TAUBERIAN THEOREMS CONCERNING POWER SERIES WITH NON-NEGATIVE COEFFICIENTS

D. BORWEIN* (London, Ontario)

1. Introduction

Suppose throughout that \( \{a_n\} \) is a sequence of non-negative number, that

\[
s_n := \sum_{k=0}^{n} a_k
\]

and that

\[
0 < f(x) := \sum_{k=0}^{\infty} a_k x^k < \infty \text{ for } 0 < x < 1.
\]

Hardy and Littlewood [4, Theorem 10] have proved the following theorem.

**Theorem H–L.** If

\[
f(x) \sim (1-x)^{-\rho} L(x) \text{ as } x \to 1-,
\]

where \( \rho \geq 0 \) and \( L(1 - \frac{1}{u}) \) is a logarithmico-exponential function such that

\[
u^{-\delta} \ll L \left(1 - \frac{1}{u} \right) \ll u^\delta,
\]

then

\[
s_n \sim \frac{n^\delta}{\Gamma(\rho+1)} L \left(1 - \frac{1}{n} \right).
\]

* This research was supported in part by Natural Sciences and Engineering Research Council of Canada.
See [3] for definitions and properties of logarithmico-exponential functions. Examples of logarithmico-exponential functions satisfying the above conditions are given by

\[ L \left(1 - \frac{1}{n}\right) := (\log u)^a (\log \log u)^b \ldots, \]

where \(c_1, c_2, \ldots\) are real numbers. Theorem H–L is Tauberian in nature in that it yields information about the asymptotic behavior of \(s_n\) from the asymptotic behavior of \(f(z)\).

The primary object of this note is to supply a simple and straightforward proof of the following generalization of Theorem H–L.

**Theorem 1.** (i) Suppose

\[ \lim_{n \to \infty} \frac{f(z^n)}{f(z)} = \lambda > 0 \quad \text{for } m = 2 \text{ and } m = 3. \]

Then

\[ f(z) = (1 - z)^{-\varphi(z)} \]

where \(\varphi = -\log \lambda \geq 0\) and, for all \(t \geq 1\),

\[ \frac{\varphi(t)}{\varphi(1)} = 1. \]

Moreover

\[ s_n \sim \frac{n^\rho}{\Gamma(p+1)} \left(1 - \frac{1}{n}\right) = \frac{1}{\Gamma(p+1)} f \left(1 - \frac{1}{n}\right) \]

and

\[ s_{n+1} \sim s_n \quad \text{and} \quad \lim_{n \to \infty} \frac{s_n}{s_{n-1}} = \lambda > 0 \quad \text{for } m = 2 \text{ and } m = 3. \]

(ii) Conversely, (2) implies (1).

If follows from Theorem 1.8 in [5] that the integers 2, 3 in (1) can be replaced by any two positive numbers \(p, q \neq 1\) such that \(\log \rho\) is irrational. It was proved in [2] that

\[ \lim_{n \to \infty} \frac{f(z^n)}{f(z)} = \lambda > 0 \]

alone does not imply (1) when \(\lambda < 1\), though (1) and (2) are equivalent when \(\lambda = 1\). Part (i) of Theorem 1 can be deduced from Karamata’s Tauberian theorem and a known result about regularly varying functions (see Theorems 2.3 and 1.8 in [5]). We give an alternate proof which is more direct and more elementary, not involving in particular, the extended continuity theorem for Laplace-Stieltjes transforms on which the proof of Karamata’s theorem is based. Part (ii) of Theorem 1 is interesting in that it shows that (1) and (2) are in fact equivalent.

**2. Preliminary results**

**Theorem 2.** Suppose \(b_n \geq 0\) for \(n = 0, 1, \ldots\),

\[ t_n := \sum_{k=0}^{n} b_k, \quad \text{and} \quad g(x) := \sum_{k=0}^{\infty} b_k x^k < \infty \quad \text{for } 0 < x < 1. \]

If (1) holds and \(\frac{g(x)}{f(x)} \to \lambda\) as \(x \to 1^–\), then \(\frac{t_n}{n} \to \lambda\).

