\lambda_n - \lambda_{n+1}) \lambda_n^{-\delta} K(\lambda_n) \text{ for } n = 2, 3, \ldots. \]

where \( H \) is a positive constant, and that the Dirichlet series \( a(x) \) is convergent for all \( x > 0 \). Suppose also that
\[ \lim_{x \to \infty} x^{-\alpha} a(x) = 0, \]
and that
\[ \lim_{x \to \infty} x^{-\alpha} a(x) = m^\tau \text{ for } m = 2 \text{ and } m = 3. \]

Then
\[ a(x) = x^\tau L \left( \frac{1}{x} \right) \text{ for } x > 0, \]

where \( L(x) \) is a function (defined for \( x > 0 \)) satisfying
\[ \lim_{x \to \infty} L(x) = 1 \text{ for all } t > 0, \]

and
\[ A_n \sim \frac{1}{\Gamma(\rho+1)} \left( \frac{1}{\lambda_n} \right) = \frac{\lambda_n^\delta}{\Gamma(\rho+1)} L(\lambda_n). \]

Proof. Suppose without loss of generality that \( b_1 := 1 \) and \( a_1 := -H \). Let
\[ b_n := (\lambda_n - \lambda_{n+1}) \lambda_n^{-\delta} K(\lambda_n) \text{ for } n = 2, 3, \ldots. \]

Then, by Lemma 2, the Dirichlet series \( b(x) \) is convergent for all \( x > 0 \), and
\[ b(x) \sim \Gamma(\rho) x^{-\alpha} K \left( \frac{1}{x} \right) \text{ as } x \to 0^+. \]

The result now follows from Theorem 6.

Remarks. In view of Theorem 18 in [9], the integers 2, 3 in (1), (14) and (15) can be replaced by any pair of positive numbers \( p, q \) such that \( \log_2 p \) is irrational. It is known (see [9], p. 49) that the hypothesis

(\( H \)) \( L(x) \) is a positive continuous function on \((0, \infty)\) such that, for some \( c > 0 \) and every \( \delta > 0, x^\delta L(x) \) is increasing and \( x^{-\alpha} L(x) \) is decreasing on \((c, \infty)\)

implies that
\[ \lim_{x \to \infty} \frac{L(x)}{x^\alpha} = 1 \text{ for all } t > 0. \]

Thus the hypotheses on \( L(x) \) in Lemmas 2 and 3 can be replaced by (\( H \)), and the hypotheses on \( K(x) \) in the corollary of Theorem 6 can be replaced by (\( H \)) with \( K(x) \) in place of \( L(x) \).

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.
REFERENCES


**Asymptotic behaviour of the $H$-transform in the complex domain**

BY RICHARD D. CARMICHAEL

Wake Forest University, Winston-Salem, NC 27109, U.S.A.

AND RAM S. PATHAK

Banaras Hindu University, Varanasi 221005, India

(Received 20 November 1986)

**Abstract**

Abelian theorems for the $H$-transform of functions and generalized functions are obtained as the complex variable of the transform approaches zero or infinity in a wedge domain in the right half plane. Quasi-asymptotic behaviour (q.a.b.) of the $H$-transformable generalized functions is defined. A structure theorem for generalized functions possessing q.a.b. is proved and is applied to obtain the asymptotic behaviour of the $H$-transform of generalized functions having q.a.b. The theorems are illustrated by examples.

1. Introduction

Asymptotic behaviour of transforms of functions is studied in the books by Boas [1], Doetsch [6], Sneddon [19] and Widder [23], where considerations of the Fourier, Laplace, and Stieltjes transforms are given. Theorems obtaining the behaviour of the transform from that of the defining function are called Abelian theorems; these theorems relate the initial value ($t = 0$) and final value ($t = \infty$) of a function $f(t)$ to the asymptotic behaviour of its transform $F(s)$ ($|s| \to 0$ or $|s| \to \infty$). Such theorems are useful in solving boundary value problems.

Zemanian [25] obtained Abelian theorems for the Hankel and $K$-transforms of functions and then extended his results to the corresponding transforms of distributions in the sense of Schwartz [18]. Jones [9] has discussed at length the asymptotic behaviour of transforms of generalized functions in his sense. The asymptotic behaviour of transforms of certain Schwartz distributions has been given by Zayed [24], but Zayed’s theorems do not cover our general transform. Following the technique of Zemanian [25] many authors have obtained Abelian theorems for more general transforms of functions and generalized functions; for example we note Joshi and Saxena [10], Miura [15], Pathak [16] and Saxena [17]. However, these authors confined their attention to transforms with real variables only.

Carmichael and Milton [4] obtained Abelian theorems for the distributional Stieltjes transform when the variable $s$ of the transform was complex. They allowed $s$ to tend to zero or infinity in the wedge domain $|s| = \sigma_1 + i\sigma_2; \sigma_1 > 0, |\sigma_2| < K \sigma_1$. They also used Lojasiewicz’s definition of limit of a distribution and thus extended the work of Lavoine and Miura [11]. Takači [21] applied the idea of quasi-asymptotic