CESARO AND BOREL-TYPE SUMMABILITY
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ABSTRACT. Though summability of a series by the Cesàro method $C_p$ does not in general imply its summability by the Borel-type method $(B, \alpha, \beta)$, it is shown that the implication holds under an additional condition.

1. Introduction. Suppose throughout that $\sum_{n=0}^{\infty} a_n$ is a series with partial sums $s_n := \sum_{k=0}^{n} a_k$, and that $\alpha > 0$ and $\alpha N + \beta > 0$ where $N$ is a nonnegative integer. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $(B, \alpha, \beta)$ to $s$ if

$$\alpha e^{-x} \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \to s \quad \text{as } x \to \infty.$$ 

The Borel-type summability method $(B, \alpha, \beta)$ is regular, and $(B, 1, 1)$ with $N = 0$ is the standard Borel summability method $B$.

We shall also be concerned with the Cesàro summability method $C_p$ ($p > -1$) and the Valiron method $V_\alpha$ defined as follows:

$$\sum_{n=0}^{\infty} a_n = s(C_p) \quad \text{if } c_n^p := \frac{s_n^p}{(n+p)} \to s \quad \text{as } n \to \infty,$$

where

$$s_n^p := \sum_{k=0}^{n} \left( \frac{n-k+p-1}{n-k} \right) s_k;$$

$$\sum_{n=0}^{\infty} a_n = s(V_\alpha) \quad \text{if } \left( \frac{\alpha}{2\pi n} \right)^{1/2} \sum_{k=0}^{\infty} \exp \left( -\frac{\alpha (n-k)^2}{2n} \right) s_k \to s \quad \text{as } n \to \infty.$$ 

Consider the series $\sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty} n^{a-1} \exp(A in^a)$ where $A > 0$ and $0 < a < 1/2$. It is known [5, p. 213] that this series is summable $C_p$ for every $p > 0$ but is not convergent. However, since $a_n \sim o(n^{-1/2})$, it follows by the Borwein Tauberian Theorem [1, Theorem 1] that the series is not summable $(B, \alpha, \beta)$ for any $\alpha$ and $\beta$. This example shows that, in general, summability $C_p$ does not imply summability $(B, \alpha, \beta)$. The following theorem indicates how to strengthen the $C_p$ summability hypothesis in order to ensure summability $(B, \alpha, \beta)$.

THEOREM 1. Suppose that $p$ is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$.

The special case $\alpha = \beta = 1$, $p = 1$ of Theorem 1 has been proved by Hardy [5, Theorem 149]. Hardy and Littlewood [4, §3] proved that the condition

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\[ c_n^p = s + o(n^{-1/2}) \] is not sufficient for the summability of \( \sum a_n \) by the Borel method. Hyslop [7, Theorem VIII] has obtained a more general result than Hardy, namely the case \( \alpha = \beta = 1 \) of Theorem 1. More recently, Swaminathan [10] has proved Theorem 1 with \( p = 1 \) and \( (B, \alpha, \beta) \) summability replaced by the more general \( F(a, q) \) summability introduced by Meir [9].

2. Preliminary results.

**Lemma 1** [8, Lemma 7]. Let \( m < x_0 < n - 1 \) where \( m, n \) are integers and let the nonnegative function \( f(x) \) be increasing on \([m, x_0]\) and decreasing on \([x_0, n]\). Then

\[
\sum_{k=m}^{n} f(k) \leq \int_{m}^{n} f(x)dx + f(x_0).
\]

**Lemma 2** [2, Theorem 3]. Suppose that \( s_n = O(n^r) \) where \( r \geq 0 \). Then \( \sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta) \) if and only if \( \sum_{n=0}^{\infty} a_n = s(V_\alpha) \).

**Theorem 2** (Cf. [6, Theorem 2]). Suppose that \( p \) is a nonnegative integer and that \( c_n^p = s + o(n^{-p/2}) \) as \( n \to \infty \). Then \( \sum_{n=0}^{\infty} a_n = s(V_\alpha) \).

**Proof.** Suppose, as we may without loss of generality, that \( s = 0 \). Let \( v_n(x) := \exp(-\alpha(n-x)^2/2n) \) and denote the \( p \)th difference of \( v_n(k) \) by \( \Delta^p v_n(k) \), so that

\[
\Delta^p v_n(k) = \sum_{r=0}^{p} \binom{p}{r} (-1)^r v_n(k+r).
\]

Applying Abel’s partial summation formula \( p < m \) times, we have that

\[
\sum_{k=0}^{m} s_k v_n(k) = \sum_{k=0}^{m-p} s_k^p \Delta^p v_n(k) + \sum_{r=0}^{p-1} s^r_{m-r} \Delta^r v_n(m-r).
\]

Letting \( m \to \infty \) and applying the limitation theorem for Cesaro summability [5, Theorem 46], we see that

\[
F(n) := \sum_{k=0}^{\infty} s_k v_n(k) = \sum_{k=0}^{\infty} s_k^p \Delta^p v_n(k).
\]

In order to prove the theorem we must show that \( F(n) = o(n^{1/2}) \). Since, by the hypothesis, \( s_k^p = o(k^{p/2}) \) as \( k \to \infty \) and \( k^{p/2} \Delta^p v_n(k) = o(n^{1/2}) \) as \( n \to \infty \), it suffices to show that

\[
G(n) := n^{-1/2} \sum_{k=0}^{\infty} k^{p/2} |\Delta^p v_n(k)|
\]

is bounded.

