Biorthogonal System in Approximation Theory

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This is partly a joint work with Prof. B. Wohlmuth.
Let $\mathbb{M} \subset \mathbb{N}$ be an index set, $\{p_n\}_{n \in \mathbb{M}}$ be a subset of an inner product space $H$ equipped with the inner product $\langle \cdot, \cdot \rangle$. This subset is called an orthogonal system if

$$\langle p_n, p_m \rangle = c_n \delta_{mn},$$

where $c_n$ is a non-zero constant and $\delta_{mn}$ is a Kronecker symbol

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{else}. \end{cases}$$

Examples: trigonometric functions, orthogonal wavelets and polynomials, etc.
Let \( \{p_n\}_{n \in \mathbb{M}} \) and \( \{q_n\}_{n \in \mathbb{M}} \) be two subsets of an inner product space \( H \), where \( H \) is equipped with the inner product \( <, \cdot, > \). These two subsets are said to form a biorthogonal system if

\[
<p_n, q_m> = c_n \delta_{mn},
\]

where \( c_n \) is a non-zero constant and \( \delta_{mn} \) is a Kronecker symbol.

Examples: biorthogonal polynomials, biorthogonal wavelets, etc.
Biorthogonal System

Let \( \{ p_n \}_{n \in \mathbb{M}} \) and \( \{ q_n \}_{n \in \mathbb{M}} \) be two subsets of an inner product space \( H \), which is equipped with the inner product \( <, \cdot, > \). Let \( \{ p_n \}_{n \in \mathbb{M}} \) and \( \{ q_n \}_{n \in \mathbb{M}} \) form a biorthogonal system. Then if

\[
f = \sum_{n \in \mathbb{M}} a_n p_n,
\]

\[
a_n = \frac{1}{c_n} < f, q_n >.
\]

Solving a linear system can be reduced to finding a biorthogonal system [Brezinski, 93].
The finite element method is the most popular method for solving partial differential equations. Finite elements are special kinds of splines.

1. Consider a variational problem: find \( u \in V \) such that

\[
a(u, v) = f(v) \quad \text{for all} \quad v \in V,
\]

where \( V \) is a subspace of a Hilbert space, and \( a(\cdot, \cdot) \) is a bilinear form and \( f \) is a linear form.

2. The finite element method for this problem is obtained by replacing the infinite dimensional space \( V \) by a finite dimensional one.

3. The finite dimensional space \( V_h \) is constructed by using a triangulation of the given domain, where we want to solve our problem.
Let $\Omega \subset \mathbb{R}^d$ be a domain (closed and bounded region). Let $\mathcal{T}_h$ be a partition of $\Omega$ into smaller subdomains (intervals, triangles, quadrilaterals, tetrahedra, hexahedra, etc.). The finite element method is characterized by defining a set of basis functions on $\mathcal{T}_h$:

- Each basis function is associated with a point in the domain.
- The size of support of each basis function is of order of the size of a typical subdomain.
- Thus the finite support size is a distinguishing feature of the finite element approach.
Let $\{\phi_1, \cdots, \phi_n\}$ be the set of finite element basis functions on the mesh $\mathcal{T}_h$ and $\mathcal{G}$ be the set of points in $\Omega$ where these basis functions are associated. A finite element basis function is called **nodal** if its value is one at its associated point and zero at other points in $\mathcal{G}$.

**Finite element basis functions in 1D**

**A finite element basis function in 2D**

**A hanging node**
Finite Element Space

The global finite element space is formed by the following process:

- A set of local basis functions are defined on a reference element
- A mapping is computed which maps the reference element to the subdomain
- The basis functions on the reference element are mapped by this mapping to compute the basis functions on the subdomain
- Then global basis functions are computed by glueing these mapped basis function together
In many problems, we have to project a quantity of interest onto a continuous finite element space. Examples are gradient reconstruction, mortar finite elements, mixed formulation of biharmonic, Darcy and elasticity equations. The projection of $\sigma_h$ onto $S_h$ can be expressed as the weak constraint:

$$\int_{\Omega} u_h \mu_h \, dx = \int_{\Omega} \sigma_h \mu_h \, dx, \quad u_h \in S_h, \: \mu_h \in M_h$$

Algebraic constraint (abusing the notation): $u_h = M^{-1} \sigma_h$, $M$ is a Gram matrix

Orthogonal projection is obtained by sing the same discrete space for $u_h$ and $\mu_h$
The space for $u_h$ is $H^1$-conforming, but it suffices to have $L^2$-conforming space for $\mu_h$.

If $S_h$ contains the piecewise polynomial space of degree $p$, it is enough that $M_h$ spans the piecewise polynomial space of degree $p - 1$.