**Proof.** The result is evidently true if \(f(z)\) tends to a finite limit as \(x \to 1^–\). Suppose therefore that \(f(z) \to \infty\) as \(x \to 1^–\).

Case (i): \(a_n > 0\) for \(n = 0, 1, \ldots\). This case follows immediately from the theorem in [2].

Case (ii): \(a_n \geq 0\) for \(n = 0, 1, \ldots\). Let

\[ f^{*}(z) := f(z) + e^x, \quad g^{*}(x) := g(x) + e^x \]

and define \(a_0^*, a_1^*, b_0^*, b_1^*\) in the obvious way. Then \(a_0^* > 0\) for \(n = 0, 1, \ldots\), and, since \(f(z) \to \infty\) as \(x \to 1^–\), (1), is satisfied with \(f^*\) in place of \(f\). Further

\[ \frac{g^{*}(x)}{f^{*}(x)} \to \lambda \quad \text{as } x \to 1^– \quad \text{if and only if} \quad \frac{g^{*}(x)}{f^{*}(x)} \to \lambda \quad \text{as } x \to 1^–, \]

and

\[ \frac{t_n^{*}}{n^{*}} \to \lambda \quad \text{if and only if} \quad \frac{t_n}{n} \to \lambda. \]

Case (ii) now follows from Case (i). \(\square\)

**Lemma 1.** If (1) holds, then, for \(m = 1, 2, \ldots\) and \(\rho = -\log \lambda \geq 0\),

\[ \lim_{x \to 1^–} \frac{f(z^m)}{f(z)} = m^\rho \]

and, for every \(\epsilon \in (0, 1)\),

\[ \lim_{n \to \infty} \frac{s_n}{f(z^m)} = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}. \]

**Proof.** The result is evidently true with \(\rho = 0\) if \(f(z)\) tends to a finite limit as \(x \to 1^–\). Suppose therefore that \(f(z) \to \infty\) as \(x \to 1^–\). It has been
shown in [2] that this together with (1) implies the first conclusion. Further, when $\rho > 0$,
\[(m+1)^{-\rho} = \int_0^1 t^m d\chi(t) \quad \text{with} \quad \chi(t) := \frac{1}{\Gamma(\rho)} \int_0^1 (-\log u)^{m-1} du.
\]
It was proved in [1] that the above implies that, when $\rho = 0$,
\[
\lim_{n \to \infty} \frac{s_n}{f(c^{1/n})} = 1,
\]
and, when $\rho > 0$,
\[
\lim_{n \to \infty} \frac{s_n}{f(c^{1/n})} = \frac{1}{c} \int_0^1 t^{-\rho} (-\log t)^{m-1} dt = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}.
\]

The next lemma has been proved in essence in [1].

**Lemma 2.** If $s_{n+1} \sim s_n$ and $\lim_{n \to \infty} \frac{s_n}{f(c^{1/n})} = \lambda > 0$ where $m$ is a positive integer, then
\[
\lim_{\alpha \to 1^+} \frac{f(\alpha^{m})}{f(\alpha)} = \lambda.
\]

### 3. Proof of Theorem 1

(1) The first conclusion has been proved in [2]. To establish the asymptotic expression for $s_n$ observe that, given $\gamma > 1$,
\[
e^{-n/\log n} < 1 - \frac{1}{n} < e^{-1/n}
\]
for $n$ sufficiently large. Hence for such $n$
\[
\frac{s_n}{f(e^{-n/\log n})} \geq \frac{s_n}{f(1 - 1/n)} \geq \frac{s_n}{f(e^{-1/n})}
\]
and so, by Lemma 1,
\[
\frac{\gamma'}{\Gamma(\rho+1)} \geq \lim_{n \to \infty} \frac{s_n}{f(1 - 1/n)} \geq \lim_{n \to \infty} \frac{s_n}{f(1 - 1/n)} \geq \frac{1}{\Gamma(\rho+1)}.
\]
Since $\gamma' \to 1$ as $\gamma \to 1^-$, it follows that

"\textit{Acta Mathematica Hungarica} 50, 1992"