ON STRONG GENERALIZED HAUSDORFF SUMMABILITY

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Introduction

For a series $\sum_0 a_k$, let $s_n = \sum_k a_k$. Let $Q = \{q_{n,k}\}$ $(n, k = 0, 1, \ldots)$ be a matrix and let

$$\sigma_n = Q(s_n) = \sum_k q_{n,k} s_k.$$ 

The series $\sum_0 a_k$ is said to be summable $Q$ to $s$ if $\sigma_n$ exists for $n = 0, 1, \ldots$ and tends to $s$ as $n$ tends to infinity. In this case we write $s_n \rightarrow s(Q)$. The symbol $P$ is reserved for matrices $\{p_{n,k}\}$ with $p_{n,k} \geq 0$, and $I$ denotes the identity matrix. We now recall the definition of strong summability introduced by Borwein [1].

Strong summability. A series $\sum_0 a_k$ is said to be summable $[P, Q]_\beta$ ($\beta > 0$) to $s$ if $\sum_{k=0}^n p_{n,k} |\sigma_k - s|^{\beta}$ exists for $n = 0, 1, \ldots$ and tends to zero as $n$ tends to infinity. In this case we write $s_n \rightarrow s[P, Q]_\beta$.

For summability methods $V$ and $W$, the notation $V \subseteq W$ means that any series summable $V$ to $s$ is also summable $W$ to $s$. The notation $V \approx W$ means that both $V \subseteq W$ and $W \subseteq V$.

Generalized Hausdorff matrices. Suppose throughout that $\lambda = \{\lambda_n\}$ is a sequence of real numbers with

$$\lambda_0 \equiv 0, \quad \inf \lambda_n > 0 \quad \text{and} \quad \sum_{n=0} \lambda_n = \infty.$$ 

Let $\Omega$ be a simply connected region that contains every positive $\lambda_n$, and suppose, for $n = 0, 1, \ldots$, that $\Gamma_n$ is a positively sensed Jordan contour lying in $\Omega$ and enclosing every $\lambda_k \in \Omega$ with $0 \leq k \leq n$. Suppose that $f$ is holomorphic in $\Omega$ and that $f(\lambda_0)$ is defined even when $\lambda_0 \in \Omega$. Define

$$\lambda_{n,k} = \begin{cases} -\lambda_{k+1} \cdots \lambda_n \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(z) dz}{(\lambda_k - z) \cdots (\lambda_n - z)} + \delta_k & \text{for } 0 \leq k \leq n \\ 0 & \text{for } k > n \end{cases}$$

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where \( \delta_k = f(\lambda_0) \) if \( k = 0 \) and \( \lambda_0 \in \Omega \), and \( \delta_k = 0 \) otherwise. Here and elsewhere we observe the convention that products like \( \lambda_{k+1} \cdots \lambda_n = 1 \) when \( k = n \). Denote the triangular matrix \( \{\lambda_{n,k}\} \) by \((\lambda; f)\). This is called a generalized Hausdorff matrix. The set of all such matrices is denoted by \( \mathcal{H}_\lambda \).

For \( \alpha \) any real number, the generalized Hausdorff matrix \( H_\alpha \) is defined to be the matrix \((\lambda; f)\) with \( f(z) = (z+1)^{-\alpha} \). For \( \alpha > -1 \), the generalized Cesàro matrix \( C_\alpha \) is defined to be the matrix \((\lambda; f)\) with

\[
f(z) = \frac{\Gamma(\alpha+1)\Gamma(z+1)}{\Gamma(\alpha+z+1)}.
\]

These reduce to the standard Hölder and Cesàro matrices when \( \lambda_n = n \). (See [1].)