We want to utilize these two properties to construct a space $M_h$ so that basis functions for $S_h$ and $M_h$ form a biorthogonal system.

We get an oblique projection.
$S_h$ is a finite element space, and we call $M_h$ the biorthogonal (or dual) space.

Biorthogonal space $M_h \iff M$ is diagonal.

If $M$ is diagonal:

- The projection is easy.
- Static condensation $\implies$ positive definite system.
- Modification of nodal basis and nested spaces $\implies$ $\mathcal{V}$- or $\mathcal{W}$-cycle multigrid.
- Nonlinear contact problems (variational inequality) $\implies$ Non-penetration can be realized pointwise.
Some Notations

- \( V_h^p \): \( H^1 \)-conforming finite element space of degree \( p \) on a line
- \( \Phi_p := \{ \varphi_1^p, \ldots, \varphi_{p+1}^p \} \): Set of local finite element basis functions of degree \( p \) on the reference edge \( I = [-1, 1] \) using lexicographical ordering

[\( \varphi_1^p \) \( \varphi_2^p \) \( \varphi_3^p \) \( \cdots \) \( \varphi_{p+1}^p \)]

- \( M_h^p \): Dual space spanned by biorthogonal basis functions of degree \( p \)
- \( \Psi_p := \{ \psi_1^p, \ldots, \psi_{p+1}^p \} \): Set of local biorthogonal basis functions of degree \( p \)

\[ \int_I \psi_i^p(s) \varphi_j^p(s) \, ds = \delta_{ij} \int_I \varphi_j^p(s) \, ds \]

Special interest for mortar, Darcy, biharmonic and elasticity mixed finite elements:

\[ V_h^{p-1} \subset M_h^p \]
First approach: Lagrange nodal FE. Optimal a priori estimates only for $p = 1$ and $p = 2$.

Second approach: Lagrange hierarchical FE. No nodal property. Existence of optimal biorthogonal base. BUT [Oswald et al. 01] larger support ($\geq 3$ edges).

Third approach: Gauss–Lobatto nodal FE. Optimal biorthogonal spaces for a finite element space of any order with equal support.

Next slide: examples of these three types of basis functions $\{\phi_1^p, \ldots , \phi_m^p\}$ for $p = 2, 3, 4$. Here $m = p + 1$. 
Finite Element Basis Functions on the Reference Edge
There are two types of basis functions in one dimension.

- Two basis functions associated with the vertices
- \( p - 1 \) inner basis functions

The glueing condition does not affect the inner basis functions. It only affects the two vertex basis functions.
\( \Psi_p \) and \( \Phi_p \) span the space of polynomials of degree \( p \), say \( \mathcal{P}_p(I) \). Let us regard \( \Psi_p \) and \( \Phi_p \) as column vectors with an abuse of notation.

\[
\Phi_p = [\phi_1^p, \cdots, \phi_{p+1}^p]^T, \quad \Psi_p = [\psi_1^p, \cdots, \psi_{p+1}^p]^T.
\]

Since \( \Psi_p = \{\psi_1^p, \cdots, \psi_{p+1}^p\} \) also spans a polynomial space of degree \( p \), there exists a matrix \( N^p \) with

\[
N^p \in \mathbb{R}^{p \times p+1}
\]

such that

\[
\Phi_{p-1} = N^p \Psi_p.
\]

Local space \( \Psi_p \) contains the polynomial space of degree \( p \), but the global space may not contain even a piecewise polynomial space of degree \( p - 1 \).
Lemma

\[ V_h^{p-1} \subset M_h^p \text{ if and only if } \]

\[
\begin{align*}
n_{1,1}^p &= n_{p,p+1}^p \quad \text{and} \quad n_{p,1}^p = n_{1,p+1}^p = 0, \\
n_{i,1}^p &= n_{i,p+1}^p = 0 \quad \text{for all} \quad 2 \leq i \leq p - 1,
\end{align*}
\]

where \( n_{i,j}^p \) is the \((i, j)\)-th entry of the matrix \( N^p \).
Analytic Condition

1. If the nodal points $x_1^p, \ldots, x_{p+1}^p$ are symmetric, these conditions reduce to

$$\varphi_1^p \in \text{span}\{\varphi_{2}^{p-1}, \ldots, \varphi_{p}^{p-1}\} \perp \text{ and } \varphi_{p+1}^p \in \text{span}\{\varphi_{1}^{p-1}, \ldots, \varphi_{p-1}^{p-1}\} \perp.$$

2. If we define $\varphi_1^p = c_1(1-x)L_p'(x)$, and $\varphi_{p+1}^p = c_2(1+x)L_p'(x)$, then the above conditions are satisfied ($L_p$ is the Legendre polynomial of degree $p$).

