A Tauberian theorem concerning Dirichlet series

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Abstract

It is shown that under certain general Tauberian conditions the asymptotic relationship

\[ \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \sim s \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \quad \text{as} \quad x \to 0+ \]

between two Dirichlet series implies the asymptotic relationship

\[ \sum_{k=1}^{n} a_k s_k \sim s \sum_{k=1}^{n} a_k \]

1. Introduction

Suppose throughout that \( \lambda := \{\lambda_n\} \) is a strictly increasing unbounded sequence of real numbers with \( \lambda_1 \geq 0 \), and that \( a := \{a_n\} \) is a sequence of non-negative numbers with \( a_1 > 0 \). Let

\[ A_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}. \]

Suppose also that \( A_n \to \infty \), and that the Dirichlet series \( a(x) \) is convergent for all \( x > 0 \).

Let \( \{s_n\} \) be a sequence of real numbers,

\[ t_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k s_k \quad \text{and} \quad \sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}. \]

The weighted mean summability method \( M_a \) and the Dirichlet series method \( D_{\lambda,a} \) (see [2]) are defined as follows:

\[ s_n \to s(M_a) \quad \text{if} \quad t_n \to s; \]
\[ s_n \to s(D_{\lambda,a}) \quad \text{if} \quad \sigma(x) \to s \quad \text{as} \quad x \to 0+. \]

When \( \lambda_n := \infty \) the method \( D_{\lambda,a} \) reduces to the power series method \( J_a \) (as defined in [1] for example). Since \( A_n \to \infty \) both methods are regular (i.e. \( s_n \to s \) implies \( s_n \to s(M_a) \) and \( s_n \to s(D_{\lambda,a}) \)), and also \( s_n \to s(M_a) \) implies \( s_n \to s(D_{\lambda,a}) \) (see [2], theorem 1). The purpose of this paper is to prove the following Tauberian converse of the latter result:
THEOREM. Suppose that
\[ \lambda_{n+1} = A_{n}, \]
where \( \lambda_{n}/A_{n} \to 1 \) when \( \lambda_{n}/A_{n} \to 1 \) as \( n \to \infty \),
(1)
and
\[ \sigma_{n}A_{n} = -H(\lambda_{n}-\lambda_{n+1})A_{n}, \]
(2)
where \( H \) is a positive constant, and that \( \sigma_{n} = s(D_{A_{n}}) \). Then \( \sigma_{n} = s(M_{A_{n}}) \).

A version of the theorem with (2) replaced by
\[ \sigma_{n}A_{n} = O(\lambda_{n}-\lambda_{n+1})A_{n}, \]
is known ([2], theorem 3). This latter result is a special case of the theorem since, as is easily shown, (2) is in fact a consequence of (1) and
\[ \sigma_{n}A_{n} = O(\lambda_{n}-\lambda_{n+1})A_{n}. \]
Thus the theorem also holds with (2) replaced by (4). The special case \( \lambda_{n} = n \) of the theorem has been proved by Tietz ([4], theorem 1). His paper has references to previously proved special cases.

2. Preliminary results

LEMMA 1. Suppose that (1) and (2) hold. Then
(i) \( a(x)/a(2x) = O(1) \) as \( x \to 0+ \), and
(ii) \( a(1)/\lambda_{n} = O(1/A_{n}) \).

Proof. The lemma follows directly from results proved elsewhere ([3], lemmas 5(i), 3 and 4).

LEMMA 2. Suppose that (1), (2) and (3) hold, and that \( \sigma(x) = O(1) \) as \( x \to 0+ \). Then \( \sigma_{n} = O(1) \).

Proof. Arguing as in the first half of the proof of [2], theorem 3 with \( c = 1/e \) but using lemma 1(i) (instead of \( a(x)/a(2x) = 1 \) as \( x \to 0+ \)), we deduce that
\[ \frac{1}{a(1)/\lambda_{n}} \sum_{k=0}^{n} \sigma_{k} = O(1), \]
and hence, by Lemma 1(ii), that \( \sigma = O(1) \).

LEMMA 3. Suppose that (1), (2) and (3) hold, and that \( \sigma_{n} = O(1) \). Then
\[ \lim \inf (\sigma_{n}-1) \geq 0 \quad \text{when} \quad \lambda_{n}/A_{n} \to 1, \quad n \to \infty. \]

Proof. Let \( K = \sup_{n \to \infty} x_{A_{n}} \). Since
\[ \sigma_{n} - 1 = \frac{1}{A_{n}} \sum_{k=0}^{n} \frac{\sigma_{k}}{A_{n}} = \frac{1}{A_{n}} \sum_{k=0}^{n} \frac{\sigma_{k}}{A_{n}}, \]
we have, by (3),
\[ \sigma_{n} - 1 = \frac{1}{A_{n}} \sum_{k=0}^{n} \frac{\sigma_{k}}{A_{n}} = \frac{1}{A_{n}} \sum_{k=0}^{n} \frac{\sigma_{k}}{A_{n}} = -H \sum_{k=0}^{n} \frac{\lambda_{k+1} - \lambda_{k}}{A_{n}} \left( \frac{A_{n}}{A_{n}} - 1 \right) \]
\[ \geq -H \left( \frac{\lambda_{n+1}}{A_{n}} - 1 \right) - K \left( \frac{A_{n}}{A_{n}} - 1 \right) \quad \text{for} \quad m > n > 0. \]

In view of (1) and (2), (5) follows from (6).
Finally, by a theorem proved elsewhere ([3], theorem 6), it follows from (1), (7), (9) and (11) that $t_n \to s$, i.e. $s_n \to s(M_d)$.

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REFERENCES