### Preliminary results

For \( 0 < t \leq 1 \), let \( \lambda_{n,k}(t) \) denote the value of \( \lambda_{n,k} \) obtained from (1) with \( f(z) = t^z \), and let \( \lambda_{n,k}(0) = \lambda_{n,k}(0+) \). Let

\[
D_0 = (1 + \lambda_0) d_0 = 1;
\]

\[
D_n = \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n) d_n \quad \text{for} \quad n \geq 1.
\]

Then, (see [3]),

\[
\int_0^1 \lambda_{n,k}(t) \, dt = \frac{d_k}{D_n} \quad \text{for} \quad 0 \leq k \leq n.
\]

If

\[
f(z) = \int_0^1 t^z \, d\chi(t) \quad \text{with} \quad \chi \in \text{BV}
\]

where \( \text{BV} \) is the space of functions of bounded variation on the closed interval \([0, 1]\), then

\[
\lambda_{n,k} = \int_0^1 \lambda_{n,k}(t) \, d\chi(t).
\]

It follows that

\[
C_1(s_n) = \frac{1}{D_n} \sum_{k=0}^n d_k s_k
\]

so that

\[
s_n - C_1(s_n) = C_1(\lambda_n a_n).
\]

If \( f \) satisfies (2), \( \chi(1) - \chi(0) = 1 \) and \( \chi(0+) - \chi(0) \), then \( X = (\lambda; f) \) is regular, i.e. \( s_n \to s(X) \) whenever \( s_n \to s \). (See [2; Theorem 1].)

Lemma 2 of [2] shows that if \( g \) and \( h \) are holomorphic in \( \Omega \) and defined at \( \lambda_0 \), \( X = (\lambda; g) \) and \( Y = (\lambda; h) \), then

\[
XY = YX = (\lambda; gh).
\]
Lemma 1 of [3] shows that if \( X=(\lambda; f) \) with \( f \) satisfying (2), \( \bar{X}=(\lambda; \bar{f}) \) with \( \bar{f}(z) = \int_0^1 t^\alpha |d\bar{X}(t)| \), and \( \beta \geq 1 \), then, for any sequence \( \{w_n\} \),

\[
|X(w_n)^\beta| \leq M^{\beta-1} \bar{X}(|w_n|^\beta)
\]

where \( M = \int_0^1 |d\bar{X}(t)| \).

From (4) it can be seen that \( H_a H_\delta = H_{a+\delta} \) for all real \( \alpha, \delta \). Theorem 2 of [2] shows that

\[
C_\alpha \simeq H_\alpha \quad \text{for} \quad \alpha > -1.
\]

(See also [5] and [6].) Thus

\[
C_\alpha C_\delta \simeq C_{\alpha+\delta} \quad \text{for} \quad \alpha > -1, \quad \delta > -1, \quad \alpha + \delta > -1.
\]

### Some theorems on strong summability

The first theorem generalizes Theorem 5 in [1].

**THEOREM 1.** Suppose \( Q \) is a matrix, \( P \) is a regular matrix in \( \mathcal{H}_2 \), and \( X=(\lambda; f) \) where \( f(z) = \int_0^1 t^\alpha |dX(t)| \) with \( X \in \mathcal{BV}, \chi(1)-\chi(0)=1 \) and \( \chi(0+)=\chi(0) \). Then, for \( \beta \geq 1 \), \( [P, Q]_\beta \subseteq [P, XQ]_\beta \).

**PROOF.** Let \( \bar{X}=(\bar{x}_{n,k})=(\lambda; \bar{f}) \) where \( \bar{f}(z) = \int_0^1 t^\alpha |d\bar{X}(t)| \). Since \( \chi \in \mathcal{BV} \) and \( \chi(0+)=\chi(0) \), it follows that \( \lim_{n \to \infty} \bar{x}_{n,k} = 0 \) for \( k=0, 1, \ldots \) and \( \sup_{n \to \infty} \sum_{k=0}^n |\bar{x}_{n,k}| < \infty \). (See [2, Theorem 1].) Hence \( \bar{X}(u_n) \to 0 \) whenever \( u_n \to 0 \). (See [4, Theorem 4].)