3. If $S_p := \{-1 =: x_1^p < x_2^p < \cdots < x_{n+1}^p =: 1\}$ be the zeros of polynomial $(1-x^2)L_p'(x)$, then $S_p$ is the set of Gauss–Lobatto nodes of order $p$. 

\[
p = 3
\]
Biorthogonality in Finite Elements

Example: \( p = 3 \)

\[
\begin{bmatrix}
1 & \frac{1+\sqrt{5}}{10} & \frac{1-\sqrt{5}}{10} & 0 \\
0 & \frac{4}{5} & \frac{4}{5} & 0 \\
0 & \frac{1-\sqrt{5}}{10} & \frac{1+\sqrt{5}}{10} & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
\frac{11}{15} & \frac{2}{5} & -\frac{1}{5} & 0 \\
\frac{4}{15} & \frac{4}{5} & \frac{4}{5} & \frac{4}{15} \\
0 & -\frac{1}{5} & \frac{2}{5} & \frac{11}{15} \\
\end{bmatrix}
\]

\[ \mathcal{N}^3_{\text{Gauss-Lobatto}} \]

\[ \mathcal{N}^3_{\text{Lagrange}} \]

\( V^2_h \not\subseteq M^3_h \)

\( V^2_h \subset M^3_h \)

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Biorthogonal System in Approximation Theory
Analytic Condition

Gauss–Lobatto nodes $\implies$ there exists a Quadrature formula exact for all polynomials of degree $\leq 2p - 1$

$$\int_{I} \varphi_{l}^{P}(\hat{s}) \varphi_{i}^{p-1}(\hat{s}) \, d\hat{s} = \sum_{j=1}^{p+1} w_{j}^{p} \varphi_{l}^{P}(x_{j}^{p}) \varphi_{i}^{p-1}(x_{j}^{p}) = 0, \quad \begin{cases} l = 1, 2 \leq i \leq p \\ l = p + 1, 1 \leq i \leq p - 1 \end{cases}$$

Theorem

$V_{h}^{p-1} \subset M_{h}^{p}$ if and only if the finite element basis of $V_{h}^{p}$ which defines $M_{h}^{p}$ is based on the Gauss–Lobatto points.

$\implies$ Optimal a priori estimates for mortar finite elements, biharmonic, Darcy and elasticity equations.
Biorthogonal basis functions for cubic and quartic finite element spaces

$p = 3$

$p = 4$
If a finite element space has a tensor product structure, the biorthogonal basis functions can be constructed by using the tensor product construction. This includes meshes of $d$-paralleloptopes.

In simplicial meshes, the lowest order case is straightforward. The biorthogonal basis with such optimal approximation property does not exist for the quadratic case. Relax the notion and use quasi-biorthogonality.

The situation for serendipity elements is similar.
Numerical Results for Biharmonic Equation

We want to find \( u \in H^2_0(\Omega) \) such that \( \int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} f \, v \, dx \), \( v \in H^2_0(\Omega) \) in \( \Omega := (0, 1)^2 \). Here we put \( \phi = \Delta u \), and get the weak form using the clamped boundary condition

\[
\int_{\Omega} \phi \psi \, dx = \int_{\Omega} \Delta u \psi \, dx = - \int_{\Omega} \nabla u \cdot \nabla \psi \, dx.
\]

Table: Discretization errors in different norms for the clamped boundary condition

<table>
<thead>
<tr>
<th>level</th>
<th># elem.</th>
<th>( |u - u_h|_{0,\Omega} )</th>
<th>( |u - u_h|_{1,\Omega} )</th>
<th>( |\Delta u - \phi_h|_{0,\Omega} )</th>
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<tr>
<td>0</td>
<td>32</td>
<td>5.34290e-01</td>
<td>6.32693e-01</td>
<td>6.32041e-01</td>
</tr>
<tr>
<td>1</td>
<td>128</td>
<td>3.26972e-01</td>
<td>0.71</td>
<td>4.01635e-01</td>
</tr>
<tr>
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<td>512</td>
<td>1.30302e-01</td>
<td>1.33</td>
<td>1.89139e-01</td>
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<tr>
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<td>3.99107e-02</td>
<td>1.71</td>
<td>8.32646e-02</td>
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<tr>
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<tr>
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<tr>
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<td>524288</td>
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<td>1.99</td>
<td>4.70081e-03</td>
</tr>
</tbody>
</table>
Conclusion:
- The importance of biorthogonality is highlighted
- The biorthogonal system using nodal finite element space of degree $p$ is constructed
- The approximation property of the biorthogonal system is analyzed

Future work:
- Extend the idea to other splines: e.g., splines with higher smoothness
- Quasi-biorthogonality may be a key where biorthogonality is not possible

Thank you