It is familiar that \( \Delta^p v_n(k) = (-1)^p v_n^{(p)}(k+c) \) for some \( c \in [0, p] \). Hence there is a \( \theta = \theta(n,k) \in [0, p] \) such that

\[
|\Delta^p v_n(k)| \leq |v_n^{(p)}(k+c)|.
\]
Since \( v_n^{(p)}(x) = v_n(x) \sum_{0 \leq r \leq p/2} b_r (n-x)^{p-2r} n^{r-p} \), where the \( b_r \)'s are constants, we get from (1) and (2) that
\[
G(n) = O \left( \sum_{0 \leq r \leq p/2} |b_r| n^{-r-1/2} \sum_{k=0}^{\infty} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) \right).
\]
Therefore to establish that \( G(n) \) is bounded it is enough to show that, for \( 0 \leq r \leq p/2 \) and \( 0 \leq \theta \leq p \),
\[
H(n) := \sum_{k=0}^{\infty} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) = O(n^{p-r+1/2}).
\]
Write
\[
H(n) = \left\{ \begin{align*}
&\sum_{k=0}^{n-p-1} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) \\
&+ \sum_{k=n-p}^{n} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) \\
&+ \sum_{k=n+1}^{\infty} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) \end{align*} \right.
\]
Since \(|n - k - \theta| \leq 2p\) for \( 0 \leq \theta \leq p \) and \( n - p \leq k \leq n \), and \( 0 < v_n(k + \theta) \leq 1 \), it is immediate that
\[
\sum_{2} = O(n^{p/2}).
\]
Next, setting \( f(x) := x^{p-2r} \exp(-\alpha x^2/2n) \) and applying Lemma 1, we have that
\[
\sum_{1} \leq \sum_{k=0}^{n-p-1} k^{p/2} (n - k)^{p-2r} v_n(k + p) \leq \sum_{k=p}^{n-1} k^{p/2} (n - k + p)^{p-2r} v_n(k) \\
\leq M n^{p/2} \sum_{k=p}^{n-1} f(n - k) \leq M n^{p/2} \sum_{k=1}^{n} f(k) \\
\leq M n^{p/2} \int_{1}^{n} f(x) dx + M C n^{p/2} \left( \frac{(p-2r)n}{\alpha} \right)^{p/2-r}
\]
where \( M := (1 + p)^{p-2r} \) and \( C := \exp(r - p/2) \). Letting \( u = \alpha x^2/2n \), we get that
\[
\sum_{1} = O \left( n^{p-r+1/2} \int_{0}^{\infty} u^{(p-1)/2} e^{-u} du \right) + O(n^{p-r}) = O(n^{p-r+1/2}).
\]
Further, with \( M \) and \( f(x) \) as above and \( g(x) := x^{3p/2-2r} \exp(-\alpha x^2/2n) \), we see that
\[
\sum_{3} \leq \sum_{k=n+1}^{\infty} k^{p/2} (k - n + p)^{p-2r} v_n(k) \\
\leq M \left( \sum_{k=n+1}^{2n} + \sum_{k=2n+1}^{\infty} \right) k^{p/2} (k - n)^{p-2r} v_n(k) \\
\leq M (2n)^{p/2} \sum_{k=1}^{n} f(k) + M 2^{p/2} \sum_{k=n+1}^{\infty} g(k) := \sum_{3.1} + \sum_{3.2}.
\]
As above \( \sum_{3,1} = O(n^{p-r+1/2}) \). And finally, as \( n \to \infty \),

\[
\sum_{3,2} = O \left( \int_0^\infty g(x)dx \right) + o(1) = O \left( n^{3p/4-r+1/2} \int_{\alpha n/2}^\infty u^{3p/4-r-1/2}e^{-u} du \right) + o(1) = o(1).
\]

Thus,

\[
(6) \quad \sum_3 = O(n^{p-r+1/2}) + o(1) \quad \text{as} \quad n \to \infty.
\]

It now follows from (3)–(6) that \( H(n) = O(n^{p-r+1/2}) \). This completes the proof. \( \Box \)

3. **Proof of Theorem 1.** The limitation theorem for Cesàro summability [5, Theorem 46] implies that \( s_n = o(n^p) \). Therefore, by Theorem 2 and Lemma 2, we have that \( \sum_{n=0}^\infty a_n = s(B, \alpha, \beta) \). \( \Box \)

4. **Related results.** The methods of Euler \( E_\delta \), Meyer-Konig \( S_\delta \), and Taylor \( T_\delta \) \((0 < \delta < 1)\) are defined as follows:

\[
\sum_{n=0}^\infty a_n = s(E_\delta) \quad \text{if} \quad \sum_{k=0}^{n} \binom{n}{k} \delta^k (1-\delta)^{n-k} s_k \to s \quad \text{as} \quad n \to \infty;
\]

\[
\sum_{n=0}^\infty a_n = s(S_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_k \to s \quad \text{as} \quad n \to \infty;
\]

\[
\sum_{n=0}^\infty a_n = s(T_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_{n+k} \to s \quad \text{as} \quad n \to \infty.
\]

These methods, as well as the Borel-type and Valiron methods, are contained in the \( F(a,q) \) family of methods mentioned in the introduction. The following theorem generalizes Swaminathan’s result [10], via Theorem 2 and [3, Satz III], for the Euler, Meyer-Konig, and Taylor methods.

**THEOREM 3.** Suppose that \( p \) is a nonnegative integer and that \( c_n^p = s+o(n^{-p/2}) \) as \( n \to \infty \). Then for \( 0 < \delta < 1 \), the series \( \sum_{n=0}^\infty a_n \) is summable to \( s \) by the \( E_\delta \), \( S_\delta \), and \( T_\delta \) methods.

**REFERENCES**


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