Let \( \{s_n\} \) be a sequence, \( \sigma_n=X(s_n) \) and \( w_n=s_n-s \). In view of the regularity of \( X \) we have \( \sigma_n-s=X(w_n)+\varepsilon_n \) where \( \varepsilon_n \to 0 \). From (4) and (5) it follows that

\[
P(|X(w_n)|^\beta) \leq M^{\beta-1} \bar{X}(|w_n|^\beta) = M^{\beta-1} \bar{X} P(|s_n-s|^\beta)
\]

where \( M = \int_0^1 |d\bar{X}(t)| \). Next, by Minkowski’s inequality,

\[
(P(|X(w_n)|^\beta + \varepsilon_n^\beta))^{1/\beta} \leq (P(|X(w_n)|^\beta))^{1/\beta} + (P(|\varepsilon_n|^\beta))^{1/\beta}.
\]

Suppose now that \( P(|s_n-s|^\beta) \to 0 \). Then, by (8), \( P(|X(w_n)|^\beta) \to 0 \) so that, by (9), \( P(|\sigma_n-s|^\beta) \to 0 \). Hence \( [P, I]_\beta \subseteq [P, X]_\beta \), from which it follows that \( [P, Q]_\beta \subseteq [P, XQ]_\beta \). □

The next two theorems generalize corollaries to Theorem 7 in [1].
THEOREM 2. If \( X \in \mathcal{H}_2 \) and \( \beta \geq 1 \), then necessary and sufficient conditions for a series \( \sum_{n=0}^{\infty} a_n \) to be summable \([C_1, X]_{\beta}\) to \( s \) are that it be summable \( C_1 X \) to \( s \) and that \( \lambda_n a_n \to 0[C_1, C_1 X]_{\beta} \).

PROOF. It follows from Theorem 1 in [1] that \( \sum_{n=0}^{\infty} a_n \) is summable \([C_1, X]_{\beta}\) to \( s \) if and only if it is summable \( C_1 X \) to \( s \) and summable \([C_1, (I-C_1) X]_{\beta}\) to 0. Further, by (3) and (4),

\[
(I-C_1)X(s_n) = X(s_n - C_1(s_n)) = C_1 X(\lambda_n a_n).
\]

The result follows. \( \square \)

In conformity with notation introduced earlier (see [1; p. 123]), the generalized strong Cesàro method \([C_1, C_{\alpha-1}]_{\beta}\) will be denoted by \([C, \alpha]_{\beta}\) and the generalized strong Hölder method \([H_1, H_{\alpha-1}]_{\beta}\) by \([H, \alpha]_{\beta}\). We require the following known result (see [8]).

**Lemma 1.** Let \( g(z) = \frac{\Gamma(\delta+1)\Gamma(z+1)(z+1)^{\delta}}{\Gamma(z+\delta+1)} \), \( \delta > -1 \).

Then both \( g(z) \) and \( 1/g(z) \) can be expressed as Mellin transforms of the form

\[
\int_{0}^{1} t^{z-1} \, d\chi(t) \quad \text{with} \quad \chi \in BV, \chi(1) - \chi(0) = 1 \quad \text{and} \quad \chi(0+) = \chi(0).
\]

**Theorem 3.** If \( \alpha > 0 \) and \( \beta \geq 1 \), then necessary and sufficient conditions for a series \( \sum_{n=0}^{\infty} a_n \) to be summable \([C, \alpha]_{\beta}\) to \( s \) are that it be summable \( C_\alpha \) to \( s \) and that \( \lambda_n a_n \to 0[C, \alpha+1]_{\beta} \).

PROOF. It follows from Theorem 2 that \( \sum_{n=0}^{\infty} a_n \) is summable \([C, \alpha]_{\beta}\) to \( s \) if and only if it is summable \( C_1 C_{\alpha-1} \) to \( s \) and \( \lambda_n a_n \to 0[C_1, C_1 C_{\alpha-1}]_{\beta} \). Next, it follows from Lemma 1 with \( \delta = \alpha - 1 \) and Theorem 1 that \( \lambda_n a_n \to 0[C_1, C_1 C_{\alpha-1}]_{\beta} \) if and only if \( \lambda_n a_n \to 0[C_1, H_\alpha]_{\beta} \). Applying Lemma 1 and Theorem 1 again, we see that \( \lambda_n a_n \to 0[C_1, C_1 C_{\alpha-1}]_{\beta} \) if and only if \( \lambda_n a_n \to 0[C_1, C_\alpha]_{\beta} \). This together with (7) yields the result. \( \square \)

The above theorem suggests the following extension of the definition of \([C, \alpha]_{\beta}\) to the case \( \alpha = 0 \): \( \sum_{n=0}^{\infty} a_n \) is summable \([C, 0]_{\beta}\) to \( s \) if the series is convergent with sum \( s \) and \( \sum_{k=0}^{n} d_k |\lambda_k a_k|^\beta = o(D_n) \). When \( \lambda_n = n \), this definition reduces to the one given by Hyslop [7].

The next theorem is an analogue of the equivalence relation (6) for strong summability.

**Theorem 4.** For \( \alpha > 0 \), \( \beta \geq 1 \), \([C, \alpha]_{\beta} \simeq [H, \alpha]_{\beta} \).
Proof. The case \( e = 0 \) follows from Theorem 2 and the definition of \([C, 0]_p\). Suppose therefore that \( e > 0 \). By Theorem 3, \( \sum_{0} a_n = s[C, a] \) if and only if \( \sum_{0} a_n = s(C_1) \) and \( \lambda_n a_n \to 0[C_1, C_0]_p \). Further, by Theorem 2, \( \sum_{0} a_n = s[H, a]_p \) if and only if \( \sum_{0} a_n = s(H_1) \) and \( \lambda_n a_n \to 0[H_1, H_0]_p \). The result now follows from (6), Lemma 1 and Theorem 1. \( \square \)

**Generalized Hausdorff matrices associated with \( L^p \) functions**

Let \( L^p \) denote the function space \( L^p(0, 1) \). In this section we deal with Hausdorff matrices \((\lambda; f)\) with \( f(z) = \int_0^1 t^z \phi(t) \, dt \) where \( \phi \in L^p \) for some \( p > 1 \). An ordinary Hausdorff matrix \( \{x_{n,k}\} \) satisfies these conditions if and only if \( \sum_{k=0}^n |x_{n,k}|^p \leq M(n+1)^{1-p} \) for \( n = 0, 1, \ldots \) where \( M \) is independent of \( n \). (See [4, Theorem 215].)

The following lemma is needed for the proof of Theorem 5.

**Lemma 2.** Let \( \phi \in L^p \) with \( p > 1 \). Let \( X = (\lambda; f) \) and \( X^{(p)} = (\lambda; f^{(p)}) \) where \( f(z) = \int_0^1 t^z \phi(t) \, dt \) and \( f^{(p)}(z) = \int_0^1 t^z |\phi(t)|^p \, dt \). If \( \mu > \beta \geq 1 \) and \( 1/p = 1/\mu - 1/\beta \), then for any sequence \( \{w_n\} \),

\[
|X(w_n)|^\mu \leq M^{\mu(1-1/\beta)}(C_1(|w_n|^\beta))^{\mu/\beta - 1} X^{(p)}(|w_n|^\beta)
\]

where \( M = \int_0^1 |\phi(t)|^p \, dt \).

Proof. Let \( f_n(t) = \sum_{k=0}^n \lambda_{n,k}(t) w_k \) where \( 0 \leq t \leq 1 \). Then, by Hölder's inequality,

\[
|f_n(t)|^\beta \leq \sum_{k=0}^n \lambda_{n,k}(t) |w_k|^\beta.
\]

(See [3, (8)].) Hence

\[
\int_0^1 |f_n(t)|^\beta \, dt \leq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t) \, dt = \frac{1}{D_n} \sum_{k=0}^n d_k |w_k|^\beta = C_1(|w_n|^\beta)
\]

and

\[
\int_0^1 |\phi(t)|^p |f_n(t)|^\beta \, dt \leq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t) |\phi(t)|^p \, dt = X^{(p)}(|w_n|^\beta).
\]
It follows, by Hölder's inequality, that

\[ |X(w_n)| = \left| \int_0^1 \varphi(t) f_n(t) \, dt \right| \leq \left( \int_0^1 |\varphi(t)|^p \, dt \right)^{1-1/\beta} \left( \int_0^1 |f_n(t)|^\beta \, dt \right)^{1/\beta - 1/\mu} \left( \int_0^1 |\varphi(t)|^p |f_n(t)| \, dt \right)^{1/\mu} \leq M^{1-1/\beta} \left( C_1(|w_n|^\beta)^{1-1/\mu} \right)^{1/\mu} \]

The following theorem generalizes Theorem 10 in [1].

**Theorem 5.** Let \( \mu > \beta \geq 1 \), \( 1/p = 1 + \mu - 1/\beta \). Let \( X = (\lambda; f) \) where \( f(z) = \int_0^1 t^p \varphi(t) \, dt \) with \( \varphi \in L^p \) and \( \int_0^1 \varphi(t) \, dt = 1 \). Then, for any matrix \( Q \), \( [C_1, Q]_\mu \subseteq [C_1, XQ]_\mu \).

The theorem remains valid when \( \mu = \infty \) (with \( 1/p = 1 - 1/\beta \) if \( \lambda > 1 \) and \( p = \infty \) if \( \beta = 1 \)) provided \( [C_1, XQ]_\infty \) is interpreted to mean \( XQ \).

**Proof.** We use the notation introduced in Lemma 2, and note that \( X \) is regular and \( X^{(p)}(v_n) \to 0 \) whenever \( v_n \to 0 \). Suppose that \( s_n \to s[C_1, Q]_\mu \), and let \( \sigma_n = Q(s_n) \), \( w_n = \sigma_n - s \), and \( v_n = C_1(|w_n|^p) \).

(i) Suppose \( \mu \) is finite. By hypothesis, \( v_n \to 0 \) and hence, by Lemma 2,

\[ C_1(|X(w_n)|^\mu) \leq K C_1 X^{(p)}(|w_n|^p) = K X^{(p)}(v_n) \to 0 \]

where \( K = M^{p(1 - 1/\beta)} \sup v_n^{1/\beta} \). Also, by the regularity of \( X \), we have \( X(\sigma_n) - s = X(w_n) + \varepsilon_n \) where \( \varepsilon_n \to 0 \). Thus, by Minkowski's inequality,

\[ (C_1(|X(\sigma_n) - s|^p)^{1/\mu} \leq (C_1(|X(w_n)|^p)^{1/\mu} + (C_1(|\varepsilon_n|^p)^{1/\mu} \to 0, \]

i.e. \( s_n \to s[C_1, XQ]_\mu \).

(ii) Suppose now that \( \mu = \infty \). By Hölder's inequality,

\[ |X(w_n)|^\beta = \left| \int_0^1 f_n(t) \varphi(t) \, dt \right|^\beta \leq m \int_0^1 |f_n(t)|^\beta \, dt \]

where \( m = M^{\beta - 1} \) if \( \beta > 1 \) and \( m = \text{ess sup} \varphi(t) \) if \( \beta = 1 \). Since (10) holds under the operative hypotheses, it follows that

\[ |X(w_n)|^\beta \leq m C_1(|w_n|^\beta) = m v_n \to 0, \]

and hence that \( s_n \to s(XQ) \). \( \square \)

**Theorem 6.** Let \( q > 1/\beta - 1/\mu \) where \( \mu \geq \beta \geq 1 \). Then, for any matrix \( Q \), \([C_1, Q]_\mu \subseteq [C_1, C_2 Q]_\mu \).
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Theorem 6. Let $\gamma = \alpha + 1/\beta - 1/\mu$ where $\mu \geq \beta \geq 1$ and $\alpha$ is any real number. Then $[H, \gamma] \subseteq [H, \mu]$. Proof. Applying first Theorem 6 and then Theorem 1 together with Lemma 1, we get

$$[H, \alpha] = [H_1, H_{\alpha-1}] = [H_1, C_{\gamma} H_{\alpha-1}] = [H_1, H_{\gamma-1}] = [H, \gamma].$$

Remark. It is known that, in the special case $\lambda_n = n$, Theorem 6 also holds when $\gamma = 1/\beta - 1/\mu$ and Theorem 7 when $\gamma = \alpha + 1/\beta - 1/\mu$. (See [1] and the references there given.) Whether the same is true for more general $\lambda_n$ is an open question.

References


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